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String production as a result of thermal fluctuations

J. M. Guerra, Jr. and G. C. Marques
Instituto de Física, Universidade de São Paulo, São Paulo, São Paulo, Brazil

S. J. Rodrigues
Departamento de Matemática, Universidade de Brasília, Brasília, Distrito Federal, Brazil
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Based on the analysis of the free energies of topological defects, we develop further the study of phase transitions in field theory at finite temperature. In the case of strings we have shown how one can get, in the dilute-gas approximation, explicit expressions for the length of the string as well as the density contrast in terms of the free energy per unit length of the string. In the high-temperature limit one can get explicit expressions for all relevant quantities up to the one-loop approximation. When applied to the SO(10) model we get good phenomenological results. In particular we derive, in a simple manner, the scale-independent Zel'dovich spectrum with the right order of magnitude.

I. INTRODUCTION

The large-scale structure of the Universe is a feature whose explanation requires the existence, in some stage of its evolution, of small fluctuations in the density of an otherwise homogeneous universe. The scenarios proposed for explaining this structure evokes processes, for the origin of the inhomogeneities, that take place soon after the moment of creation of the Universe. The inhomogeneities acted as seeds in the formation of structures that become effective only after the decoupling of photons from ordinary matter.

The most relevant question in this context, and as yet not settled, is the one related to the nature of the inhomogeneities. One possibility is the existence of massive particles, different from ordinary baryons, in various stages of the Universe. Fluctuations in the density of these particles in space act as seeds responsible for structure formation. This possibility has been exploited by a very large number of researchers.[1]

Among the physical processes that give rise to inhomogeneities we would like to emphasize the role of phase transitions. As pointed out by Kibble[2] topological defects arise in cosmological phase transitions making them natural candidates for seeds that emerge in the very early Universe. Among these defects it seems that strings are one of the best candidates. Strings may give rise to density perturbations from which galaxies evolve.[3,4]

In this paper we will be concerned with the relevance of defects in phase transitions and their possible role in symmetry restoration. We will be concerned only with topologically stable defects.

Our motivation for the study of the role of topological defects in phase transitions is twofold: in the first place, because this study, in the context of finite-temperature field theory, has never been explored in a systematic way, and in the second place, because one can argue, as will be shown later, that phase transitions might be induced by the condensation of defects as a result of thermal fluctuations. When one takes into account the role of defects one gets a novel picture for the phase transition. This picture might be relevant, in field theory at finite temperature, to the understanding of the large-scale structure of the Universe.

There are two basic questions to be answered in any string-based scenario for structure formation. These questions are the loop formation (sizes, structure, and density) and the string evolution. In Vilenkin's approach,[5] for instance, the formation process of a small closed loop seems to play a crucial role. The evolution of strings has been studied by Kibble[6] and Vilenkin.[3]

In this paper we will be concerned only with string formation in the early Universe. Our analysis differs from previous ones in the fact that we have studied string formation as a result of thermal fluctuations. We will show that thermal fluctuations induce the production of a large number of strings. As a matter of fact, above a critical temperature (T_c) even infinite strings (of the Nielsen-Olesen type) can be produced. Under these circumstances the system goes to a new phase for $T > T_c$. In this phase there is condensation of strings. Our conclusion is that thermal fluctuations are extremely relevant in any string-driven structure formation.

The thermodynamical argument on which we have based our argument is the so-called Kosterlitz-Thouless picture of phase transition.[7] As a matter of fact in their classical paper Kosterlitz and Thouless were analyzing spin configurations called vortices which are configurations analogous to infinite strings studied here.

The plan of our paper is the following. In Sec. II we establish the general framework. In Sec. III we give formal expressions, in field theory at finite temperatures, for the free energy of topological defects. These expressions are fairly simple in the high-temperature limit. In Sec. IV we consider the case of strings. We determine the free energy of the Nielsen-Olesen string in the high-temperature limit and determine the condensation temperature of such strings. In Sec. V we give the results for the SO(10) model. Section VI is particularly relevant for cosmological implications since in this section we give explicit expressions for the length of strings and the contrast density in the dilute-gas approximation. When applied to cosmology our results seem to lead to observationally compatible results for the contrast density induced by strings, since it gives results with the proper order of magnitude. Furthermore the contrast density is practically temperature (time) independent. That is, we get Zel'dovich's "constant curvature" spectrum.

II. FREE ENERGY OF TOPOLOGICAL DEFECTS

In order to study the problem of symmetry breaking we shall employ the so-called variational method.[8] Let φ_i be a solution (herewith we will call them background-field configurations) of the following variational problem:

$$\left.\frac{\delta\Gamma(\varphi,T)}{\delta\varphi}\right|=0 , \tag{2.1}$$

where Γ is a group-invariant functional depending also on external parameters that we are labeling by T. Let us assume for the moment that $\Gamma(\varphi_i,T)$ represents the free energy of the system in the presence of the background field φ_i. Let us designate by φ_V the solution of (2.1) independent of x (the vacuum of the theory).

The study of phase transitions can then be pursued by analyzing the difference between free energies associated to different backgrounds (one of them we take as the vacuum), that is, we analyze the difference

$$F(\varphi_i,T)\equiv\Gamma(\varphi_i,T)-\Gamma(\varphi_V,T) . \tag{2.2}$$

In the case of the field theory at finite temperature, T is the temperature and, as we shall see in the following, Γ is the effective action defined by

$$\Gamma\equiv\sum_n\frac{1}{n!}\int\cdots\int dx_1\cdots dx_n\Gamma^{(n)}(X_1\cdots X_n)\varphi(X_1)\varphi(X_2)\cdots\varphi(X_n) , \tag{2.3}$$

where $\Gamma^{(n)}$ is the one-particle-irreducible Green's functions of the theory.

In the case in which, as in the semiclassical method, φ_i corresponds to a topological defect (φ_D), then we define

$$F(\varphi_D,T)\equiv\Gamma(\varphi_D,T)-\Gamma(\varphi_V,T) \tag{2.4}$$

as the free energy associated to a topological defect.

We would like to stress the difference between the approach that we propose here and the so-called effective-potential method. From (2.2) it is very easy to see the difference. In fact, the effective potential is a thermodynamic functional analogous to $F(\varphi_D,T)$ except for the substitution $\varphi_D\to\overline{\varphi}$ where $\overline{\varphi}$ is the generic space-time-independent field configuration. That is,

$$V_{\text{eff}}(\overline{\varphi})\equiv\frac{T}{L^3}[\Gamma(\overline{\varphi})-\Gamma(\varphi_V)] . \tag{2.5}$$

Like in the phenomenological Landau theory of phase transition,[9] one looks for the extrema, Eq. (2.1), of the thermodynamic potential $V_{\text{eff}}(\overline{\varphi},T)$. Assume that at T_0 the absolute minimum of V is at $\overline{\varphi}_0$. The different minima of theories with spontaneous symmetry varies with T. Assume that at some temperature T_1 other minimum (φ_i) becomes equal to that at φ_0, i.e., $V_{\text{eff}}(\varphi_0,T_0)=V_{\text{eff}}(\varphi_1,T_1)$ at T_1 and becomes the new absolute minimum when $T<T_1$. This situation describes a first-order phase transition.

The point that we would like to stress is that the effective potential gives a description of the phase transition in terms of space-time-independent field-theoretical configurations. We believe that a better description is achieved by computing the free energies associated to other background fields.[10] In this section we will analyze the expression of the Gibbs energies for nontrivial background fields.

