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Generalized Atkin-Lehner symmetry

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Atkin-Lehner symmetry was proposed several years ago as a mechanism for obtaining a vanishing one-loop cosmological constant in nonsupersymmetric superstring models, but for models formulated in four-dimensional spacetime this symmetry cannot be realized. We therefore investigate various means of retaining the general Atkin-Lehner idea without having strict Atkin-Lehner symmetry. We first explicitly construct non-Atkin-Lehner-symmetric partition functions which not only lead to vanishing cosmological constants but which also avoid a recent proof that Atkin-Lehner-symmetric partition functions cannot arise from physically viable string models in greater than two dimensions. We then develop a systematic generalization of Atkin-Lehner symmetry, basing our considerations on the use of non-Hermitian operators as well as on a general class of possible congruence subgroups of the full modular group. We find that whereas in many instances our resulting symmetries reduce to either strict Atkin-Lehner symmetry or symmetries closely related to it, in other cases we obtain symmetries of a fundamentally new character. Our results therefore suggest possible new avenues for retaining the general Atkin-Lehner "selection rule" approach for obtaining a vanishing one-loop cosmological constant.

I. INTRODUCTION

The fixing of a zero of energy, or equivalently the theoretical calculation of the cosmological constant, is a problem modern physics has faced for over seventy years.[1,2] Observations of small galaxy clusters and of gravitational redshifts have enabled us to provide an experimental upper bound on the energy density of a "flat" spacetime:[1,2]

$$\frac{|\langle T_{00}\rangle_{\rm flat}|}{c^2} \le 10^{-29}\ {\rm g/cm}^3 \ ; \qquad (1.1)$$

this in turn enables us to place a corresponding limit on the cosmological constant Λ:

$$\begin{aligned} |\Lambda| &= 8\pi\frac{G^2\hbar}{c^7}\ |\langle T_{00}\rangle_{\rm flat}|(M_P c^2)^2 \\ &\le 4.86\times 10^{-122}(M_P c^2)^2 \ , \end{aligned} \qquad (1.2)$$

where $M_P \equiv \sqrt{\hbar c/G} \cong 1.22\times 10^{19}\ {\rm GeV}/c^2$ is the Planck mass. The cosmological constant, which is taken to be identically zero in all but certain extreme models of the present-day Universe, is therefore one of the most experimentally well-constrained numbers in physics.

No theory, on the other hand, has been able to provide an adequate explanation for the apparent vanishing of the cosmological constant. In principle, this explanation would seem to await the development of a quantum field theory of gravity, so that Λ might be properly treated as a suitably renormalized vacuum expectation value of the stress-energy field operator. In the absence of such a theory, many different alternative approaches have been investigated,[2] yet none of these has yielded a completely satisfactory and compelling argument either. Since superstring theory[3] is to date the only theory providing a natural (albeit first-quantized) unified quantum theory of gravitation, it therefore behooves us to investigate what light superstring theory may shed upon the matter.

Superstring theories with spacetime supersymmetry trivially have vanishing one-loop cosmological constants: the contributions to the cosmological constant from bosonic states in the theory are precisely canceled by those from their fermionic (superpartner) states. In fact, general arguments exist[4] indicating that the presence of supersymmetry guarantees a vanishing cosmological constant when higher-loop contributions are included as well. However, at present-day energies supersymmetry is badly broken (if it was ever present at all), and thus we would *a priori* expect a cosmological constant at the Planck scale. The fact that constraints such as (1.2) continue to hold even in the absence of spacetime supersymmetry indicates that some other suppression mechanism may well be operating; furthermore, string theories with non-zero cosmological constants cannot represent true ground states or vacua of a string field theory (because of a nonzero dilaton one-point correlation function), and hence such theories are not even well defined. One mechanism that has been suggested for constraining the cosmological constant without supersymmetry involves string theories having equal numbers of massless bosonic and fermionic degrees of freedom;[5] for such theories the cosmological constant experiences exponential suppression. However, the vast majority of known, well-formulated string theories do not have this property. Another approach[6] involves investigating the dependence of the cosmological constant on the values of various background field expectation values for toroidally compactified heterotic string theories; it is found that the cosmological constant does indeed take extremal values when the compactified theories have enhanced gauge symmetries (thereby hinting at some sort of symmetry argument restricting the cosmological constant), but these extremal values are not necessarily zero.

Perhaps the suggestion receiving the greatest attention has been that of Moore, involving Atkin-Lehner symmetry.[7] The underlying mathematical motivation can be summarized as follows (details are provided in Sec. II). In string theory the cosmological constant Λ is (to first order) determined by the value of the one-loop string diagram, the integral of the one-loop string partition function Z over conformally inequivalent tori:

$$\Lambda \propto \int_{\mathcal{F}} \frac{d^2\tau}{{\tau_2}^2}\, Z(\tau_1,\tau_2) \ . \tag{1.3}$$

Here τ is the complex parameter specifying the torus (with $\tau \equiv \tau_1 + i\tau_2$, $\tau_i \in \mathbf{R}$), and $\mathcal{F} \equiv \mathcal{F}(\Gamma)$ is the fundamental domain of the full modular group ($\mathcal{F}$: $|\tau| \geq 1$, $\tau_2 > 0$, $-\frac{1}{2} \leq \tau_1 \leq \frac{1}{2}$). Recalling that the Petersson inner product of two modular functions f and g of weight k with respect to the full modular group is[8-10]

$$\langle f|g\rangle \equiv \int_{\mathcal{F}} \frac{d^2\tau}{{\tau_2}^2}\, {\tau_2}^k\, \overline{f} g \ , \tag{1.4}$$

we see that if we can in some sense associate Z with ${\tau_2}^k \overline{f} g$, then a vanishing cosmological constant will result from the orthogonality of f and g with respect to the Petersson inner product. The vanishing of the cosmological constant could thus be understood as a selection rule, a manifestation of some underlying (i.e., Atkin-Lehner) symmetry inherent in the partition function Z.

This approach has the advantage of ensuring an exactly vanishing one-loop cosmological constant in the absence of supersymmetry as the result of a symmetry argument rather than as the result of a prearranged fine-tuning. It also has the advantage that this Atkin-Lehner symmetry can appear manifestly in expressions resembling the partition functions of self-consistent tachyon-free superstring models formulated in arbitrary (even) spacetime dimensions D. However, although such models with $D = 2$ have been constructed,[7] the many attempts at constructing such $D \geq 4$ models displaying Atkin-Lehner symmetries have all failed;[11-13] in fact, Balog and Tuite have recently constructed a general proof[14] showing essentially that the off-shell tachyons required for a model with an Atkin-Lehner symmetry are inconsistent with Lorentz invariance and proper spin statistics.

In this paper, therefore, we investigate the various ways in which we might somehow retain the Atkin-Lehner "selection rule" mechanism for obtaining a vanishing cosmological constant, yet avoid some of the problems associated with strict Atkin-Lehner symmetry. We pursue essentially two lines of thought.

First, we investigate adding to an Atkin-Lehner symmetric partition function other terms which themselves lack an Atkin-Lehner symmetry but which manage to leave the one-loop integral (1.3) unchanged. We refer to this as "hiding" the Atkin-Lehner symmetry. Some of these added terms have the property that they are purely imaginary functions of τ: although such terms cannot by themselves comprise the partition function of a self-consistent heterotic string model (as we will prove), we will see that when added to an Atkin-Lehner-symmetric partition function they can produce a total partition function with the correct candidate behavior. We also propose other such terms which are not purely imaginary functions, but which succeed as well in leaving the one-loop cosmological constant unchanged. We also show that inclusion of these extra terms restores the proper tachyons to these new partition functions, thereby averting the Balog-Tuite proof. We therefore find that we can explicitly construct many new partition functions in which Atkin-Lehner symmetry is thus hidden, and which may well correspond to physically sensible (but as yet unconstructed) superstring models.

