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Existence and stability of nontopological fermion strings

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We demonstrate the existence of a class of nontopological fermion string solutions. The fermion configurations exist because of a nontrivial coupling between the fermions and a real scalar field, and their long, thin regions of false vacuum are supported against collapse by the pressure of massless particles trapped in the interior, rather than by the supporting pressure given by Pauli's exclusion principle in white dwarfs and neutron stars. Our solutions, even with angular momentum, appear unstable to perturbations in the scalar fields.

In recent years, the active interplay between particle physics and cosmology has made frequent use of scalar fields as driving the dynamics and the formation of structure in the early Universe. Phase transitions of quantum fields in the early Universe may produce very thin tubes of false vacuum, known as cosmic strings.[1,2] Topological defects demand the presence of internal symmetries, whose associated charges are absolutely conserved regardless of the dynamics. On the other hand, nontopological solitons (NTS's) occur in theories with a continuous symmetry and therefore a conserved Noether charge carried by fields confined to a finite region of space. Previous investigations of NTS's have, for the most part, concentrated on theories with global symmetries. The simplest example of a NTS is the so-called Q ball that can appear in a U(1)-invariant theory with a single complex scalar field that has nonlinear self-interactions.[3] The NTS's found in the literature all possess spherical symmetry.[3–5] Recently, Copeland, Kolb, and Lee[6] (CKL) investigated the possible existence of nontopological string solutions, analogous to the topological-inspired vortex solutions.[7] Such solutions possess a cylindrical symmetry rather than the spherical symmetry of Friedberg, Lee, and Sirlin[4,5] (FLS).

Friedberg and Lee[5] extended their studies of nontopological soliton solutions to include also the fermion field. The interaction between a scalar field and a set of fermion fields is investigated by decomposing the total Hamiltonian H into a sum of two terms: $H=H_{\rm qcl}+H_{\rm corr}$, where $H_{\rm qcl}$ denotes the quasiclassical part and $H_{\rm corr}$ the quantum correction. General theorems have been given for $H_{\rm qcl}$ concerning the existence of solutions, the general properties of such solitons, and the condition under which the lowest-energy state of $H_{\rm qcl}$ is a soliton solution, not the usual plane-wave solution.

In this paper we investigate the possible existence of nontopological fermion string solutions analogous to those obtained by Copeland, Kolb, and Lee[6] except, in our model, there is a spin -$\frac{1}{2}$ field ψ and a real scalar field σ. Such fermionic configurations exist because of a nontrivial coupling between the fermions and the real scalar field, and their long thin regions of false vacuum are supported against collapse by the pressure of massless particles trapped in the interior, rather than by the supporting pressure given by Pauli's exclusion principle in white dwarfs and neutron stars. Our fermionic configuration is a nondissipative solution with cylindrical symmetry to the classical field equations that, for fixed charge Q, represents the field configuration with lower energy than the free-particle solutions and, hence, is stable against decay into free particles. However, the strings appear unstable to forming spheres when we allow the charge to migrate along the string by perturbing the scalar field solutions, even if we introduce a current along the string or give it some angular momentum since there is no topological reason for their stability unlike the case of cosmic strings. We discuss the properties of Higgs-type bosons, including the possibility that the nontrivial coupling between the fermions and the Higgs-type fields may cause the fermions to decay and make the strings unstable.

In the model we consider in the following, the interaction between the real scalar field σ and the fermion field ψ is

$$-f\bar{\psi}\psi\sigma \ ,$$

where f is the coupling constant, and $\bar{\psi}$ is the adjoint of ψ, making $\bar{\psi}\psi$ a Lorentz scalar. Let the fermion mass (in the normal vacuum) be m. For simplicity we assume

$$m-f\sigma_0=0 \ ,$$

so that the fermion has a zero effective mass in the false vacuum. Therefore the expectation values of the Higgs-boson fields modify the masses of other fields.