In the following we will justify expressions that give the (Gibbs) free energies for nontrivial backgrounds.[11] Although, within the one-loop approximation, our expressions give results that are by now standard and can be found in text books,[12] we present this derivation due to the fact that it is fairly general and is an extension, to finite temperature, of the background-field method.[13]

Assume that φ_b is a generic field configuration and let us compute the thermodynamical properties of the system in the presence of such a background field. This should be inferred from the functionals $\tilde{Z}[J,\varphi_b]$ and $\tilde{W}(J,\varphi_b)$ defined by

$$\tilde{Z}[J,\varphi_b]=e^{-\tilde{W}(J,\varphi_b)}=\int D[\varphi]\exp\left[-S[\varphi-\varphi_b]+\int_0^\beta d\tau\int d^3\mathbf{x}\,J(x)\varphi(x)\right] . \tag{2.6}$$

By means of a change of variables one can write

$$\tilde{Z}[J,\varphi_b]=e^{-W(J)}\exp\left[\int_0^\beta d\tau\int d^3\mathbf{x}\,J(x)\varphi_b(x)\right]=Z[J]\exp\left[\int_0^\beta d\tau\int d^3\mathbf{x}\,J(x)\varphi_b(x)\right] , \tag{2.7}$$

where $Z[J]=\tilde{Z}(J,0)$ and $W(J)=\tilde{W}(J,0)$ stands for the above functionals evaluated without the background field. As is well known $Z(J)$ and $W(J)$ are the functional generators of the disconnected and connected Green's functions, respectively. $\tilde{Z}(J,\varphi_b)$ and $\tilde{W}(J,\varphi_b)$ stand for the same functionals in the presence of the background φ_b.

From (2.6) and (2.7) it follows that

$$\tilde{W}[J,\varphi_b]=W[J]-\int_0^\beta d\tau\int d^3\mathbf{x}\,J(x)\varphi_b(x) . \tag{2.8}$$

Since $\tilde{\varphi}=\delta\tilde{W}[J,\varphi_b]/\delta J$, the expected value of the field in the presence of J and φ_b, we can write

$$\tilde{\Gamma}[\tilde{\varphi},\varphi_b]\equiv\tilde{W}[J,\varphi_b]-\int_0^\beta d\tau\int d^3\mathbf{x}\frac{\delta(\tilde{W})}{\delta J(x)}J(x)\ , \tag{2.9}$$

where $\tilde{\Gamma}[\tilde{\varphi},\varphi_b]$ is the background-field effective action.

By substituting (2.8) into (2.9) it follows that

$$\tilde{\Gamma}[\tilde{\varphi},\varphi_b]\equiv W[J]-\int_0^\beta d\tau\int d^3\mathbf{x}\,J(x)[\tilde{\varphi}(x)+\varphi_b]\ . \tag{2.10}$$

Consequently if one differentiates (2.8) with regard to J one obtains

$$\frac{\delta W}{\delta J}=\tilde{\varphi}+\varphi_b\ . \tag{2.11}$$

From (2.11) one gets the relationship

$$\tilde{\varphi}=\overline{\varphi}-\varphi_b\ . \tag{2.12}$$

If one substitutes (2.12) into (2.10) one then obtains

$$\begin{aligned}\tilde{\Gamma}[\tilde{\varphi},\varphi_b]&=W[J]-\int_0^\beta d\tau\int d^3\mathbf{x}\,J(x)\overline{\varphi}(x)\\&\equiv\Gamma(\overline{\varphi})\equiv\Gamma[\tilde{\varphi}+\varphi_b]\ .\end{aligned} \tag{2.13}$$

Expression (2.13) is well known within the context of the background-field method—that is, the generating functional for the theory in the presence of the background can be obtained from the generating functional without the background field computed just by making the replacement $\overline{\varphi}\to\tilde{\varphi}+\varphi_b$.

The free energy in the presence of the background field is

$$\begin{aligned}F(\beta,\varphi_b)&\equiv\lim_{J\to 0}\tilde{W}[J,\varphi_b]\\&\equiv\lim_{J\to 0}\left[\tilde{\Gamma}[\tilde{\varphi},\varphi_b]+\int_0^\beta d\tau\int d^3\mathbf{x}\,J(x)\tilde{\varphi}\right]\ .\end{aligned} \tag{2.14}$$

Finally, one notes that if φ_b is a particular solution of the classical equation (2.1),

$$\left.\frac{\delta\Gamma}{\delta\varphi}\right|_{\varphi=\varphi_c}=0\ , \tag{2.15}$$

that is

$$\varphi_b=\varphi_c=\overline{\varphi}\ , \tag{2.16}$$

then in the limit $J\to 0$ Eq. (2.16) leads to $\tilde{\varphi}=0$. Under this circumstance it follows from (2.13) and (2.14) that

$$F(\beta,\varphi_c)=\Gamma[\varphi_c]\ ; \tag{2.17}$$

i.e., the free energy of the system in the presence of the background field φ_c satisfying the classical equation (2.15) is given by the effective action computed at this configuration. If Γ is computed at the zero-loop level, (2.15) corresponds to the classical Euler-Lagrange equations. This is precisely the situation that we are interested in the semiclassical approximation.

III. FORMAL EXPRESSION FOR DEFECT FREE ENERGY

The partition function for a given gauge theory, whose Euclidean Lagrangian density is L, may be expressed as a function integral[14]

$$Z(\beta)=N^{-1}(\beta)\oint[D\varphi]\exp\left[-\int_0^\beta d\tau\int d^3\mathbf{x}[L-J(x)\varphi(x)]\right]\times\text{gauge-fixing terms}\ , \tag{3.1}$$

where τ is the Euclidean time; φ stands for all fields in the theory and the integral over the fields is subject to the following boundary condition in φ:

$$\varphi(\mathbf{x},0)=\varphi(\mathbf{x},\beta)\quad\text{for bosonic fields}$$

and

$$\varphi(\mathbf{x},0)=-\varphi(\mathbf{x},\beta)\quad\text{for fermionic fields}\ .$$

N is a normalization constant which may be chosen such that $Z(\infty)=1$.

The free energy of the system is defined through the equations[14]

$$F(\beta,J)=-\beta^{-1}\ln Z\ , \tag{3.2}$$

$$M(x,J)\equiv M_J(x)=-\frac{\delta(\beta F)}{\delta J(x)}\ , \tag{3.3}$$

$$\Gamma(\beta,M_J)=F(\beta,J)+\beta^{-1}\int_0^\beta d\tau\int d^3\mathbf{x}\,M_J(x)J(x)\ . \tag{3.4}$$

$\Gamma(\beta,M_J)$ is the generating functional of one-particle-irreducible Green's functions and is the free energy of the field configuration M_J. The effective-potential method analyzes Γ for constant field configurations M_J in order to obtain the phase diagram of the model.[14]

One can define the free energies of the different types of topological defects[10,11,15] by

$$F_M=-\beta^{-1}\ln\left[\frac{Z_M}{Z_V}\right]\ , \tag{3.5}$$

$$F_S=-\frac{\beta^{-1}}{L}\ln\left[\frac{Z_S}{Z_V}\right]\ , \tag{3.6}$$

and

$$F_W=-\frac{\beta^{-1}}{L^2}\ln\left[\frac{Z_W}{Z_V}\right]\ , \tag{3.7}$$

where F_W, F_S, and F_M are, respectively, the free energy for domain walls, strings, and magnetic monopoles. Usually a given model does not exhibit all the three different topological defects, so one must consider only the relevant ones. Z_M, Z_S, and Z_W stands for the partition function of the system evaluated when one imposes boundary conditions that force the existence of a magnetic monopole, string, and domain-wall defect in the system, while Z_V is the partition function obtained using topologically trivial boundary conditions (vacuum sector). L is the size of the system.

The various thermodynamical functions can be written, in the one-loop approximation, as shown in the Sec. II, as differences of the effective action of the theory evaluated at certain field configurations. Let $\Gamma(\varphi)$ be the effective action of the theory and φ_V be the constant field configuration associated to the vacuum of the theory. In terms of the effective action one can write the effective potential

$$V_{\mathrm{eff}} \equiv \frac{T}{L^3}[\Gamma(\bar{\varphi})-\Gamma(\varphi_V)] , \tag{3.8}$$

where the overbar stands for constant field configurations.