Second, we investigate generalizing Atkin-Lehner symmetry itself, utilizing the same basic orthogonality argument but working with a much wider class of operators in the space of modular forms. In particular, the strict Atkin-Lehner operators on which the symmetry is based are usually taken to be *Hermitian* with respect to the Petersson inner product, yet we demonstrate that this requirement is unnecessary for the general argument to proceed. We therefore work with an essentially unrestricted class of (not necessarily Hermitian) operators, and consider an equally broad class of congruence subgroups with respect to which our partition functions can undergo coset decomposition. We find that in many cases our resulting symmetry either reduces to or implies strict Atkin-Lehner symmetry, and thus in these cases we obtain (if nothing else) a more complete picture of the mathematical implications of strict Atkin-Lehner symmetry. However, in other cases our resulting symmetry is of a fundamentally new character, and might therefore (either in its direct form or "hidden") provide a new mechanism for obtaining a vanishing cosmological constant. At present we are unable to construct expressions which both resemble superstring partition functions and display these symmetries; work in this area is continuing. The generality of our overall investigation, however, suggests that any such approach towards obtaining a vanishing one-loop cosmological constant will involve symmetries of either the strict or our generalized Atkin-Lehner variety.

This paper is organized as follows. First, in Section II, we give a brief review of Atkin-Lehner symmetry as originally proposed by Moore,[7] emphasizing only those points that will be relevant for our generalizations and establishing notational conventions. Then, in Section III, we discuss our method for "hiding" Atkin-Lehner symmetry, and after quickly reviewing the essential points of the Balog-Tuite proof[14] we demonstrate that our proposed partition functions succeed in avoiding its conclusions. In Section IV, we then discuss the implications of relaxing some of the restrictions of strict Atkin-Lehner symmetry, and thereby lay the groundwork for all of our generalizations. Finally, in Section V, we explore the ramifications of our generalized Atkin-Lehner symmetry, outlining those cases in which it reduces to or complements strict Atkin-Lehner symmetry and those cases in which it is fundamentally new. In an Appendix we list some formulas, derivations, and results which are used in the text.

II. BRIEF REVIEW OF ATKIN-LEHNER SYMMETRY

In this section we briefly review the important features of Atkin-Lehner symmetry as introduced by Moore,[7] stressing only those aspects relevant to our future generalizations. Complete discussions and details can be found in Ref. 7. Details concerning the modular group and the theory of modular forms can be found in any of the standard references;[8-10] some definitions are also included in the Appendix.

Atkin-Lehner operators are operators acting within and hence preserving the space [denoted $M_k(\Gamma')$] of those functions f which are weight k modular forms with respect to a congruence *subgroup* Γ' of the full modular group Γ. For all operators $\alpha = \binom{a\ b}{c\ d} \in \mathrm{GL}(2,\mathbf{Q})$ which have positive determinant, satisfy $\alpha^2 \propto \mathbf{1}$, and *normalize* the subgroup Γ' (i.e., for all $g \in \Gamma'$ there exists an $h \in \Gamma'$ such that $g\alpha = \alpha h$), the corresponding Atkin-Lehner operators $T_{[\alpha]}$ are defined

$$T_{[\alpha]}\, f(\tau) \;\equiv\; f[\alpha](\tau) \;\equiv\; (\det\alpha)^{k/2}\,(c\tau+d)^{-k}\, f(\alpha\tau)\,. \tag{2.1}$$

The idempotence of α implies that the Atkin-Lehner operators $T_{[\alpha]} \equiv [\alpha]$ will square to $\mathbf{1}$ when operating on forms with even weight, and hence will have eigenvalues ± 1. We can rewrite the Petersson inner product (1.4) in a form suitable for modular functions f and g of a subgroup Γ' by performing a coset decomposition:

$$\langle f|g\rangle_k \;\equiv\; \frac{1}{[\Gamma:\Gamma']}\int_{\mathcal{F}} \frac{d^2\tau}{{\tau_2}^2}{\tau_2}^k \sum_{i=1}^{[\Gamma:\Gamma']} \overline{f}[\gamma_i]g[\gamma_i]\;; \tag{2.2}$$

here $\mathcal{F}$ is still the fundamental domain of the *full* modular group, $[\Gamma:\Gamma']$ is the index of Γ' in Γ, and γ_i are the arbitrarily chosen right coset representatives of a right-coset decomposition of Γ' in Γ. It then follows that as long as the Γ'-invariant function $F \equiv \overline{f}g$ ($= F[\beta]$ for all $\beta \in \Gamma'$) satisfies some technical constraints having to do with convergence and the cancellation of tachyonic poles (so that the inner product *exists*), the Atkin-Lehner operator $T_{[\alpha]}$ will be Hermitian with respect to this inner product:

$$\langle f|g\rangle \;=\; \langle\, f \mid (T_{[\alpha]})^2 g\,\rangle \;=\; \langle\, T_{[\alpha]}f \mid T_{[\alpha]}g\,\rangle\,. \tag{2.3}$$

Thus, if f and g are chosen to be Atkin-Lehner eigenfunctions with opposite eigenvalues (or simply if F has Atkin-Lehner eigenvalue -1), then their Petersson inner product (2.2) will vanish. Therefore making the association

$$Z \;\equiv\; {\tau_2}^k \sum_i F[\gamma_i] \;=\; {\tau_2}^k \sum_i \overline{f}g[\gamma_i]\,, \tag{2.4}$$

we see that this vanishing of the Petersson inner product implies the vanishing of the (one-loop) cosmological constant. Note that this association does not require that the sum over coset transforms ${\tau_2}^{-k}Z$ itself be an Atkin-Lehner eigenfunction; in fact, this can never occur, for if $\gamma_i\alpha = \alpha\gamma_i'$ then for $\alpha \notin \Gamma$ we find $\gamma_i' \notin \Gamma$ – in other words, γ_i' cannot be coset representatives of Γ' in Γ. [Of course this association *does* require that Z and $-T_{[\alpha]}Z \equiv -{\tau_2}^k T_{[\alpha]}({\tau_2}^{-k}Z)$ have the same vanishing *integral* over fundamental domain. We note also that if $\alpha \in \Gamma$, then the only Z which is odd under $T_{[\alpha]}$ is identically zero and corresponds to a string model with spacetime supersymmetry. In this case, then, the Atkin-Lehner mechanism works by default.] One usually chooses the subgroup Γ' to be any of the (nonprincipal) congruence subgroups $\Gamma_0(N)$ of Γ:

$$\Gamma_0(N) \;\equiv\; \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \;\text{ with } c \overset{N}{=} 0 \right\} \tag{2.5}$$

where $\overset{N}{=}$ signifies equality modulo N, and takes the corresponding Hermitian operator $\alpha \notin \Gamma$ to be $\alpha_N : \tau \to -1/(N\tau)$.

One can alternatively understand this mechanism by writing (2.2) as

$$\langle f|g\rangle_k \;=\; \frac{1}{[\Gamma:\Gamma']}\int_{\mathcal{F}(\Gamma')} \frac{d^2\tau}{{\tau_2}^2}\,{\tau_2}^k\,\overline{f}g\,. \tag{2.6}$$

The integrand, by design, is odd under the transformation α: $\tau \to \alpha\tau$, whereas the measure is invariant under this transformation and the fundamental domain $\mathcal{F}(\Gamma')$ is transformed into itself. This integral must therefore vanish. Note that in this argument the transformation is taken to be α, *not* $[\alpha]$; the integrand (*including* ${\tau_2}^k$) is odd under α because $\overline{f}g$ itself is assumed odd under $[\alpha]$. (Transforming ${\tau_2}^k$ under α provides the factor $(\det\alpha)^k\,|c\tau+d|^{-2k}$ needed to compensate for the difference between α and $[\alpha]$ acting on $F \equiv \overline{f}g$.) Equation (2.4) then stands as written for reconstructing the full partition function.