To illustrate the basic mechanism, consider the following example of a nontopological string. The theory contains an additive quantum number N (as the baryon number) carried by a spin-$\frac{1}{2}$ field ψ, with elementary field quantum having $N=\pm 1$. In addition, there is a real scalar field σ. Take the self-interaction of σ to be the typical degenerate vacuum form (in units $\hbar=c=1$):

$$U(\sigma)=\tfrac{1}{2}\mu^2\sigma^2\left[1-\frac{\sigma}{\sigma_0}\right]^2 , \qquad (1)$$

where $\mu=g\sigma_0$ is the mass of σ. We may take $\sigma=0$ to be the normal vacuum state and $\sigma=\sigma_0$ the false (or degenerate) vacuum. The string contains an interior in which $\sigma \simeq \sigma_0$, a shell of width $\sim\mu^{-1}$, over which σ changes from σ_0 to 0, and an exterior that is essentially the normal vacuum. The Hamiltonian density $\mathcal{H}$ of the system is

$$\mathcal{H}=\tfrac{1}{2}\pi^2+\tfrac{1}{2}(\nabla\sigma)^2+U(\sigma)+\psi^\dagger(-i\boldsymbol{\alpha}\cdot\nabla+\beta m-f\beta\sigma)\psi , \tag{2}$$

where

$$[\pi(\mathbf{r},t),\sigma(\mathbf{r}',t)]=-i\delta^3(\mathbf{r}-\mathbf{r}') ,$$
$$\{\psi(\mathbf{r},t),\psi^\dagger(\mathbf{r}',t)\}=\delta^3(\mathbf{r}-\mathbf{r}') , \tag{3}$$

and α and β are 4×4 Dirac matrices, $\mathbf{r}=(\rho,\theta,z)$ and $\rho=\sqrt{x^2+y^2}$.

Let the fermion mass in the normal vacuum be m. For simplicity we assume

$$m-f\sigma_0=0 , \tag{4}$$

so that the fermion has a zero effective mass in the false vacuum. We introduce an operator $\chi(\mathbf{r},t)$, defined by

$$\chi(\mathbf{r},t)=\sigma(\mathbf{r},t)-\sigma_c(\mathbf{r}) , \tag{5}$$

where $\sigma_c(\mathbf{r})$ is a time-independent c-number function that satisfies

$$\sigma_c\to 0 \quad \text{as } \rho\to\infty ; \tag{6}$$

the detailed form of σ_c is yet to be determined.

It is convenient to expand the operator $\psi(\mathbf{r},t)$ in terms of a complete set of orthonormal c-number time-independent spinor functions $u_l(\mathbf{r})$ and $v_l(\mathbf{r})$:

$$\psi(\mathbf{r},t)=\sum_{l=1}^{\infty}[a_l(t)u_l(\mathbf{r})+b_l^\dagger(t)v_l(\mathbf{r})] , \tag{7}$$

where u_l and v_l are determined by

$$[-i\boldsymbol{\alpha}\cdot\nabla+\beta(m-f\sigma_c)]\times\begin{cases}u_l\\ v_l\end{cases}=\epsilon_l\times\begin{cases}u_l ,\\ -v_l ,\end{cases} \tag{8}$$

in which the subscript $l=1,2,\ldots$ is arranged so that $0<\epsilon_1\le\epsilon_2\le\cdots$. By using (3), one sees that the operators a_l and b_l all anticommute, while

$$\{a_l,a_m^\dagger\}=\{b_l,b_m^\dagger\}=\delta_{lm} . \tag{9}$$

In terms of χ, π, a_l, and b_l, the total Hamiltonian H may be written as a sum of two terms, a quasiclassical part $H_{\rm qcl}$ and a quantum correction $H_{\rm corr}$:

$$H=\int\mathcal{H}d^3r=H_{\rm qcl}+H_{\rm corr} , \tag{10}$$

where

$$H_{\rm qcl}=\int[\tfrac{1}{2}(\nabla\sigma_c)^2+U(\sigma_c)]d^3r+\sum_{l=1}^{\infty}\epsilon_l(a_l^\dagger a_l+b_l^\dagger b_l) , \tag{11}$$

and

$$H_{\rm corr}=\int\{\tfrac{1}{2}\pi^2+[-\nabla^2\sigma_c+U'(\sigma_c)-f\psi^\dagger\beta\psi]\chi+\tfrac{1}{2}(\nabla\chi)^2+\tfrac{1}{2}U''(\sigma_c)\chi^2+\frac{1}{3!}U'''(\sigma_c)\chi^3+\frac{1}{4!}U''''(\sigma_c)\chi^4\}d^3r-\sum_{l=1}^{\infty}\epsilon_l+\text{counterterms} . \tag{12}$$

The possible existence of string configurations can be seen by assuming cylindrical symmetric solutions in the equation of the scalar field. We use trial solutions with the cylindrical symmetry

$$\sigma_c=\begin{cases}\sigma_0 & \text{for } \rho\le R ,\\ \sigma_0\exp[-\mu(\rho-R)] & \text{for } \rho\ge R .\end{cases} \tag{13}$$