Whereas for the defects that we are concerned in grand unified theories (GUT's) (monopole, string, and wall) one has[11]

$$F_M=[\Gamma(\varphi_M)-\Gamma(\varphi_V)] , \tag{3.9}$$

$$F_S=\frac{1}{L}[\Gamma(\varphi_S)-\Gamma(\varphi_V)] , \tag{3.10}$$

and

$$F_W=\frac{1}{L^2}[\Gamma(\varphi_W)-\Gamma(\varphi_V)] ; \tag{3.11}$$

that is, all thermodynamical parameters can be written as differences between the effective action computed at some special field-theoretical configurations and those associated with the vacuum of the theory. These special field-theoretical configurations, within the semiclassical scheme, are the defects associated with the classical solutions to the Euler-Lagrange equations of the model.

The general structure of $\Gamma[\beta,\varphi_D(x)]$ is

$$\Gamma[\beta,\varphi_D(x)]=\sum_{n=1}\frac{1}{n!}\prod_{j=1}^{n}\left[\int_0^\infty d\tau_j\int d^3x_j\varphi_D(x_j)\right]\Gamma^{(n)}(\tau_1\mathbf{x}_1,\ldots,\tau_n\mathbf{x}_n) , \tag{3.12}$$

where $\Gamma^{(n)}(\tau_1\mathbf{x}_1,\ldots,\tau_n\mathbf{x}_n)$ are the one-particle-irreducible Green's functions, φ_D stands for the fields associated with the defect. If one uses the Fourier transform of $\Gamma^{(n)}$, given by

$$\Gamma^{(n)}(\tau_1\mathbf{x}_1,\ldots,\tau_n\mathbf{x}_n)=\beta^{-n}\prod_{j=1}^{n}\sum_{n_j=-\infty}^{\infty}\int\frac{d^3\mathbf{k}_j}{(2\pi)^3}\tilde{\Gamma}^{(n)}(\omega_1\mathbf{k}_1,\ldots,\omega_n\mathbf{k}_n)\exp\left[-i\sum_{l=1}^{n}(\omega_l\tau_l+\mathbf{k}_l\cdot\mathbf{x}_l)\right] , \tag{3.13}$$

where $\omega_l=2\pi l\beta^{-1}$, and remembering that translational symmetry allows us to set

$$\tilde{\Gamma}^{(n)}(\{\omega_i\mathbf{k}_i\})=6(2\pi)^3\delta\left[\sum\omega_i\right]\delta^3\left[\sum k_i\right]\bar{\Gamma}^{(n)}(\{\omega_i\mathbf{k}_i\}) \tag{3.14}$$

then, for static field configurations (those with which we will be concerned in this paper), the general structure of $\Gamma(\beta,\varphi_D)$ is

$$\Gamma(\beta,\varphi_D)=\beta\sum_{n=1}^{\infty}\frac{1}{n!}\prod_{j=1}^{n}\int d^3\mathbf{k}_j\tilde{\varphi}(-\mathbf{k}_j)\bar{\Gamma}^{(n)}(\{\mathbf{k}_j,\omega_j=0\})\delta^3\left[\sum\mathbf{k}_j\right] . \tag{3.15}$$

The graphs that contribute to $\bar{\Gamma}^{(n)}$ will involve sums over the discrete ω_j which, once performed, yield a term independent of temperature plus one which has the full T dependence. This separation can always be implemented if one uses identities of the form

$$\beta^{-1}\sum_{n=-\infty}^{\infty}\frac{1}{\left[\dfrac{2n\pi}{\beta}\right]^2+z^2}=\frac{1}{2z}+\frac{1}{z(e^{\beta z}-1)} . \tag{3.16}$$

One can then split $\bar{\Gamma}^{(n)}$ into two parts

$$\bar{\Gamma}^{(n)}(\{\mathbf{k}_i,\omega_i=0\})=\bar{\Gamma}_0^{(n)}(\{\mathbf{k}_i\})+\bar{\Gamma}_T^{(n)}(\{\mathbf{k}_i\},\omega_i=0) , \tag{3.17}$$

where the second term contains all the T dependence. The general structure of this dependence can be inferred by making a change in all internal momenta integration variables. This change is just a replacement $\mathbf{p}\to\mathbf{p}'=\mathbf{p}\beta$. After this scaling in the internal momenta one can predict, from pure dimensional analysis, that $\bar{\Gamma}_T^{(n)}(\{\mathbf{k}_i,\omega_i=0\})$ has the structure[14]

$$\bar{\Gamma}_T^{(n)}(\{\mathbf{k}_i\},\omega_i=0)=\sum_{\gamma_n}T^{d(\gamma_n)}G_{\gamma_n}\left[\frac{\mathbf{k}_i}{T},\frac{m}{T}\right] , \tag{3.18}$$

where $d(\gamma_n)$ is the superficial degree of divergence of a graph γ_n contributing to $\bar{\Gamma}$ and G_{γ_n} is dimensionless. Putting (3.15), (3.17), and (3.18) together, we have

$$\Gamma(\beta,\varphi_D)=\Gamma_0(\varphi_D)+\sum_{n=1}^{\infty}\frac{1}{n!}\prod_{j=1}^{n}\int d^3\mathbf{k}_j\tilde{\varphi}_D(-\mathbf{k}_j)\sum_{\gamma_n}T^{d(\gamma_n)}G_{\gamma_n}\left[\frac{\mathbf{k}_j}{T},\frac{m}{T}\right]\delta^3\left[\sum\mathbf{k}_j\right], \quad (3.19)$$

where $\Gamma_0(\varphi_D)$ is the effective action computed at the background field φ_D at zero temperature.

Using (3.9)–(3.11) and (3.19), the free energies of the various topological defects can then be written as

$$F^D(\beta)=[\Gamma_0(\varphi_D)-\Gamma_0(\varphi_V)]\frac{1}{L^\alpha}$$
$$-\frac{1}{L^\alpha}\left\{\sum_{n=1}^{\infty}\frac{1}{n!}\prod_{j=1}^{n}\int d^3\mathbf{k}\,\tilde{\varphi}_D(-\mathbf{k}_j)\delta^3\left[\sum\mathbf{k}_j\right]\sum_{\gamma_n}T^{d(\gamma_n)}G_{\gamma_n}\left[\frac{\mathbf{k}_j}{T},\frac{m}{T}\right]-\sum_{\gamma_n}\frac{1}{n!}\varphi_V^n\sum_{\gamma_n}T^{d(\gamma^n)}G_{\gamma_n}\left[0,\frac{m}{T}\right]L^3\right\}, \quad (3.20)$$

where α is an index that, in accordance with (3.9)–(3.11), runs from 0 to 2.

To get a formal series for the free energy from any solution associated with a particular defect, we just introduce it in (3.20).

Just for the sake of completeness, we write the expression for the effective potential. From (3.8) and (3.19) it follows that

$$V_{\text{eff}}(\bar{\varphi})=\frac{1}{V}[\Gamma_0(\bar{\varphi}_D)-\Gamma_0(\varphi_V)]$$
$$+\sum_{n=1}^{\infty}\frac{1}{n!}(\bar{\varphi}^n-\varphi_V^n)\sum_{\gamma_n}T^{d(\gamma_n)}G_{\gamma_n}\left[0,\frac{m}{T}\right]. \quad (3.21)$$

From expression (3.20) one can see that, in the high-temperature limit, the leading contributions comes from graphs that have higher superficial degrees of divergence. As we will show in the next example, these graphs up to a given order in the semiclassical expansion, are easy to isolate.

IV. THE STRING FREE ENERGY (Ref. 16)

In this section we will be concerned with strings at finite temperature.[16] The ones we will be discussing were obtained explicitly, in the U(1)-gauge model with spontaneous symmetry breakdown, by Nielsen and Olesen.[17]

Within the classical context, the energy per unit length associated to the string is positive. Since the ones we will consider are infinitely extended, one might argue that such a structure cannot be present in the system, as its cost, in energy, is infinite. Quantum effects, however, might change this picture. In fact, as has been pointed out by Bricmont and Fröhlich,[15] there is a certain region in parameter space for which the cost in energy is zero. This implies that one has reached, for these values of the parameters, another phase of the theory—the one in which the condensation of defects takes place. We will show that, for high enough temperatures, the defect free energy becomes zero, thus signaling a transition temperature.