Let us illustrate how this works in practice by restricting our attention to the simplest case $N = 2$ with partition functions of heterotic strings formulated in light-cone gauge for $D \leq 10$ spacetime dimensions. If our remaining degrees of freedom are carried by free world-sheet fermions obeying periodic or antiperiodic boundary conditions around the two noncontractable loops of the torus,[15] then our partition functions will be built from theta functions of half-integer characteristics (i.e., the Jacobi theta functions) and will take the form

$$Z \;=\; {\tau_2}^k\Delta^{-1}\overline{\Delta}^{-1/2} \sum_s \left(a_s \prod_{i=2}^{4} {\vartheta_i}^{n_i^{(s)}}\,\overline{\vartheta_i}^{\overline{n_i}^{(s)}} \right)\,. \tag{2.7}$$

Here Δ is the weight-12 modular form $[\eta(\tau)]^{24}$ (where η is the Dedekind eta function), a_s, $n_i^{(s)}$, and $\overline{n_i}^{(s)}$ are integers, the n's must be non-negative, and for all terms s we must have

$$\sum_{i=2}^{4} n_i^{(s)} = 24+2k$$

and (2.8)

$$\sum_{i=2}^{4} \overline{n_i}^{(s)} = 12+2k \, .$$

This sum over terms includes the sum over coset transforms (2.4). These weights k of the relevant modular forms are related to the spacetime dimension D by

$$k = 1 - D/2 \, . \tag{2.9}$$

Definitions and relevant properties of the Jacobi theta functions and Dedekind eta function are contained in the Appendix; more detailed discussions of these functions and their applications can be found in the literature.[16,17]

The first step, then, in constructing such a partition function Z with an $N=2$ Atkin-Lehner symmetry is to construct a $\Gamma_0(2)$-invariant F which we write in the form

$$F = \left(\frac{\mathcal{M}}{\Delta\overline{\Delta}^{1/2}}\right)\left(\frac{F_0}{\mathcal{M}}\right) , \tag{2.10}$$

where $\mathcal{M}$ is defined to be

$$\mathcal{M} \equiv (\vartheta_3\vartheta_4)^4(\overline{\vartheta_3\vartheta_4})^2 \tag{2.11}$$

and where F_0 is an expression to be determined (a sum of terms each with $24+2k$ powers of ϑ and $12+2k$ powers of $\overline{\vartheta}$). The reason for doing this $\mathcal{M}$-decomposition is that each factor in parentheses in (2.10) must separately have well-defined transformation properties under $[\alpha_2]$; this follows from the observation (see the Appendix)

$$\left(\frac{\mathcal{M}}{\Delta\overline{\Delta}^{1/2}}\right)[\alpha_2] = \left(\frac{\mathcal{M}}{\Delta\overline{\Delta}^{1/2}}\right) = \frac{8}{(\eta\,\eta[\alpha_2])^8(\overline{\eta}\,\overline{\eta}[\alpha_2])^4} \, . \tag{2.12}$$

Thus, $F_1 \equiv F_0/\mathcal{M}$ must itself be antisymmetric under $[\alpha_2]$ in order to obtain a net odd Atkin-Lehner symmetry for F. Note that since $\alpha_2{}^2 \propto \mathbf{1}$, for even D any such F_1 must be of the form

$$F_1 = F_2 - F_2[\alpha_2] \, ; \tag{2.13}$$

this is because $[\alpha_N{}^2] = (-1)^k$ when operating on forms of weight k, and because F_1 and F_2 must be functions of even total weight $14-D$. For example, for $D=4$ it can be shown that the choice $F_2 = 8\,(\vartheta_3\vartheta_4)^7\,\overline{\vartheta_2}^4\,\overline{\vartheta_3\vartheta_4}$ leads [via (2.13), (2.10), and (2.4)] to a total Z which could correspond, in principle, to a tachyon-free model.[7] (No such self-consistent model has been constructed, however.[11,12])

III. HIDDEN ATKIN-LEHNER SYMMETRY

In this section we investigate various means of "hiding" Atkin-Lehner symmetry by adding to an Atkin-Lehner-symmetric expression various terms which themselves not only lack any symmetry but also make vanishing contributions of their own to the one-loop cosmological constant. We then proceed to examine the "pole structures" (cusp divergence behaviors) of our new total partition functions, and relate these results to the proof of Balog and Tuite[14] that Atkin-Lehner symmetry is inconsistent with general physical principles. The results of this section suggest various ways to avoid the suppositions of their proof and thereby retain in part the Atkin-Lehner mechanism; in particular, we actually construct $D=4$ partition functions which lead to vanishing one-loop cosmological constants, pass our tests for physical consistency, and yet have no Atkin-Lehner symmetry.

One immediate way of generating such extra terms which make no contribution to the cosmological constant is to exploit the observation that the Petersson inner product of two modular functions is necessarily real, provided these functions are modular with respect to a congruence subgroup of the full modular group and provided that they and their relevant coset transforms have q expansions (where we take $q \equiv e^{2\pi i\tau}$ throughout). To see this, we let f and g be these modular functions with respect to $\Gamma' \subset \Gamma$, and write our inner product in the form (2.2). If we assume not only that f and g have q expansions, but that all $f[\gamma_i]$ and $g[\gamma_i]$ have q expansions as well, then it is straightforward to see that the real part of the integrand will be even in τ_1 and the imaginary part will be odd in τ_1. Since the fundamental domain $\mathcal{F} = \mathcal{F}[\Gamma]$ can always be chosen to be symmetric in τ_1, this means that $\langle f|g\rangle \in \mathbf{R}$. (Of course, if we work backwards with our inner product expressed in the form (2.6), we can conclude that $\mathcal{F}[\Gamma']$ can always be chosen to be symmetric in τ_1 as well.) Thus, provided the needed q expansions exist, we have the result

$$\langle f \mid g\rangle = \langle g \mid f\rangle \, . \tag{3.1}$$

In the case that f and g are built from Jacobi theta functions, these q expansions certainly exist. We therefore conclude that the inner product of such modular functions must necessarily be real.

This fact that the cosmological constant must be real gives us the freedom to add arbitrary purely imaginary functions having q expansions to a total partition function without changing the corresponding one-loop vacuum energy. Thus, the Atkin-Lehner mechanism discussed in the previous section can in principal be generalized to hold for partition functions which lack any overt total Atkin-Lehner symmetry, but which differ from Atkin-Lehner symmetric partition functions by precisely such purely imaginary functions [so that $\mathrm{Re}(F[\alpha_N]) = -\mathrm{Re}(F)$ continues to hold even though $F[\alpha_N] \neq -F$]. This is significant, particularly because we can construct such purely imaginary functions which are already in the $D \leq 10$ heterotic-string form (2.7) that we are considering—i.e., any function of the form

$$\tau_2{}^k\ \Delta^{-1}\overline{\Delta}^{-1/2}\ \sum_s\ a_s\left(\ \prod_{i=2}^{4}\vartheta_i{}^{n_i{}^{(s)}+4}\,\overline{\vartheta_i}{}^{\overline{n_i}^{(s)}} - \prod_{i=2}^{4}\vartheta_i{}^{\overline{n_i}^{(s)}+4}\,\overline{\vartheta_i}{}^{n_i{}^{(s)}}\ \right) \tag{3.2}$$

with

$$\sum_{i=2}^{4} n_i{}^{(s)}\ =\ \sum_{i=2}^{4}\overline{n_i}^{(s)}\ =\ 14-D\qquad\forall\ s \tag{3.3}$$

is purely imaginary. That (3.2) is purely imaginary follows from rewriting it in the form