The fermion field ψ is confined to the interior of the string which is the false vacuum. For the treatment of a gas of fermions whose interactions are neglected, the Fermi-Dirac statistics must be used. The restriction of the Pauli exclusion principle on fermions will only permit one particle per elementary state. Thus at $T=0$, we would expect

$$\bar{n}(\epsilon)=\begin{cases}1 & \text{if } \epsilon<\epsilon_F ,\\ 0 & \text{if } \epsilon>\epsilon_F ,\end{cases} \tag{14}$$

where ϵ_F is the Fermi energy. The number and kinetic energy per unit length are

$$\frac{dN}{d\eta}=\frac{V}{2\pi}\epsilon_F^2 , \tag{15}$$

$$\frac{dE_k}{d\eta}=\frac{V}{3\pi}\epsilon_F^3 , \tag{16}$$

where η is a dimensionless scale $\eta\equiv z\mu$. We have

$$\frac{dE_k}{d\eta}=\frac{2\sqrt{2}}{3R}\left[\frac{dN}{d\eta}\right]^{3/2} . \tag{17}$$

The shell contains a surface energy, and the surface energy per unit length is

$$\frac{dE_s}{d\eta}=\frac{1}{\mu}\int_0^\infty[\tfrac{1}{2}(\nabla\sigma_c)^2+U(\sigma_c)]2\pi r\,dr . \tag{18}$$

Substituting Eq. (14) into (18), we have

$$\frac{dE_s}{d\eta}=\frac{7\pi R}{12}\sigma_0^2+\frac{31\pi}{144\mu}\sigma_0^2 . \tag{19}$$

The larger N in the string is, the larger the radius R is, so for $R\gg\mu^{-1}$ we can ignore the last term in Eq. (19) and write an upper bound for the total energy density:

$$\left[\frac{dE}{d\eta}\right]_{\rm true}\le\left[\frac{dE}{d\eta}\right]_{\rm trial}\simeq\frac{2\sqrt{2}}{3R}\left[\frac{dN}{d\eta}\right]^{3/2}+2\pi Rs\mu^{-1} , \tag{20}$$

where $s=\frac{7}{24}\mu\sigma_0^2$. By using

$$\frac{d}{dR}\left[\frac{dE}{d\eta}\right]=0\ , \tag{21}$$

we obtain the mass per unit length:

$$\frac{dM}{d\eta}=\left[\frac{16\sqrt{2}\pi s}{3\mu}\right]^{1/2}\left[\frac{dN}{d\eta}\right]^{3/4}\ . \tag{22}$$

Because the exponent of $dN/d\eta$ is <1, where $dN/d\eta$ is large, the string mass density $dM/d\eta$ is always less than that of the free particle solution, and that ensures its stability against decay into free particles. Equation (22) should be compared with the energy density of the plane-wave solution for N free particles:

$$\left[\frac{dE}{d\eta}\right]_{\text{free}}=m\frac{dN}{d\eta}\ . \tag{23}$$

The nontopological string solution is stable against decay into free particles when it is formed with a lower energy density than the free-particle distribution; comparing Eqs. (22) and (23), this occurs when

$$\frac{dN}{d\eta}\geq 2\pi^2\left[\frac{4}{m}\right]^4\left[\frac{s}{\mu}\right]^2=\left[\frac{dN}{d\eta}\right]_c\ , \tag{24}$$

where $m=f\sigma_0$, $s=7/24\mu\sigma_0^2$. Thus if the number density at formation is larger than $(dN/d\eta)_c$, then it is energetically favorable to form fermion strings.

Under a Lorentz transformation along the z axis we have the relation between the original z-independent string in a frame (z',t') and the Lorentz-boosted coordinate frame (z,t):

$$z'=\gamma(z-vt),\quad t'=\gamma(t-vz)\ , \tag{25}$$

where $\gamma=(1-v^2)^{-1/2}$, $v=dz/dt$: The energies in the two frames are also related by

$$E'=\gamma(E-vP_z)\ , \tag{26}$$

where P_z is the particle momentum in the z direction. Equivalence of z-dependent and z-independent solutions is obtained by taking $v=k/\omega$, in Eq. (25). Thus we can always Lorentz boost our k-dependent solution to an inertial frame where there is no such dependence, and use our results of Eq. (24).