We would like to stress that our approach is particularly convenient when we are interested in the production of strings as a result of thermal fluctuations within the usual framework of field theory at finite temperature (and consequently thermal equilibrium). The picture that we propose for the role of strings in phase transitions is inspired in the so-called Kosterlitz-Thouless picture of phase transition.[7] In their classical paper they have dealt with vortices which, in the model studied by Kosterlitz-Thouless, are configurations analogous to strings.

In the case of strings, the argument of Kosterlitz and Thouless can be stated in the following way:[7] the free energy per unit length associated to a given string can be written as

$$F_{\text{string}}(T)=M-TS(T), \quad (4.1)$$

where M is the mass per unit length at zero temperature and $S(T)$, to be identified later, is an entropy term.

Assuming that M is positive (as in the case at classical level) there is no chance, for low temperatures, for the appearance of strings in the system since the energy cost for the introduction of a string in the system is infinite. However as the temperature increases, as will be shown later, the entropy term takes over the mass term, so that for a critical temperature we will have

$$F_{\text{string}}(T_c)=0. \quad (4.2)$$

That is, the cost in energy for introducing one infinite string is zero. In this way for temperatures high enough there is a condensation of strings. The system goes to a phase in which there is a condensate of strings. This entails a new phase of the system.

In the case of the string solution one writes, from (3.10) and (3.20),

$$F_{\text{string}}(T)=[\Gamma_0(\varphi_S)-\Gamma_0(\varphi_V)]\frac{1}{L}-\frac{1}{L}\left\{\sum_{n=1}^{\infty}\frac{1}{n!}\prod_{j=1}^{n}\int d^3\mathbf{k}\,\tilde{\varphi}_S(-\mathbf{k}_j)\delta^3\left[\sum \mathbf{k}_j\right]\sum_{\gamma_n}T^{d(\gamma_n)}G_{\gamma_n}\left[\frac{\mathbf{k}_j}{T},\frac{m}{T}\right]\right.$$

$$\left.-\sum_{\gamma_n}\frac{1}{n!}\varphi_V^n\sum_{\gamma_n}T^{d(\gamma^n)}G_{\gamma_n}\left[0,\frac{m}{T}\right]L^3\right\}. \tag{4.3}$$

In the high-temperature limit ($T \gg w_j^0, |\mathbf{k}_j|, m$) the expression above depends only on the zero-momentum character of G_{γ_n}. One can then write, in this limit, F_{string} under form (4.1) where M_S is the energy per unit length of the string,

$$M_S \equiv \frac{1}{L}[\Gamma_0(\varphi_S)-\Gamma_0(\varphi_V)] , \tag{4.4a}$$

and

$$-TS(T)\underset{T\to\infty}{\simeq}\frac{1}{L}\left[\sum_{n=1}^{\infty}\frac{1}{n!}\prod_{j=1}^{n}\int d^3\mathbf{k}\,\tilde{\varphi}_S(-\mathbf{k}_j)\delta^3\left[\sum \mathbf{k}_j\right]\sum_{\gamma_n}T^{d(\gamma_n)}G_{\gamma_n}(0,0)-\frac{1}{n!}\varphi_V^n\sum_{\gamma_n}T^{d(\gamma_n)}G_{\gamma_n}(0,0)L^3\right] . \tag{4.4b}$$

Within the one-loop approximation the graphs with higher superficial divergence will dominate in the high-T limit. The superficial divergence has index two, so that, in the high-temperature limit one can predict the general structure of the free energy per unit length, independently of the model, as

$$F_{\text{string}}(T)=M+T^2\sum g_i(0,0)A_i , \tag{4.5}$$

where A_i is a constant that depends on the model, M is the classical energy per unit length, and $g_i(0,0)$ is the ith contribution of a graph that has superficial degree of divergence 2.

In this way one can predict that, independently of the model

$$T_c=\left[\frac{M}{-\sum g_i(0,0)A_i}\right]^{1/2} . \tag{4.6}$$

In order to give explicit examples, in field theory, of the structure predicted by (4.5) we will work with two models. The first model is scalar electrodynamics at finite temperature. Its Lagrangian density is written as

$$L=-\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu}+|D_\mu\phi|^2-\lambda(\phi^*\phi-\phi_V^2)^2 . \tag{4.7}$$

The partition function for the theory may be expressed as a functional integral involving the Euclidean Lagrangian density L_E and imaginary times, $\tau=it$:

$$Z(\beta)=N^{-1}\oint[d\phi][dA_\mu]\exp\left[-\int_0^\beta d\tau\, d^3x\, L_E\right] \times \text{gauge-fixing terms} , \tag{4.8}$$

$$L_E=-\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu}+|D_\mu\phi|^2+\lambda(\phi^*\phi-\phi_V^0)^2,\quad A_4=iA_0 . \tag{4.9}$$

The integral over the fields is subject to boundary conditions in τ:

$$\phi(\mathbf{x},0)=\phi(\mathbf{x},\beta),\quad A_\mu(\mathbf{x},0)=A_\mu(\mathbf{x},\beta) .$$

The normalizing constant N is such that $N(\infty)=1$.

In order to fix the gauge we have to add to L_E in (4.8) a gauge-fixing term. In the Fermi gauge we just add a term $-(1/2\alpha)(\partial_\mu A_\mu)^2$. We shall see that, although the individual graphs depends on α (that is, on the gauge) the free energy per unit length is independent of α (that is, gauge independent).

Vortices are static solutions of the classical equations and that have finite energy per unit length. For static solution, the Euler-Lagrange equation associated with the model (4.9) in the Coulomb gauge ($\nabla\cdot A=0$) is

$$(\nabla^2-2e^2\phi^*\phi)\mathbf{A}=-i\phi^*\nabla\phi , \tag{4.10}$$

$$(\nabla+ie\,\mathbf{A})^2=2\lambda\phi(\phi^*\phi-\phi_V^2) . \tag{4.11}$$

By employing cylindrical coordinates (r,θ,z) and looking for a z-independent solution with cylindrical symmetry one writes

$$\mathbf{A}(r,\theta)\equiv\hat{\theta}A(r)\equiv\hat{\theta}\frac{n}{er}[1-F(r)] , \tag{4.12}$$

$$\phi(r,\theta)=\rho(r)e^{in\theta} , \tag{4.13}$$

where n is the topological charge associated with the classical solution and should be an integer number in order that

$$\phi(\theta)=\phi(\theta+2\pi) . \tag{4.14}$$

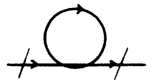
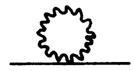
The asymptotic conditions imposed by the finite-energy requirement are

$$F(r)\underset{r\to\infty}{\to}0 , \tag{4.15}$$

$$\rho(r)\underset{r\to\infty}{\to}\phi_V . \tag{4.16}$$

In the high-temperature limit the graphs with highest superficial degree of divergence can be easily identified. So that, in the high-temperature limit the expression for the string free energy can be written in the Fermi gauge, as

$$F_{\text{string}} = -\frac{T}{L}[\Gamma(\bar{\phi}, A_r) - \Gamma(\phi_V, 0)]$$

$$\equiv \frac{T}{L}\Bigg[S(\bar{\phi}, \bar{A}_\mu) - S(\phi_V, 0) - \Bigg($$

$$+$$

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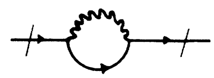
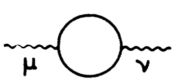

$$\Bigg) \int_0^\beta d\tau \int d^3\mathbf{x} (|\bar{\phi}|^2 - \phi_V^2)$$

$$-\Bigg($$


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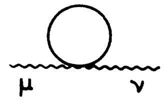
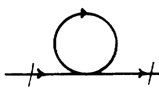
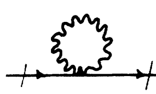


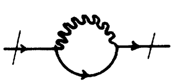
$$\Bigg) \int_0^\beta d\tau \int d^3\mathbf{x}\, \bar{A}_\mu \bar{A}_\mu \Bigg] , \qquad (4.17)$$

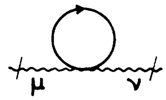
where S is the classical action.