$$\left(\frac{\tau_2{}^k}{|\eta|^{2D-4}}\right)\ \left(\frac{\vartheta_2\vartheta_3\vartheta_4}{\eta^3}\right)^4\ \sum_s\ a_s\left[\ \prod_{i=2}^{4}\left(\frac{\vartheta_i}{\eta}\right)^{n_i{}^{(s)}}\left(\frac{\overline{\vartheta_i}}{\overline{\eta}}\right)^{\overline{n_i}^{(s)}} - \prod_{i=2}^{4}\left(\frac{\vartheta_i}{\eta}\right)^{\overline{n_i}^{(s)}}\left(\frac{\overline{\vartheta_i}}{\overline{\eta}}\right)^{n_i{}^{(s)}}\right]\ ; \tag{3.4}$$

the first term in parentheses is manifestly real, the summation is manifestly purely imaginary, and the second term in parentheses [according to Eq. (A2) of the Appendix] is equal to 16 and hence is purely real as well. [In fact, if we construct such $D\le 10$ heterotic string models by fermionizing $10-D$ of the bosonic degrees of freedom to become free world-sheet fermions obeying periodic or antiperiodic boundary conditions around the noncontractable loops on the torus,[15] then we may interpret the first term in parentheses in (3.4) as the purely real partition function of the remaining $D-2$ (light-cone gauge) bosonic degrees of freedom, and interpret the second term and summation as potentially arising from the fermionic degrees of freedom in the theory. The sum over s would then be a sum over the sectors in the theory, or the sum over fermion boundary conditions needed for modular invariance. We remark in this context that Schellekens, who examined the partition functions of fermionic string theories obtained through lattice constructions, also managed to obtain such purely imaginary functions among those yielding vanishing cosmological constants.[13] We are *not*, of course, claiming that (3.2) itself is the partition function of such a self-consistent heterotic string model—indeed, we will be able to prove **that no such model can exist—but rather that such a term may be added at will to existing partition functions without destroying the overt interpretable form of the function.] Note that in the language of (3.1), adding (3.2) to a given partition function amounts to adding**

$$\sum_s a_s\ (\ \langle f_s|g_s\rangle\ -\ \langle g_s|f_s\rangle\ )\ =\ 0 \tag{3.5}$$

to its corresponding one-loop vacuum energy, where

$$f_s\ =\ \Delta^{-1/2}\ \prod_{i=2}^{4}\vartheta_i{}^{\overline{n_i}^{(s)}}\ ,\qquad g_s\ =\ \Delta^{-1}\ (\vartheta_2\vartheta_3\vartheta_4)^4\ \prod_{i=2}^{4}\vartheta_i{}^{n_i{}^{(s)}}\ . \tag{3.6}$$

We thus are able to judiciously replace various terms within the F of (2.10) with their complex conjugates, thereby destroying the strict Atkin-Lehner symmetry of F and yet still retaining the Atkin-Lehner mechanism. In fact, if F originally has a strict Atkin-Lehner symmetry with $N=2$, then proceeding as in previous sections we find that any term in $F_1\equiv F_0/\mathcal{M}$ of the form

$$\frac{\vartheta_2{}^4}{(\vartheta_3\vartheta_4)^2}\ \prod_{i=2}^{4}\vartheta_i{}^{n_i{}^{(s)}}\overline{\vartheta_i}{}^{\overline{n_i}^{(s)}} \tag{3.7}$$

[with the n's and $\overline{n}$'s satisfying (3.3)] may have its contribution to F freely replaced by the complex conjugate of that contribution. [Note that (3.7) requires that the relevant terms in F_1 must contain at least *four* powers of ϑ_2.] As far as the Atkin-Lehner mechanism is concerned, this freedom exists separately within any coset transform: the relevant terms within each $F[\gamma_i]$ may be replaced independently for any i. Of course, the requirement that the total Z be modular-invariant forces such replacements to happen jointly for corresponding terms across all coset transforms simultaneously.

However, a simple argument can now be used to greatly constrain (though not entirely eliminate) the applicability of this idea. For any given partition function Z, a Taylor expansion

$$Z\ =\ \tau_2{}^k\ \sum_{m,n} a_{mn}\ \overline{q}^m\ q^n \tag{3.8}$$

allows us to easily extract the net particle degeneracies a_{mn} that the corresponding model must have. Here m and n are respectively the contributions to the total energy of the particle (or state) from the left-moving and right-moving excitations, a_{mn} is the net number of allowed states having this energy distribution and surviving the Gliozzi-Scherk-Olive (GSO) projections (i.e., surviving the sum over sectors or boundary conditions), and $q\equiv e^{2\pi i\tau}$. (By "net number" we mean the number of such spacetime bosonic states minus the number of such spacetime fermionic states. Also note that this number actually tallies individual degrees of freedom; a graviton, for example, is a state with $m=n=0$ and will increase a_{00} by 2, one count for each helicity.) Clearly, states with $m\ne n$ do not satisfy the level-matching conditions and are "off shell" (they still contribute, of course, to the cosmological constant with a dependence on the absolute value $|m-n|$), and states with $m+n<0$ are tachyonic. It is then clear that replacing any term with its complex conjugate is equivalent to interchanging m and n in its corresponding Taylor expansion. Thus, models whose partition functions differ by terms of the form (3.2) have precisely the same on-shell particle degeneracies.

One of the major constraints involved in a search for Atkin-Lehner-symmetric partition functions is the requirement that the corresponding model have no on-shell tachyonic states (i.e., that $a_{mn}=0$ for all $m=n<0$); otherwise, the Petersson inner products do not converge (and the model is flawed from a physical standpoint as well). This constraint is sufficiently restrictive that for $N=2$ and $D=4$, only one partition function free of on-shell tachyons has been constructed[7]; it is built from theta functions with half-integer characteristics. Similarly, for $N=3$ and $D=4$, again only one such partition function has been constructed;[12] it built from theta functions with third-integer characteristics. We therefore see that our freedom to add purely imaginary functions such as (3.2) to Z (and thereby destroy its overt Atkin-Lehner symmetry) will not allow us to evade this constraint and create new Z's which are tachyon-free.

We therefore conclude that this freedom, while potentially yielding new partition functions with vanishing cosmological constants, can be fruitfully applied only to those Atkin-Lehner symmetric partition functions which are already free of on-shell tachyons. In particular, we limit our attention (for $D=4$) to the partition function proposed by Moore:[7]

$$F_1 = -F_1[\alpha_2] = 8\,\vartheta_3{}^7\vartheta_4{}^7\,\overline{\vartheta_2}^4\,\overline{\vartheta_3\vartheta_4} - \vartheta_2{}^{14}\,\overline{\vartheta_2}^2\,\overline{\vartheta_3}^2\,\overline{\vartheta_4}^2 \tag{3.9}$$

with $F_{\rm AL}\equiv \mathcal{M}F_1/(\Delta\overline{\Delta}^{1/2}) = -F_{\rm AL}[\alpha_2]$. Recalling that we can apply this idea to only those terms in F_1 for which ϑ_2 appears with power ≥ 4 [see (3.7)], we construct the purely imaginary function

$$\begin{aligned} I \;\equiv\; \Delta^{-1}\overline{\Delta}^{-1/2}\{\;& \vartheta_2{}^{14}\vartheta_3{}^4\vartheta_4{}^4\,\overline{\vartheta_2}^2\,\overline{\vartheta_3}^4\,\overline{\vartheta_4}^4 \\ & -\vartheta_2{}^6\vartheta_3{}^8\vartheta_4{}^8\,\overline{\vartheta_2}^{10}\,\} \end{aligned} \tag{3.10}$$

and consider working with $F^{(1)}\equiv F_{\rm AL}+I$. Observe that since the phases obtained under $[S]$ and $[T]$ are the same for a given term and its complex conjugate (provided $n_2 \stackrel{4}{=} \overline{n_2}$), the $\Gamma_0(2)$ invariance of $F_{\rm AL}$ ensures the $\Gamma_0(2)$ invariance of $F^{(1)}$. The corresponding entire partition function $Z^{(1)}$ is therefore still obtained from (2.4). By construction, a model with partition function $Z^{(1)}$ will have a vanishing one-loop vacuum energy and cosmological constant, yet $Z^{(1)}$ itself will have no Atkin-Lehner symmetry. Note that we may indeed consider the more general series of partition functions

$$F^{(k)} \;=\; F_{\rm AL} \;+\; k\,I \quad (k\in\mathbf{Z})\;; \tag{3.11}$$

all corresponding $Z^{(k)}$'s will lead to zero cosmological constant, and only for $k=0$ will a corresponding model have an Atkin-Lehner symmetry. We remark that in principle *any* purely imaginary function I can be added to $F_{\rm AL}$ at this stage; we have chosen I as in (3.10) merely in order to remove explicitly the contribution of the second term in (3.9) when forming $F^{(1)}$, and thereby erase all overt signs of Atkin-Lehner symmetry. (Later we will limit our possibilities for I.)