Next, we investigate the stability of these solutions to arbitrarily small perturbations in the fields. By using Eqs. (15) and (16), we obtain that the average kinetic energy $\bar{\epsilon}$ of a fermion is

$$\bar{\epsilon}=\tfrac{2}{3}\epsilon_F=\tfrac{2}{3}\left[\int\psi_F^\dagger H_F^2\psi_F\right]^2\ , \tag{27}$$

where ϵ_F is the Fermi energy and the normalized wave function ψ_F satisfies

$$H_F\psi_F=\epsilon_F\psi_F\ , \tag{28}$$

where

$$H_F=-i\boldsymbol{\alpha}\cdot\nabla+\beta m-f\beta\sigma_c\ , \tag{29}$$

and

$$H_F^2=-\nabla^2+(m-f\sigma_c)^2-g\rho_z\boldsymbol{\sigma}\cdot(\nabla\sigma_c)\ . \tag{30}$$

We define

$$\Phi=N^{1/2}\psi_F\ , \tag{31}$$

then the string energy is given by the minimum of the functional $E(\Phi,\sigma_c)$ at a fixed N:

$$E(\Phi,\sigma_c)=N^{1/2}\left[\int\Phi^\dagger H_F^2\Phi\, d^3r\right]^{1/2}+\int[\tfrac{1}{2}(\partial_0\sigma_c)^2+\tfrac{1}{2}(\nabla\sigma_c)^2+U(\sigma_c)]d^3r\ . \tag{32}$$

Under arbitrary perturbations

$$\Phi\to\tilde{\Phi}=\Phi+\delta\Phi(\mathbf{r},t),\quad \sigma\to\tilde{\sigma}=\sigma+\delta\sigma(\mathbf{r},t)\ . \tag{33}$$

We have

$$\delta E=\tfrac{1}{3}N^{1/2}\left[\int\Phi^\dagger H_F^2\Phi\, d^3r\right]^{-1/2}\int d^3r\{\Phi^\dagger H_F^2(\delta\Phi)+(\delta\Phi^\dagger)H_F^2\Phi+\Phi^\dagger[2f(f\sigma_c-m)(\delta\sigma_c)-g\rho_z\boldsymbol{\sigma}\cdot\nabla(\delta\sigma_c)]\Phi\}$$
$$+\int d^3r[(\partial_0\delta\sigma_c)(\partial_0\sigma_c)+(\nabla_i\delta\sigma_c)(\nabla_i\sigma_c)+U'_{\sigma_c}\delta\sigma_c]\ . \tag{34}$$

The requirement of charge conservation, $\delta N=0$, becomes

$$\delta N=\int d^3r[(\delta\Phi^\dagger)\Phi+\Phi^\dagger(\delta\Phi)]=0\ . \tag{35}$$

Upon substitution into Eq. (34) and after integrating by parts,

$$\delta E=\tfrac{1}{3}N^{1/2}\left[\int\Phi^\dagger H_F^2\Phi\, d^3r\right]^{-1/2}\int d^3r[(\Phi^\dagger H_F^2-\Phi^\dagger\epsilon_F^2)\delta F]+\int d^3r[-(\nabla_i\nabla^i)\sigma_c+U'_{\sigma_c}]\delta\sigma_c\ , \tag{36}$$

where $U'_{\sigma_c}=dU/d\sigma_c$. The first-order variation in E vanishes from the equations of motion as expected. We will use the fact that under a perturbation in Φ or ψ_F, the Fermi energy ϵ_F calculated from ψ_F also change: $\epsilon_F\to\epsilon_F+\delta\epsilon_F$. Since the Fermi energy formula

$$\epsilon_F=\frac{1}{N}\int\Phi^\dagger H_F\Phi\, d^3r\ , \tag{37}$$

we see how ϵ_F is to be interpreted as a function of Φ:

$$\delta\epsilon_F=\frac{\epsilon_F}{N}\int d^3r[\Phi^\dagger\delta\Phi+(\delta\Phi^\dagger)\Phi]+\frac{1}{N}\int\Phi^\dagger(\delta H_F)\Phi\, d^3r\ . \tag{38}$$

The second variation is easily obtained from Eq. (34):

$$(\delta^2 E)_N = \int d^3 r\, \Theta^\dagger H \Theta \ , \tag{39}$$

where

$$\Theta = \begin{bmatrix} \left[\dfrac{1}{3\epsilon_F}\right]^{1/2} \delta\Phi \\ \delta\sigma_c \end{bmatrix} , \tag{40}$$

and

$$H = -\nabla_i \nabla^i + \begin{bmatrix} a & b \\ c & d \end{bmatrix} , \tag{41}$$

where

$$a = (mf\sigma_c)^2 - g\rho_z \boldsymbol{\sigma}\cdot\nabla\sigma_c - \epsilon_F^2 \ ,$$