After the renormalization process we get the following results, in the high-temperature limit:

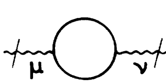

$$\equiv -\frac{\lambda}{3} T^2 , \qquad (4.18)$$


$$\equiv -e^2 \frac{(3+\alpha)}{12} T^2 , \qquad (4.19)$$


$$\equiv \frac{e^2 \alpha T^2}{12} , \qquad (4.20)$$



$$\equiv -e^2 \frac{T^2}{6} \delta_{\mu\nu} , \qquad (4.21)$$



$$\equiv +e^2 \frac{T^2}{6} \delta_{ij} , \qquad (4.22)$$

where $i,j = 1,2,3$. Substituting (4.18)–(4.22) in (4.17) we get[16]

$$F_{\text{string}} = M_S + \frac{4\lambda + 3e^2}{12} T^2 \int_0^\infty dr\, 2\pi r [\rho^2(r) - \phi_V^2] , \qquad (4.23)$$

where M_S is the zero-temperature energy per unit length that can be written, in terms of the function ρ and F defined in (4.12) and (4.13), as[16,18]

$$M_S = 2\pi \int_0^\infty r\, dr \left[\frac{1}{2} \left[\frac{n}{er} \right]^2 \left[\frac{dF}{dr} \right]^2 + \left[\frac{n}{r} \right]^2 F^2 \rho^2 + \left[\frac{d\rho}{dr} \right]^2 + \frac{\lambda}{4}(2\rho^2 - \varphi_V^2)^2 \right] . \qquad (4.24)$$

From expression (4.23) one can see that for temperatures high enough the free energy of strings becomes zero. The temperature for which this happens is

$$T_c = \frac{6 M_S}{\pi(4\lambda + 3e^2) \int_0^\infty dr\, r(\phi_V^2 - \rho^2)} . \qquad (4.25)$$

At this temperature the cost for introducing a string in the system is zero and one expects that condensation of strings takes place thus signaling a new phase. As expected, the critical temperature is gauge independent.

V. FREE ENERGY OF COSMIC DEFECTS

Let us analyze the high-temperature behavior of the free energy associated with topological defects that might be relevant to cosmology. In this context an interesting example will be to study the SO(10) model. This model exhibits, depending on the symmetry-breaking pattern, different types of defects.[19,20] Consider the symmetry-breaking patterns

$$\mathrm{SO}(10) \underset{126}{\longrightarrow} \mathrm{SU}(5) \times Z_2 \underset{45}{\longrightarrow} \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) ,$$

$$\mathrm{SO}(10) \underset{45}{\longrightarrow} \mathrm{SU}(5) \times \mathrm{U}(1) \underset{126}{\longrightarrow} \mathrm{SU}(5) .$$

In the first case one expects[19] the production, at the phase transition, of stable strings. In the second case one expects the production, at the first phase transition, of magnetic monopoles. In this way, depending on the symmetry-breakdown pattern we will have the production of different defects in the Universe.

We will show, in the following, that independently of the type of defect its free energy has the structure predicted by (4.5), that is

$$F_{\text{defect}} = M - BT^2 . \qquad (5.1)$$

In order to prove this all we have to do is to show that the effective action behaves like (5.1). Let us give an explicit example. The Lagrangian density describing the SO(10) model is

$$\mathcal{L} = -\tfrac{1}{16} T_r(G_{\mu\nu} G^{\mu\nu}) + \tfrac{1}{4} T_r(D_\mu \phi)^2 - V(\phi) , \qquad (5.2)$$

where

$$D_\mu \phi = \partial_\mu \phi - \sqrt{1/2}\, ig [A_\mu, \phi] , \qquad (5.3)$$

$$V(\phi)=\frac{a}{16}(T_r\phi^2)^2+\tfrac{1}{4}bT_r\phi^4-\tfrac{1}{4}\mu^2T_r\phi^2 \; . \tag{5.4}$$

$D_\mu\phi$ in (5.3) is then the covariant derivative and V is the potential that will lead to spontaneous breakdown of symmetry that depends, in this model, on the parameters a, b, and μ. g is the coupling constant of vector-boson fields.

The multiplet of the Higgs field ϕ is in the 45-dimensional adjoint representation. In terms of the fields $G^{ij}_{\mu\nu}$, A^{ij}_{μ}, and ϕ^{ij} one can write $G_{\mu\nu}$, A_μ, and ϕ in (5.1) as

$$G_{\mu\nu}=\sqrt{1/2}\sigma^{ij}G^{ij}_{\mu\nu} \; , \tag{5.5}$$

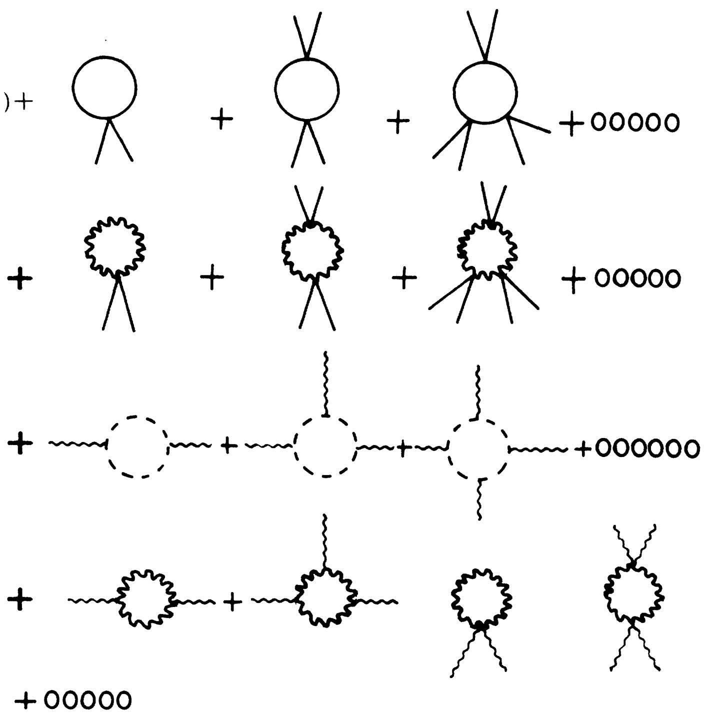
$$A_\mu=\sqrt{1/2}\sigma^{ij}A^{ij}_{\mu} \; , \tag{5.6}$$

$$\phi=\sqrt{1/2}\sigma^{ij}\phi^{ij} \; . \tag{5.7}$$

Let us analyze the behavior of the free energy associated with a classical field-theoretical configuration associated with a set of 45 Higgs fields ϕ^i_c and 45 gauge bosons A^i.

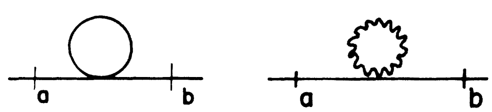
Up to one-loop approximation the free energy of the topological defect has the structure predicted in Sec. III that in the example that we are considering has the expansion

$\Gamma(\bar{\varphi}, A_\mu)=S_{\rm cl}(\bar{\varphi}, A_\mu)+$

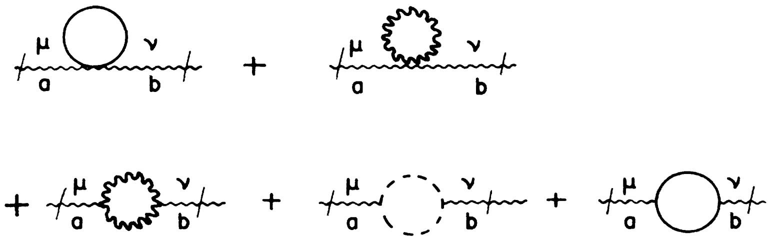


$$=S_{\rm cl}(\bar{\varphi},\bar{A}_\mu)-\frac{1}{2!}\Sigma^{ab}(T)\int_0^\beta d\tau\int d^3\mathbf{x}\,\bar{\varphi}^a_d\bar{\varphi}^b_d-\frac{1}{2!}\Pi^{ab}_{\mu\nu}(T)\int_0^\beta d\tau\int d^3\mathbf{x}\,\bar{A}^a_\mu A^b_\nu+\cdots \; . \tag{5.8}$$