We now approach the question of whether these partition functions $Z^{(k)}$ might correspond to self-consistent string models, especially in light of the recent general proof by Balog and Tuite[14] that partition functions with Atkin-Lehner symmetry cannot by themselves correspond to physically consistent string models. We will find that our $Z^{(k)}$'s are able to avoid the suppositions of the proof, and have properties which depend on k and which are suggestive of proper behavior.

The proof given by Balog and Tuite is remarkably simple and general; it involves consideration of only the "pole strengths" of the relevant partition functions, or the divergence behavior of these functions at the cusps in the fundamental domain of the congruence subgroup under consideration. Restricting our attention to $\Gamma_0(2)$, we see that there are two such $\Gamma_0(2)$-independent cusps: $\tau\to i\infty$ and $\tau\to 0$. (The full modular group fundamental domain has one cusp as $\tau\to\tau_\infty\equiv i\infty$, and in general the complete set of cusps for a given subgroup $\Gamma'\subset\Gamma$ is $\{\gamma_i\,\tau_\infty,\; i=1,\ldots,[\Gamma:\Gamma']\}$, where γ_i are the coset representatives of Γ' in Γ. Behavior at the Γ'-independent cusps is of course sufficient to describe behavior at all the cusps.) We define the "pole strength" of a given function $f(\tau)$ as $\tau\to i\infty$ (or $q\to 0$) to be the negative of the leading power of q in a Taylor-expansion in q of f, and define the pole strength as $\tau\to 0$ (or $q\to 1$) to be the pole strength of $f[S]$ as $\tau\to i\infty$. Given a function $f(\tau,\overline{\tau})$, then, we follow Balog and Tuite by indicating its four pole strengths with the notation

$$f \;\sim\; \begin{bmatrix} A & a \\ B & b \end{bmatrix}\;, \tag{3.12}$$

where A and B are the $\tau\to i\infty$ pole strengths of f and $f[S]$ respectively, and where a and b are the corresponding pole strengths as $\overline{\tau}\to(i\infty)^*$. Notice that if f is a purely imaginary function, then we must have $A=a$ and $B=b$; similarly, if f is some modular-invariant function (such as a partition function) then we must have $A=B$ and $a=b$ (because $f=f[S]$). Further examination shows as well that for F in our generic form (2.7), we have $A,B\leq 1$ and $a,b\leq\frac{1}{2}$; additionally, our pole strengths must be quantized in eighth-integers. [The upper bounds on our pole strengths come from the minimum attainable vacuum energies in any sector of a heterotic string theory, or equivalently from the normal-ordering constants in our string Hamiltonians. The quantization constraint is apparent from Eq. (A1) in the Appendix, where we are allowing for possible odd powers of ϑ_2.]

Let us now quickly review the Balog-Tuite proof that Atkin-Lehner symmetry is inconsistent with Lorentz invariance and spin statistics. In particular, we focus on the effect the operator $[\alpha_N]$ has on the pole structure of our $\Gamma_0(N)$-invariant function F. We first define the scaling matrix $X_N \equiv \binom{N\ 0}{0\ 1}$, so that $\alpha_N = SX_N$. If F itself has the pole structure given in (3.12), then since $F[\alpha_N] = F[S][X_N]$, we see that $F[\alpha_N]$ will have pole strengths NB and Nb as $q \to 0$ and $\overline{q} \to 0$ respectively. Similarly, since $\alpha_N S = SX_N S = -X_{1/N}$, we see that $F[\alpha_N S]$ will have pole strengths A/N and a/N as $q \to 0$ and $\overline{q} \to 0$ respectively. Thus, the effect of $[\alpha_N]$ on the pole structure of F is to change it as follows:

$$[\alpha_N] : \begin{bmatrix} A & a \\ B & b \end{bmatrix} \longrightarrow \begin{bmatrix} NB & Nb \\ A/N & a/N \end{bmatrix} . \tag{3.13}$$

If F is assumed invariant under $[\alpha_N]$, then we must therefore have the relations on the pole strengths $A = NB$ and $a = Nb$. However, since $A \leq 1$, this constraint implies $B \leq 1/N < 1$ for $N > 1$. There must therefore be some cancellations in the degeneracy matrix a_{mn} corresponding to F between spacetime bosonic and fermionic contributions, so that the upper limit $B = 1$ is not reached. However, for any right-moving tachyonic states with $m < 0$ (contributing to a pole strength $a > 0$), such bosonic contributions *cannot* be canceled because Lorentz invariance forbids fermionic states which are tachyonic. Hence, if $B < 1$, we must have the pole strength $a = 0$. The Atkin-Lehner relation $a = Nb$ then requires $b = 0$, so there are no poles at all in F as $\overline{\tau}$ approaches its two cusp points. This means that F, viewed as a modular function in the variable $\overline{\tau}$ with τ-dependent coefficients, is a modular *form* in $\overline{\tau}$. (See Ref. 14 for details. In particular, F is strictly speaking a modular form with respect to only a congruence *subgroup* of Γ.) Now, the relation (2.9) shows that for spacetime dimension $D > 2$, the weight k of F will be negative. Since a standard result from modular form theory asserts that there exist *no* such strict modular forms with negative weight,[8-10] we conclude that $F = 0$. Hence, any F which is assumed not only to have Atkin-Lehner symmetry but also to arise from a physically sensible model must vanish—i.e., it must correspond at least to a supersymmetric model. Thus, direct Atkin-Lehner symmetry fails as a mechanism yielding a vanishing cosmological constant in the absence of supersymmetry. We remark that this argument of Balog and Tuite is a general one, independent of the particular structure of the string theory in question. (Balog and Tuite also present a more detailed alternative proof in the case of string theories obtained through lattice compactifications.[14])

We have seen that the freedom to add purely imaginary functions allows us to build $D = 4$ partition functions $Z^{(k)}$ and thereby remove explicit Atkin-Lehner symmetry, but in order to truly change the underlying physics and avoid the above general proof, such additions must significantly change the pole structures of our partition functions. It is straightforward to calculate the pole structure of $F_{\rm AL}$ as defined through (3.9):

$$F_{\rm AL} \sim \begin{bmatrix} 1 & 1/4 \\ 1/2 & 1/8 \end{bmatrix} ; \tag{3.14}$$

note that this indeed satisfies the Atkin-Lehner pole constraints. Similarly, we can calculate the pole structure of I as defined in (3.10); we obtain

$$I \sim \begin{bmatrix} 1/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} . \tag{3.15}$$

We thus find that our general functions $F^{(k)}$ defined in (3.11) have the pole structures

$$F^{(k)} \sim \begin{cases} \begin{bmatrix} 1 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} & \text{for } k \neq 1; \\ \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} & \text{for } k = 1 . \end{cases} \tag{3.16}$$

Note that for $k = 1$, the leading B and a poles from $F_{\rm AL}$ and I precisely cancel; this explains the two different cases above. It is clear, however, that in neither case do both $\overline{\tau}$ pole strengths vanish; hence we cannot immediately conclude that $F^{(k)} = 0$. While it might have been appealing to find a purely imaginary function I leading to a total $F^{(k)}$ with pole strength $B = 1$, we can easily see that if I is to take the general form (2.7), then having $B = 1$ would require $n_4 = 0$ for some term in I. No such term can appear in I, as (3.4) indicates. (Below we will present other terms which are *not* purely imaginary and which *do* have $B = 1$.)