$$b = \Phi[2f(f\sigma_c - m) - g\rho_z \boldsymbol{\sigma}\cdot\nabla] \ ,$$

$$c = \Phi^\dagger[2f(f\sigma_c - m) - g\rho_z \boldsymbol{\sigma}\cdot\nabla] \ ,$$

$$d = 6g^2\sigma_c^2 - 6g^2\sigma_c\sigma_0 + g^2\sigma_0^2 \ .$$

The stability of a particular configuration is established by evaluating Eq. (39) or at least the sign of Eq. (39). If $(\delta^2 E)_N > 0$ this implies that the perturbation has produced a new configuration of higher energy than the original string configuration, and so the original is stable. The crucial equations for our purpose are

$$\left[\frac{2g^2}{\mu}\right](\delta^2 E)_N = \sum_i{}' c_i^2 \lambda_i \ , \tag{42}$$

where the sums exclude zero λ_i's, and the c_i are defined for an arbitrary eigenfunction Θ_i by $\Theta = \sum_i c_i \Theta_i$ where Θ_i are a complete orthonormal set of real function, and λ_i is the set of eigenvalues satisfying

$$H\Theta = -d^2\Theta / dt^2 \ . \tag{43}$$

We can use an analogous argument in Ref. 4 to demonstrate that there exists at least one negative eigenvalue for the (θ, r) component of H since the s state must have lower energy than the p state. From Eq. (42) the corresponding $(\delta^2 E)_N$ would be less than zero. Without going into a type of the nontopological soliton analysis performed, we cannot say whether there exist more than one negative eigenvalue of H.

The next step in the analysis is to investigate possible perturbations allowed by the equations of motion. If we have azimuthal symmetry in the ψ field, and the solution for ψ is independent of z, then we can have a perturbation of the form

$$\delta\psi_e(\rho,\theta,z) = \epsilon\psi_s(\rho,\theta)\cos(kz) \ , \tag{44}$$

where $|\epsilon| \ll 1$ and $\psi_s(\rho,\theta)$ is the s-state eigenfunction that has a negative eigenvalue when acted upon by H. For this type of perturbation, Eq. (42) becomes

$$\left[\frac{2g^2}{\mu}\right](\delta^2 E)_N = \sum_i{}' c_i^2(\lambda_i + k^2) \ , \tag{45}$$

where k^2 comes from the $\nabla_z \nabla^z$ term in the Laplacian of Eq. (41).

We can see that for sufficiently small k^2, i.e., long-wavelength perturbations, the eigenvalue corresponding to the s state will remain negative; hence, $(\delta^2 E)_N < 0$ for some perturbations. Only for small-wavelength perturbations along the z axis is the string solution stable. Under these perturbations, the effective string tension varies along the string. Regions with a very small string tension which have lost the number become surrounded by the regions with a high tension which have gained the number. The result is that the low tension regions become pinched off and spherical solutions form with an intrinsic size $\sim k^{-1}$.

If we consider the rotating nontopological strings, the result is to increase the radius of string R and decrease the kinetic energy $dE_k/d\eta$. Since there is still a translational invariance in the ρ-θ plane, the negative eigenvalue λ still exists. However, the magnitude of the eigenvalue λ decreases, because of the θ dependence in the solution and the $\nabla_\theta \nabla^\theta$ term in the Laplacian of Eq. (41) acting on Eq. (42). The decay time goes roughly as $t_{\text{dec}} \sim 1/|\lambda| + k^2$. So for decreasing $|\lambda|$, the decay time increases. Therefore the effect of angular momentum is to mitigate, but not remove, the instability.

It has been previously demonstrated that putting a current along the string is equivalent to Lorentz boosting a string in an inertial frame with a current to a frame moving with relative velocity k/ω. Hence, we expect the physics of the perturbation analysis when there is no z dependence on ψ to follow through even when there is initially a current present.

At present, very little is known about the nature of the Higgs bosons, except that they should be massive, spin 0, and have expectation values which should modify the masses of other fields. Thus the $(dN/d\eta)_c$ for nontopological strings could be less than the above estimate, depending upon the theory. It is quite likely that the scalar fields are in fact phenomenological fields, and their interactions should be described by an effective Lagrangian. The nontrivial coupling between the fermions and the Higgs-type fields may cause the fermions to decay and make the string unstable. We will address these problems with more details in a separate publication.

We have investigated the possible existence of nontopological fermion strings. Although the energetics and equations of motion allow for them to be formed, it appears that they would be unstable to the formation of spherical solitons.

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