$S_{\rm cl}$ is the classical action associated with the background field, $\Sigma^{ab}(T)$ can be represented graphically as

$\Sigma^{ab}(T)=$  (5.9)

whereas $\Pi^{\mu\nu}_{ab}(T)$ can be represented as

$\Pi^{ab}_{\mu\nu}(T)=$  (5.10)

The wavy, solid, and dashed line stand, respectively, for the gauge bosons, Higgs bosons, and ghost fields (for the fluctuations we are working in the Landau gauge). $\Pi^{ab}_{\mu\nu}$ can be identified as the polarization tensor for zero external momenta.[21] Following our earlier prescription (3.17), we can also write

$$\Sigma^{ab}(T)=\Sigma_0^{ab}+\bar{\Sigma}_T^{ab}(\{\mathbf{K}_i\},\omega_i=0)\ , \tag{5.11}$$

$$\Pi^{ab}_{\mu\nu}(T)=\Pi^{ab}_{\mu\nu 0}+\bar{\Pi}^{ab}_{\mu\nu}(T)\ . \tag{5.12}$$

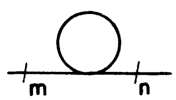
First of all one notes, looking at (5.8), the appearance of ultraviolet divergences. These, however, can be treated, as usual, by adding appropriate renormalization counterterms which are just the usual ones at zero temperature. This means that the zero-temperature renormalization scheme suffices for getting finite expression to free energies of topological defects. Substituting (5.11) into (5.8), one can obtain the topological defect free energies of the SO(10) model:

$$F_d(T)=M_d-\frac{1}{2!}\bar{\Sigma}^{ab}(T)\int_0^\beta d\tau\int d^3\mathbf{x}[\bar{\varphi}_d^a(\mathbf{x})\bar{\varphi}_d^b(\mathbf{x})-\varphi_V^2]-\frac{1}{2!}\bar{\Pi}^{ab}_{\mu\nu}(T)\int_0^\beta d\tau\int d^3\mathbf{x}\,\bar{A}^a_\mu(\mathbf{x})\bar{A}^b_\nu(\mathbf{x})+\cdots\ , \tag{5.13}$$

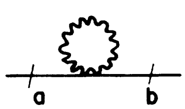
where now M_d stands for the renormalized mass of the defect at the zero-loop level, $\bar{\Sigma}^{ab}(T)$ and $\Pi^{ab}_{\mu\nu}(T)$ are given in (5.11) and (5.12), the fields $\bar{\varphi}^a_d$ and $\bar{A}^a_\mu$ are the classical field-theoretical configurations associated with the defect and the ellipsis represents contributions that are not shown in (5.13).

One could go further and write down similar expression for all the one-loop graphs for the topological structures of the SO(10) model. However, instead of doing this explicitly, we will just analyze the high-temperature limit of the free energy. In this limit, the form (4.3) is particularly useful, since the leading power in T of series (4.3) is easily obtained. Property (3.18) permits us to identify those contributions, which are the ones with higher superficial degrees of divergence. These contributions are precisely the ones we have written explicitly.

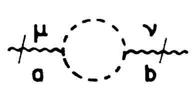
In the high-temperature limit, the graphs appearing in (5.9) and (5.10) yield



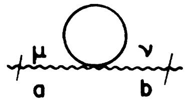
$$=-(\tfrac{47}{4}a+720b)\frac{T^2}{12}\delta^{mn}\ , \tag{5.14}$$



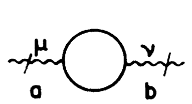
$$=-4g^2T^2\delta^{ab}\ , \tag{5.15}$$



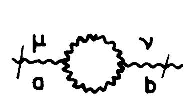
$$=\begin{cases}-\frac{2}{3}g^2T^2\delta^{ab}\delta_{\mu\nu} & \text{for } \mu,\nu=1,2,3\ ,\\ \frac{2}{3}g^2T^2\delta^{ab}\delta_{\mu\nu} & \text{for } \mu \text{ and/or } \nu=4\ ,\end{cases} \tag{5.16}$$



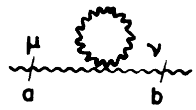
$$=-\tfrac{4}{3}g^2T^2\delta^{ab}\delta_{\mu\nu}\ , \tag{5.17}$$



$$=\begin{cases}\frac{4}{3}g^2T^2\delta^{ab}\delta_{\mu\nu} & \text{for } \mu,\nu=1,2,3\ ,\\ -\frac{4}{3}g^2T^2\delta^{ab}\delta_{\mu\nu} & \text{for } \mu \text{ and/or } \nu=4\ ,\end{cases} \tag{5.18}$$



$$=\begin{cases}4g^2T^2\delta^{ab}\delta_{\mu\nu} & \text{for } \mu,\nu=1,2,3\ ,\\ -4g^2T^2\delta^{ab}\delta_{\mu\nu} & \text{for } \mu \text{ and/or } \nu=4\ ,\end{cases} \tag{5.19}$$



$$= -\tfrac{10}{3} g^2 T^2 \delta^{ab} \delta_{\mu\nu} \,. \tag{5.20}$$

From (5.8)–(5.14), (3.11), and (3.12) we have the asymptotic expressions for $\Sigma^{ed}(T)$ and $\Pi^{cd}_{\mu\nu}(T)$:

$$\overline{\Sigma}^{cd}(T) \sim -T^2 \left[2g^2 + \frac{1}{24} \left[\frac{47a}{4} + 720b \right] \right] \delta^{cd} \,, \tag{5.21}$$

$$\overline{\Pi}^{ab}_{\mu\nu}(T) \sim -\tfrac{14}{3} g^2 T^2 \delta^{ab} \delta_{\mu\nu} \,. \tag{5.22}$$

One obtains from (5.13)–(5.22) the high-temperature behavior

$$F_d(T) = M_d + T^2 \left[2g^2 + \frac{1}{24} \left[\frac{47a}{4} + 720b \right] \right] \int \frac{d^3\mathbf{x}}{L^\alpha} \left[\sum_{a=1}^{45} \overline{\varphi}^a_d \overline{\varphi}^a_d - \varphi_v^2 \right] + \tfrac{14}{3} g^2 T^2 \int \frac{d^3\mathbf{x}}{L^\alpha} \sum_{a=1}^{45} \overline{A}^a_4 \overline{A}^a_4 \,. \tag{5.23}$$

The appearance of the term $\int d^3\mathbf{x} (\overline{A}^a_k)^2$ in the last expression seems to be a problem. First of all because it seems to be not gauge invariant and, finally, because it diverges. This implies that in order to be consistent one has to adopt a "physical" gauge for the background field. We take the background in the gauge $A^a_4 = 0$. Only in this gauge we get a finite result for the free energy of the string. This has been discussed, in the case of the Nielsen and Olesen string, in Ref. 16.

The conclusion is that, also in the SO(10) model one can predict, for the free energy associated with any defect, in the high-temperature limit:

$$F_d(T) = M_d - BT^2 \,, \tag{5.24}$$

where B is the constant that depends on the classical solution associated to the defect and M_d is (in this approximation) the classical energy (per unit length or area) of the topological defect.

VI. APPLICATION TO COSMOLOGY

In the preceding sections we have shown that for temperatures high enough there is condensation of cosmic strings. For a critical temperature there is consideration even of infinite strings.

Below the critical temperature only finite strings are allowed to exist in the system. In the following we will consider the formation of finite strings of length L and, by making simple hypotheses, we will compute their lengths as well as their distribution.