This function I in (3.10) is not the only purely imaginary function which might be added to $F_{\rm AL}$. If indeed we wish to ensure a total $F_{\rm AL} + I$ which has pole strength $a = 0$ whenever $B < 1$ (as Lorentz invariance and the absence of fermionic tachyons would demand), then we want in particular other purely imaginary functions I_i having pole strengths $a = 1/4$ so that the $a = 1/4$ pole in $F_{\rm AL}$ might possibly be canceled in the total $F_{\rm AL} + I_i$. It is a straightforward task to obtain a complete list of all two-term purely imaginary functions

$$I_i \equiv \Delta^{-1}\overline{\Delta}^{-1/2}\, \vartheta_2{}^{n_2}\vartheta_3{}^{n_3}\vartheta_4{}^{n_4}\, \overline{\vartheta_2}^{\overline{n_2}}\, \overline{\vartheta_3}^{\overline{n_3}}\, \overline{\vartheta_4}^{\overline{n_4}} \ - \text{c.c.} \tag{3.17}$$

which have pole strength $a = 1/4$. Here $\sum_{i=2}^4 n_i = 22$, $\sum_{i=2}^4 \overline{n_i} = 10$, and therefore each such function may be specified by the four powers $(n_2, n_3, \overline{n_2}, \overline{n_3})$ of the first term [apparently yielding $(66)^2 = 4356$ possibilities]. However, if we demand that all of the powers satisfy the constraint $n_i \stackrel{4}{=} \overline{n_i}$ (this is needed for consistency, for example, in the spin-structure construction of fermionic strings[15]), then remarkably there are only 11 such functions having $a = 1/4$. Each one has pole structure $A = a = 1/4$ and $B = b$ for some b, and for each, the leading terms contributing to the $a = 1/4$ pole are therefore of the form $f\,\overline{q}^{-1/4}\,(q^{p_1} + f_2\, q^{p_2} + \cdots)$ (where $f > 0$ is ensured by adjusting the overall sign of I_i, or equivalently by interchanging the powers $(n_i - 4) \leftrightarrow \overline{n_i}$ in our labeling). Table I lists these 11 purely imaginary functions, along with their corresponding b, f_1, p_1, f_2, and p_2 values (where for convenience we have defined $f_1 \equiv \log_2 f$). Note that in some cases multiple sets of

TABLE I. Purely imaginary functions I_i having pole strength $a = 1/4$.

i	Powers in first term $(n_2, n_3, \overline{n_2}, \overline{n_3})$	b	Leading pole contributions f_1	p_1	f_2	p_2
1	(6,12,2,0)	1/2	12	1/4	42	5/4
2	(6,12,2,4) or (10,8,2,4) or (10,8,2,8)	1/2	11	1/4	8	3/4
3	(10,8,2,0)	1/2	11	1/4	8	3/4
4	(14,4,2,8)	1/2	15	3/4	22	7/4
5	(14,4,2,4)	1/2	15	3/4	22	7/4
6	(14,4,2,0)	1/2	15	3/4	22	7/4
7	(10,4,2,8)	0	11	1/4	−8	3/4
8	(6,8,2,0) or (10,4,2,0) or (10,4,2,4)	0	11	1/4	−8	3/4
9	(6,10,2,2) or (10,6,2,2) or (10,6,2,6)	1/4	11	1/4	10	5/4
10	(6,9,2,1) or (10,5,2,1) or (10,5,2,5)	1/8	11	1/4	−4	3/4
11	(6,11,2,3) or (10,7,2,3) or (10,7,2,7)	3/8	11	1/4	4	3/4

powers are listed; the corresponding expressions for I_i are all equal upon application of the Jacobi identity [the first of Eqs.(A2) in the Appendix]. We also observe that $I_9[S] = -I_9$ (which explains why I_9 has $b = 1/4$), as well as the relations

$$\begin{aligned} I_1 &= I_2 + I_7 = I_4 + 2I_7 \\ &= I_3 + I_8 = I_6 + 2I_8 \\ &= I_5 + I_7 + I_8 \,. \end{aligned} \tag{3.18}$$

The function I_5 in Table I is simply the function I in (3.10); we have already seen in (3.16) that it succeeds in producing pole strength $a = 0$ for $F^{(1)} \equiv F_5^{(1)} \equiv F_{\rm AL}+I_5$. This occurs because the leading contributions to the a $=1/4$ pole strength in $F_{\rm AL}$ are $-2^{15}\,\overline{q}^{-1/4}(q^{3/4}+22q^{7/4})$, and because I_5 has precisely these f and p values but with opposite overall sign. (Note that *all* terms $\overline{q}^{-1/4}q^n$ for $n \geq 3/4$ must, and do, cancel as well, along with terms $\overline{q}^{-1/8}q^n$ for all n.) From Table I, we see that F_4 $\equiv F_{\rm AL} + I_4$ and $F_6 \equiv F_{\rm AL} + I_6$ will also have no $\overline{q}^{-1/4}q^n$ terms for $n = 3/4$, $5/4$, and $7/4$, and indeed both of these total invariants have pole strengths

$$F_4, F_6 \sim \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix} . \tag{3.19}$$

These functions, along with F_5, are also consistent with Lorentz invariance, and lead to zero cosmological constant in the absence of Atkin-Lehner symmetry.

If we wish to use any of the other functions I_i from Table I in this way, since they all have $p_1 = 1/4$ we must take some linear combination of them in order not only to cancel their $\overline{q}^{-1/4}q^{1/4}$ contributions to the $a = 1/4$ pole strength but also to retain their $\overline{q}^{-1/4}q^{3/4}$ contributions. (Note that none of these functions has $p_1 = -1/4$, since purely imaginary partition functions cannot have net on-shell particle degeneracies. The powers p_i are restricted to values $z/2+1/4$ with integer z as a consequence of our restriction $n_2 \stackrel{4}{=} \overline{n_2}$.) However, looking first at I_1 through I_8 [i.e., those functions whose powers (n_2,n_3,n_4) and $(\overline{n_2},\overline{n_3},\overline{n_4})$ are equal to (2,0,0) modulo 4], we see that various linear combinations either completely remove *all* terms with factors of $\overline{q}^{-1/4}$ (e.g., $I_8 - I_7 = I_2 - I_3$ has by itself a "pole" strength $A = a < 0$), or reproduce the successful functions $I_4 = I_2 - I_7$, $I_5 = I_2 - I_8$, $I_6 = I_3 - I_8$. Indeed, if we attempt to use the functions I_9, I_{10}, and I_{11} [which have powers (2,2,2), (2,1,3), and (2,3,1) modulo 4 respectively], we find that while we may succeed in cancelling the leading contribution(s) to the $a = 1/4$ pole strength, we cannot find suitable combinations to cancel the entire pole.

Thus, working with the expression $F_{\rm AL}$ proposed by Moore, we see that we have been able to generate three distinct $\Gamma_0(2)$-invariant functions F_4, F_5, and F_6 for which $a = 0$, for which Atkin-Lehner symmetry is hidden, and for which the corresponding one-loop cosmological constants vanish. We can then in principle further add to these expressions any arbitrary linear combinations of purely imaginary functions J_i having "pole" strengths $a \leq 0$. There are only six such two-term functions, and they are listed in Table II; their leading contributions are again of the form $2^{f_1}\,\overline{q}^{-a}\,(q^{p_1}+f_2q^{p_2})$, where these f and p parameters are listed.

However, let us remark at this point that given a model's partition function and a subgroup $\Gamma' \subset \Gamma$, the pole stengths of the Γ'-invariant F are not uniquely specified because F itself is not uniquely determined. For any given invariant F, we can construct the quantity

$$\begin{aligned} F' &\equiv \frac{1}{[\Gamma:\Gamma']-1}\,({\tau_2}^{-k}Z - F) \\ &= \frac{1}{[\Gamma:\Gamma']-1}\sum_{i=2}^{[\Gamma:\Gamma']} F[\gamma_i] \end{aligned} \tag{3.20}$$

where γ_i are the coset representatives with $\gamma_1 \equiv \mathbf{1}$ and where ${\tau_2}^{-k}Z$ is defined in (2.4). It then follows that F and F' will be invariants of the same subgroup Γ', and that

$$Z' \equiv {\tau_2}^k \sum_{i=1}^{[\Gamma:\Gamma']} F'[\gamma_i] = Z \,. \tag{3.21}$$

We thus may equivalently take F' as our fundamental Γ' invariant. However, F and F' will not necessarily have the same pole structures; in fact, for our original

TABLE II. Purely imaginary functions J_i having "pole" strengths $a \leq 0$.

i	Powers in first term $(n_2, n_3, \overline{n_2}, \overline{n_3})$	a	b	Leading contributions f_1	p_1	f_2	p_2
1	(8,10,4,2) or (12,6,4,2) or (12,6,4,6)	0	1/2	15	1/2	4	1
2	(8,8,4,0) or (12,4,4,0) or (12,4,4,4)	0	1/4	15	1/2	−4	1
3	(8,9,4,1) or (12,5,4,1) or (12,5,4,5)	0	3/8	15	1/2	16	3/2
4	(9,8,5,0) or (13,4,5,0) or (13,4,5,4)	−1/8	3/8	17	5/8	−2	9/8
5	(9,9,5,1) or (13,5,5,1) or (13,5,5,5)	−1/8	1/2	17	5/8	2	9/8
6	(10,8,6,0) or (14,4,6,0) or (14,4,6,4)	−1/4	1/2	19	3/4	22	7/4

examples we find

$$F_{\rm AL}' \sim \begin{bmatrix} 1/2 & 1/4 \\ 1 & 1/2 \end{bmatrix} ,\qquad F_5^{(k)\prime} \sim \begin{cases} \begin{bmatrix} 1/2 & 1/2 \\ 1 & 1/2 \end{bmatrix} & \text{for } k \neq 1, \\ \begin{bmatrix} 0 & 1/2 \\ 1 & 1/2 \end{bmatrix} & \text{for } k = 1 . \end{cases} \tag{3.22}$$

We therefore see that in all of our cases we now have $B = 1$; for these functions there is therefore no apparent need for any (impossible) tachyonic cancellation, or for having $a = 0$.

The foregoing discussion leads us to conclude that while we must be careful to check that $a = 0$ if $B < 1$ for *all possible* Γ' invariants, beyond this it is perhaps only the pole structure of the *full* partition function Z which should concern us. What pole structures for Z might correspond to physical heterotic string models? As we remarked above, we must have $A = B$ and $a = b$ for modular invariance. Furthermore, it is straightforward to prove that we must have $A = B = 1$, for in the spectrum of any physically sensible nonsupersymmetric string model there will always exist at least one off-shell spacetime bosonic state with right- and left-moving energies $(m, n) = (0, -1)$. (Such a state is essentially the graviton without its left-moving bosonic coordinate excitation. The assumed absence of gravitinos in the model implies the absence of such corresponding fermionic states against whose contributions to Z the contributions of this state might cancel.) Thus, our full partition function Z must at least have the pole structure

$$Z \sim \begin{bmatrix} 1 & -m \\ 1 & -m \end{bmatrix} \tag{3.23}$$

for some $m \geq -1/2$. Note that this already rules out the possibility of obtaining a heterotic string partition function Z which is itself a purely imaginary function, as claimed earlier.

States with $m = -1/2$ can have a variety of corresponding left vacuum energies n. One requirement for the physical consistency of a string model is that for all states, $m - n$ must be an integer; otherwise, integration across the fundamental domain of the full modular group in the $\tau_2 > 1$ region would fail to enforce the level-matching conditions $L_0 = \overline{L_0}$ and thereby lead to inconsistencies. Thus, since $n \geq -1$, we can possibly have $n = -1/2,\ 1/2,\ 3/2$, etc. If we assume that our model has no on-shell tachyons (otherwise the cosmological constant would diverge), then we must have $n \geq 1/2$. States with $n = 1/2$ and $m = -1/2$ can occur as spacetime Lorentz scalars or Lorentz vectors; they must of course be spacetime bosons, since Lorentz invariance requires that no spacetime fermions can have a negative right-moving energy. It is easy to construct self-consistent, tachyon-free, nonsupersymmetric string models which have such Lorentz scalar and vector states, so in general there are indeed many cases for which $m = -1/2$ in (3.23). (In fact, we remark that while it is also easy to construct models for which there are *no* such Lorentz vector states, it is fairly difficult if not impossible to avoid such Lorentz scalar states. Thus $m = -1/2$ may well be an absolute requirement for such a model.)

Calculating the pole structures corresponding to the full partition functions of our examples, we find

$$Z_{\rm AL} \sim \begin{bmatrix} 1 & 1/4 \\ 1 & 1/4 \end{bmatrix} , \tag{3.24}$$

whereas

$$Z_4^{(k)}, Z_5^{(k)}, Z_6^{(k)} \sim \begin{bmatrix} 1 & 1/2 \\ 1 & 1/2 \end{bmatrix} \quad \text{for all } k . \tag{3.25}$$

We thus see that whereas $Z_{\rm AL}$ has a very unlikely pole-structure behavior, our partition functions $Z_{4,5,6}^{(k)}$ do indeed have the pole-structure behavior required by physical consistency. Furthermore, for $k = 1$ their possible $\Gamma_0(2)$-invariants are all consistent with Lorentz invariance and the absence of fermionic tachyons. These properties are all true as well for partition functions $Z_{4,5,6}^{\{a_i\}}$ formed from $F_{4,5,6}^{\{a_i\}} \equiv F_{\rm AL} + I_{4,5,6} + \sum_{i=1}^{6} a_i J_i$.

There may also exist *other* sorts of terms which are *not* purely imaginary but which, when added to an Atkin-Lehner symmetric partition function, also leave the corresponding cosmological constant unchanged. In particular, we have conjectured elsewhere[18] that the modular-invariant expression

$$Q \equiv \tau_2{}^{-1}\Delta^{-1}\overline{\Delta}^{-1/2} \sum_{\substack{i,j,k=2\\ i\neq j\neq k}}^{4} |\vartheta_i|^4 \times \Big\{ \vartheta_i{}^4\vartheta_j{}^4\vartheta_k{}^4 \left[2\,|\vartheta_j\vartheta_k|^8 - \vartheta_j{}^8\overline{\vartheta_k}{}^8 - \overline{\vartheta_j}{}^8\vartheta_k{}^8 \right] + \vartheta_i{}^{12}\left[4\,\vartheta_i{}^8\overline{\vartheta_j}{}^4\overline{\vartheta_k}{}^4 + (-1)^i\ 13\,|\vartheta_j\vartheta_k|^8 \right] \Big\} \tag{3.26}$$

satisfies

$$\int_{\mathcal{F}(\Gamma)} \frac{d^2\tau}{\tau_2{}^2}\, Q = 0\,, \tag{3.27}$$

and thus the $i = 2$ terms within the summation in (3.26) may well be taken as a suitable $\Gamma_0(2)$-invariant function F_Q to be added to $F_{\rm AL}$. Note that neither Q nor F_Q is purely imaginary (or pointwise vanishing, for that matter), and note also that in this case F_Q (unlike I) does not by itself have a vanishing contribution to the cosmological constant. Rather, it is the *sum* of the contributions from its coset transforms, or equivalently the contribution from the total $Q \equiv \tau_2{}^{-1}\sum_{i=1}^{3} F_Q[\gamma_i]$, which together yields the vanishing change in cosmological constant. (An analysis of this function Q can be found in Ref. 18.) The functions Q and F_Q are calculated[18] to have the same pole structure:

$$F_Q \sim Q \sim \begin{bmatrix} 1 & 1/2 \\ 1 & 1/2 \end{bmatrix}\,, \tag{3.28}$$

and thus adding F_Q to $F_{\rm AL}$ will enable us once again to avoid the consequences of the Balog-Tuite proof. (In fact, we note that since Q has the same pole structure as in (3.25), and also since Q has $a_{mm} = 0$ for all $m < 0$, Q itself might well be used to construct the partition function of a physically sensible heterotic string model directly. This possibility is also discussed in Ref. 18.)