We will make only two hypotheses. The first one is that the strings that are formed below the phase transition do not differ (except for the finite length) from the strings that we have dealt with in the preceding section. More explicitly we assume that the mass (M_{string}), free energy (f_{string}), and energy (E_{string}) and other relevant physical quantities associated with a single string can be obtained from analogous quantities defined per unit length, for the infinite string ($M_s, F_s, E_s, \ldots$), by just multiplying these quantities by the length of the string (L). That is,

$$\begin{aligned} f_{\text{string}} &= LF_s \,, \\ M_{\text{string}} &= LM_s \,, \\ &\vdots \,. \end{aligned} \tag{6.1}$$

Hypothesis (6.1) can be understood on very simple grounds: it means that finite strings formed below the critical temperature are just formed above the critical temperature but, due to their instability, have broken into pieces of smaller sizes of length L. This hypothesis, that $M_{\text{string}} = LM_s$, can be used for deriving time-independent density perturbations in a very simple way.[22]

The length of the string should depend on the temperature. One can expect that as one increases the temperature, the length of the string increases. That is, thermal fluctuations induce the creation of strings of higher and higher length as the temperature increases. As a matter of fact we have seen that at the critical temperature even strings of infinite length can be produced. This means that at this temperature one should expect that

$$L(T_c) = \infty \,. \tag{6.2}$$

The other hypothesis that we will make is that the gas of strings is dilute. Under this hypothesis one can write, for the partition function (Z) of the dilute gas,[23]

$$\begin{aligned} \frac{Z}{Z_V} &= \exp\left[\frac{Z_S}{Z_V} \right] \\ &= \exp\left[\frac{1}{(2\pi)^3} \int d^3\mathbf{x} \int d^3\mathbf{p}\, e^{-\beta E_{\text{string}}} \right] \,, \end{aligned} \tag{6.3}$$

where Z_S is the partition function associated with one string.

The energy (E_{string}) of a string moving with velocity $\mathbf{V}$, in the nonrelativistic limit, is[24]

$$\begin{aligned} E_{\text{string}} &= f_{\text{string}}(T) + \frac{M_{\text{string}}}{2} \mathbf{V}^2 \\ &\equiv f_{\text{string}}(T) + \frac{p^2}{2M_{\text{string}}} \,. \end{aligned} \tag{6.4}$$

From (6.3) and (6.4) it follows that Z_S can be written as

$$\frac{Z_S}{Z_V}=Ve^{-(1/T)f_{\text{string}}(T)}\left[\frac{TM_{\text{string}}}{2\pi}\right]^{3/2} . \tag{6.5}$$

It follows from (6.5) and (6.3) that the free energy of a dilute gas of strings $[VF=-T\ln(Z/Z_V)]$ will be given by

$$F=-Te^{-(1/T)f_{\text{string}}(T)}\left[\frac{TM_{\text{string}}}{2\pi}\right]^{3/2} . \tag{6.6}$$

Although we have made use of the nonrelativistic approximation one can show that the relevant term $(TM_{\text{string}})^{3/2}$ can be obtained without resorting to this approximation. In field theory it follows by just taking into account the zero modes. This is shown in Appendix B.

By making use of our first hypothesis one can state that strings of length L, in the dilute-gas approximation, have a free energy whose expression is

$$F=-T^{5/2}e^{-LF_s(T)/T}\left[\frac{LM_s}{2\pi}\right]^{3/2} . \tag{6.7}$$

The length L is a parameter as yet unknown. However L can be determined by remembering that, for each temperature, there will be a length L favored by statistical arguments. This length L is the one that minimizes the free energy. The system produces strings whose size can be determined from the condition

$$\left.\frac{dF}{dL}\right|_{\bar{L}}=0 \quad \left[\left.\frac{d^2F}{dL^2}\right|_{L=\bar{L}}>0\right] . \tag{6.8}$$

It follows then, from (6.7), that the length of a string as a function of temperature will be

$$\bar{L}(T)=\frac{3}{2}\frac{T}{F_s(T)} . \tag{6.9}$$

At this point we would like to comment on our expression (6.9) for the length of strings produced as a result of thermal fluctuation. In the first place the length of these strings tends to zero as T goes to zero. This follows from (6.9) and expression (6.2). This is also expected on physical grounds. Finally, one can see from this expression that at the critical temperature, for which $F_s(T_c)=0$, strings of infinite length are favored. In our scheme the critical temperature T_c defined by (6.2) is the same as the critical temperature defined by (4.2). Both schemes are, then, totally compatible.

Let us turn now to the computation of the contrast density due to string production as a result of thermal fluctuations. The contrast density associated with any type of defect is defined through the expression

$$\frac{\delta\rho}{\rho}=\frac{\rho_{\text{defect}}}{\rho_{\text{total}}}=\frac{\rho_{\text{defect}}}{\rho_{\text{defect}}+\rho_{\text{elem part}}} . \tag{6.10}$$

For strings one then has

$$\frac{\delta\rho}{\rho}=\frac{\rho_{\text{string}}}{\rho_{\text{string}}+\rho_{\text{elem part}}} . \tag{6.11}$$

The energy density associated with strings whose average number is $\bar{N}(T)$ and whose mass is M_{string} is

$$\rho_{\text{string}}=\frac{\bar{N}(T)M_{\text{string}}}{V} . \tag{6.12}$$

Since $\bar{N}(T)$ is given by[21]

$$\bar{N}(T)=\frac{Z_S}{Z_V} . \tag{6.13}$$

One can write from (6.12) and (6.13), for a dilute gas of strings of length L,

$$\rho_{\text{string}}=e^{-L(T)F_s(T)/T}\left[\frac{TL(T)M_s}{2\pi}\right]^{3/2}L(T)M_s . \tag{6.14}$$

For the most favored strings, those obeying (6.9), one gets

$$\rho_{\text{string}}=\left[\frac{T}{e2\pi}\right]^{3/2}\left[\frac{3M_s}{2}\frac{T}{F_s(T)}\right]^{5/2} . \tag{6.15}$$

Expressions (6.9) and (6.15) are the main results of this section.

In the high-temperature limit, as we have argued in Sec. III, and given examples in Secs. III and IV, one can compute L and ρ_{string} explicitly. We can write, on general grounds,

$$F_s(T)=M_s-BT^2=M_s\left[1-\frac{T^2}{T_c^2}\right] , \tag{6.16}$$

where

$$T_c^2\equiv\frac{M_s}{B} . \tag{6.17}$$

Where, for the Nielsen-Olesen strings, one gets, from (4.23),

$$B=\frac{4\lambda+3e^2}{6}\pi\int_0^\infty r\,dr[\phi_V^2-\rho^2(r)] . \tag{6.18}$$

For $F_s(T)$ given by (6.16) one gets

$$L(T)=\frac{3}{2}\frac{T}{M_s\left[1-\frac{T^2}{T_c^2}\right]} \tag{6.19}$$

with T_c defined in (6.17) and

$$\rho_{\text{string}}(T)=\frac{3}{2}\left[\frac{3}{4\pi e}\right]^{3/2}\frac{T^4}{\left[1-\frac{T^2}{T_c^2}\right]^{5/2}} . \tag{6.20}$$

Since the contribution of the elementary particles can be written in terms of the number of degrees of freedom fermionic (N_F) and bosonic (N_B) as

$$\rho_{\text{elem part}}=\frac{\pi^2}{30}(N_B+\tfrac{7}{8}N_F)T^4 . \tag{6.21}$$

The contrast density will be given as

$$\frac{\delta\rho}{\rho}=\frac{1}{1+\frac{\pi^2}{30}(N_B+\frac{7}{8}N_F)\left[1-\frac{T^2}{T_c^2}\right]^{5/2}\frac{2}{3}\left[\frac{4\pi e}{3}\right]^{3/2}} . \tag{6.22}$$

For T below T_c and for $N_B+\frac{7}{8}N_F\gg 1$ the contrast density is small and can be approximated by

$$\frac{\delta\rho}{\rho}\sim\frac{1}{\frac{\pi^2}{30}(N_B+\frac{7}{8}N_F)\left[1-\frac{T^2}{T_c^2}\right]^{5/2}\frac{2}{3}\left[\frac{4\pi e}{3}\right]^{3/2}}$$
$$\sim\frac{1}{9(N_B+\frac{7}{8}N_F)\left[1-\frac{T^2}{T_c^2}\right]^{5/2}} . \tag{6.23}$$

For the SO(10) model one can write (for $T<T_c$), taking the effective degrees of freedom of the minimal SU(5) model,

$$N_B+\tfrac{7}{8}N_F=160.75 . \tag{6.24}$$

So that for $T\sim\frac{1}{3}T_c$ one gets

$$\frac{\delta\rho}{\rho}\sim 6\times 10^{-4} . \tag{6.25}$$

This result is compatible with the bounds imposed by the anisotropy of the background radiation.

Another interesting result of this section is that $\delta\rho/\rho$ is almost independent of temperature for $T<T_c$ and can be approximated by

$$\frac{\delta\rho}{\rho}\sim\frac{1}{9(N_B+\frac{7}{8}N_F)} . \tag{6.26}$$

In this way we showed that the density contrast is the scale-invariant (Zel'dovich) spectrum.

The effect of topological defects on phase transitions in the early Universe have been analyzed also by Copeland, Haws, and Rivers.[25] Their approach for getting the statistical properties of a gas of strings is different from ours.

VII. CONCLUSIONS

In this paper we have developed further an alternative method to the study of phase transitions in field theory at finite temperatures. The distinction from the usual approach, based on the effective potential, is that we deal with space-time field-theoretical configurations. Our analysis is based on the free energy associated with topological defects. In this context, the critical temperature is the one for which the free energy goes to zero.

Up to the one-loop level and in the high-temperature limit, one can easily compute the leading contribution to the free energy of strings. In the one-loop approximation only graphs with superficial degree of freedom of order 2 contribute. There are only a few of them in this approximation. We have illustrated how to compute the free energy of the Nielsen-Olesen string and the strings of the SO(10) model. We give an explicit expression for the critical temperature.

The temperature for which the free energy of the string becomes zero is critical. At this temperature there is condensation of strings since the cost in energy for introducing such an object is zero. The condensation is enterely due to thermal effects. Thermal fluctuations cannot be ignored in any string-driven mechanism responsible for the large-scale structure of the Universe. We have developed a very precise scheme for determining the free energy of topological defects and applied this scheme to the computation of the critical temperature in the high-temperature limit. The critical temperature is relevant because, in our picture, at this temperature, large (infinite) strings break down into smaller ones.

Within the dilute-gas approximation one can get extremely simple expressions for the length of strings and their density as a function of the free energy per unit length.

In temperatures just below critical, one gets very simple expressions for their lengths and for the density contrast. In particular we get, for any renormalizable model, within the semiclassical approximation the result

$$\frac{\delta\rho}{\rho}=\frac{1}{1+\frac{\pi^2}{30}(N_B+\frac{7}{8}N_F)\left[1-\frac{T^2}{T_c^2}\right]^{5/2}\frac{2}{3}\left[\frac{4\pi e}{3}\right]^{3/2}} .$$

This expression is the main result of our paper. In the first place because it shows practically no dependence with temperature (or time). That is, for $T<T_c$ one can write

$$\frac{\delta\rho}{\rho}\sim\frac{1}{9(N_B+\frac{7}{8}N_F)} .$$

One gets in this way not only the scale-independent Zel'dovich spectrum but also a totally compatible density contrast for models for which $N_B+\frac{7}{8}N_F>100$, that is, a Zel'dovich spectrum with the proper order of magnitude. This is the case of the SO(10) model.

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APPENDIX A: GENERATOR OF SO(10)

For the SO(10) model the 45 generators are given by

$$\sigma^{ij}=\frac{1}{2i}(\Gamma^i\Gamma^j-\Gamma^j\Gamma^i) . \tag{A1}$$

Γ^i are the generalized Dirac matrices constructed from the Pauli matrices $\sigma_1,\sigma_2,\sigma_3$:

$$\begin{aligned}
\Gamma_1&=\sigma_1\times\sigma_1\times 1\times 1\times\sigma_2 ,\\
\Gamma_2&=\sigma_1\times\sigma_2\times 1\times\sigma_3\times\sigma_2 ,\\
\Gamma_3&=\sigma_1\times\sigma_1\times 1\times\sigma_2\times\sigma_3 ,\\
\Gamma_4&=\sigma_1\times\sigma_2\times 1\times\sigma_2\times 1 ,\\
\Gamma_5&=\sigma_1\times\sigma_1\times 1\times\sigma_2\times\sigma_1 ,
\end{aligned} \tag{A2}$$

$$\Gamma_6=\sigma_1\times\sigma_2\times 1\times\sigma_1\times\sigma_2\ ,$$
$$\Gamma_7=\sigma_1\times\sigma_3\times\sigma_1\times 1\times 1\ ,$$
$$\Gamma_8=\sigma_1\times\sigma_3\times\sigma_2\times 1\times 1\ ,$$
$$\Gamma_9=\sigma_1\times\sigma_3\times\sigma_3\times 1\times 1\ ,$$
$$\Gamma_{10}=\sigma_2\times 1\times 1\times 1\times 1\ .$$

Γ_i obey the Clifford algebra

$$\Gamma_i\Gamma_j+\Gamma_j\Gamma_i=2\delta_{ij},\quad i,j=1,\ldots,10\ . \tag{A3}$$

The commutations relations of the 45 generators of SO(10) are given by

$$[\sigma_{ij},\sigma_{kl}]=2i(\delta_{ik}\sigma_{jl}+\delta_{jl}\sigma_{ik}-\delta_{il}\sigma_{jk}-\delta_{jk}\sigma_{il})\ . \tag{A4}$$

APPENDIX B: ZERO-MODE CONTRIBUTION

We will develop in this appendix an alternative way to obtain expression (6.6). We will make use of the semiclassical expansion as an approximative method to obtain the free energy of closed strings with radius R. By making use of the dilute-gas approximation, the partition function is given by[23]

$$Z=Z^0\exp\left[\frac{Z^1}{Z^0}\right]\ , \tag{B1}$$

where

$$Z^0=e^{-S(\phi_V)}\det{}^{-1/2}[-\nabla^2+V''(\phi_V)] \tag{B2}$$

and

$$Z^1=e^{-S(\phi_D)}\det{}^{-1/2}[-\nabla^2+V''(\phi_D)]\ . \tag{B3}$$

In Eq. (B2) ϕ_V is the vacuum of the theory, and ϕ_D, in (B3), is a field-theoretical configuration describing a topological defect at rest.

Letting

$$VF=-T\ln\frac{Z}{Z_0} \tag{B4}$$

and treating separately the zero eigenvalues, one obtains, from (B1)–(B3),

$$F=-T^{z+1}\left[\frac{S(\phi_D)}{2\pi}\right]^{z/2}\left[\frac{\det'[-\nabla^2+V''(\phi_D)]}{\det[-\nabla^2+V''(\phi_V)]}\right]^{-1/2}e^{[S(\phi_D)-S(\phi_V)]}\ , \tag{B5}$$

where the prime indicates that the zero eigenvalues of $-\nabla^2+V''(\phi_D)$ must be omitted from the determinant and z is the number of these eigenvalues, which in this paper is three (we are working with tridimensional theories).

From (6.6) and (6.7) we obtain, in the high-temperature limit,

$$F=-T^{z+1}\left[\frac{S(\phi_D)}{2\pi}\right]^{z/2}\exp(-\beta f_D)\ , \tag{B6}$$

where $S(\phi_D)$ is the classical action associated with the topological defect and f_D the free energy of one defect.

Within the finite-temperature scheme and for the string,

$$S[\phi_S]=\frac{M_{\text{string}}}{T}\ .$$

For finite strings $z=3$ since we have three translational zero modes. From (B6) one gets

$$F=-T\left[\frac{TM_s}{2\pi}\right]^{3/2}e^{-(1/T)f_{\text{string}}}\ . \tag{B7}$$

That is another way to derive (6.6).

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