Thus, there seem to be many ways in which Atkin-Lehner symmetry may be exploited to yield vanishing cosmological constants, without having partition functions exhibit Atkin-Lehner symmetry themselves. At present we are unable to construct a model having any of these partition functions, although we cannot definitively rule out this possibility.

IV. GENERALIZING ATKIN-LEHNER SYMMETRY: THE GROUNDWORK

In this section and the subsequent one, we instead undertake a more general investigation into what other types of symmetries and operators $[\alpha]$ might allow the same Atkin-Lehner mechanism to yield a vanishing one-loop vacuum amplitude. In this section we lay the groundwork for our search by systematizing our consideration of various operators $[\alpha]$ and congruence subgroups $\Gamma' \subset \Gamma$, and in the next section we investigate the implications of various choices. We shall not be concerned with pole structures or the Balog-Tuite proof in these sections, since any new symmetries we obtain can in principle also be "hidden" in the manner discussed in the previous section.

Perhaps the most important generalization of strict Atkin-Lehner symmetry, one which is briefly alluded to in Moore's original paper,[7] is the possibility of working with operators α which do *not* satisfy $\alpha^2 \propto \mathbf{1}$. The implications of this, however, are quite important. The primary motivation behind Atkin-Lehner symmetry is the analogy with selection rules in quantum mechanics: given an inner product in a Hilbert space and an operator Hermitian with respect to it, any two eigenstates of that operator with different eigenvalues will be orthogonal (have vanishing inner product). It is essential for the validity of this result that the operator be Hermitian with respect to the inner product. In the case of Atkin-Lehner symmetry, however, we are dealing with the Petersson inner product acting in the space of modular forms, and one can show that, under certain restrictions,

$$[\alpha]^\dagger = [(\det\alpha)\,\alpha^{-1}]\,. \tag{4.1}$$

(The proof will be sketched below.) Thus we see that only for $\alpha^2 \propto \mathbf{1}$ will $T_{[\alpha]} \equiv [\alpha]$ be Hermitian; for the Petersson inner product, these two properties are essentially synonymous. We are therefore in effect investigating working with non-Hermitian operators $[\alpha]$.

The crucial step in the Atkin-Lehner argument given in Section II is Eq. (2.3), and we see that the middle step in (2.3) fails to hold if $T_{[\alpha]}$ is not Hermitian (or equivalently, if it does not square to $\mathbf{1}$). However, the immediate corollary of the above result (4.1) is that under the same restrictions as for (4.1), we have

$$\langle f|g\rangle = \langle\, f[\alpha] \mid g[\alpha]\,\rangle\,. \tag{4.2}$$

(This follows from (4.1) upon considering $\langle f|g[\alpha\alpha^{-1}]\rangle$.) Thus, we see that although the middle step in (2.3) may no longer hold for non-Hermitian operators, the final step is still valid. We therefore expect the Atkin-Lehner mechanism to work with non-Hermitian operators $T_{[\alpha]} \equiv [\alpha]$ as well.

Because (4.2) is crucial to all that follows, before proceeding further we give a brief sketch of its proof independent of (4.1). This sketch will highlight the restrictions that apply; the complete proof is standard and can be found in nearly all books on modular form theory.[8] The fundamental idea here is a simple change of variables. We begin by considering $f[\alpha]$ and $g[\alpha]$ to be modular functions with respect to some subgroup $\Gamma' \subset \Gamma \equiv {\rm SL}(2,\mathbf{Z})$, and write

$$\langle f[\alpha] \mid g[\alpha]\rangle = \frac{1}{[\Gamma:\Gamma']}\int_{\mathcal{F}[\Gamma']} \frac{d^2\tau}{\tau_2{}^2}\ \left(\tau_2{}^k\ \overline{f}[\alpha]\,g[\alpha]\right) \tag{4.3}$$

where $\mathcal{F}[\Gamma']$ is the fundamental domain of Γ'. If $\alpha \in \mathrm{GL}^+(2,\mathbf{Q})$, then defining $z \equiv \alpha\tau$ we easily show that the integrand in parentheses is equal to ${z_2}^k\overline{f(z)}g(z)$ and that our modular-invariant measure can be similarly expressed with τ replaced by z. We thus have an integral completely in terms of z, and performing a change of variables back from z to τ we obtain

$$\langle f[\alpha] \mid g[\alpha]\rangle = \frac{1}{[\Gamma:\Gamma']}\int_{\alpha\mathcal{F}[\Gamma']}\frac{d^2\tau}{{\tau_2}^2}\,{\tau_2}^k\,\overline{f(\tau)}g(\tau)\ . \tag{4.4}$$

This is equal to $\langle f|g\rangle$, *provided* we can carefully choose the group Γ' so that: (1) $f[\alpha]$ and $g[\alpha]$ are modular functions with respect to it [to justify (4.3)]; and (2) f and g are simultaneously modular functions with respect to some other group Γ'' such that

$$\frac{1}{[\Gamma:\Gamma'']}\int_{\mathcal{F}[\Gamma'']} = \frac{1}{[\Gamma:\Gamma']}\int_{\alpha\mathcal{F}[\Gamma']}\ . \tag{4.5}$$

These two requirements can be satisfied[8] if we choose $\Gamma' = \alpha^{-1}\Gamma''\alpha$ with $\Gamma'' = G\cap\alpha G\alpha^{-1}$ for some *congruence* subgroup $G\subset\Gamma$. Since this implies that Γ' and Γ'' are also congruence subgroups, this is the restriction that must be placed on these groups and consequently on f and g. Since for congruence subgroups Γ and Γ'' the above choices can always be made, we conclude that (4.2) holds for all $\alpha\in\mathrm{GL}^+(2,\mathbf{Q})$, provided $f,g\in M_k(G)$ where G is a congruence subgroup of the modular group.

The rest of the Atkin-Lehner argument given above then follows as before. In particular, we can write our invariant F in the more general form

$$F = \sum_i \overline{f_i}\,g_i\ , \tag{4.6}$$

in which case our assertion is that if we have

$$\sum_i \langle f_i[\alpha] \mid g_i[\alpha]\rangle = A\sum_i\langle f_i \mid g_i\rangle \tag{4.7}$$

with $A\neq 1$ (for some Hermitian *or* non-Hermitian operator $[\alpha]$), then a model with partition function $Z \equiv {\tau_2}^k\sum_j F[\gamma_j]$ will have vanishing cosmological constant (where the γ_j form the right transversal in Γ of a congruence subgroup normalized by α). Note that we need not necessarily require that $A=-1$ (i.e., we need not require *odd* symmetry), for our operator $[\alpha]$ need not square to $\mathbf{1}$. (In the case that our partition function is built from Jacobi theta functions, however, we will see that the operator $[\alpha]$ often squares to $\mathbf{1}$ even if the matrix α itself does not.)

We take this more general form (4.6) for F in order to allow yet another generalization forbidden within strict Atkin-Lehner symmetry. Because $T_{[\alpha]}$ was assumed to be Hermitian, odd eigenfunctions were required to take the form (2.13); the "cycle" or "orbit" of the operator $T_{[\alpha]}$ was at most of length two. However, in the present case we can in principle have cycles of greater length, allowing cyclic permutations within (4.7) where, for example,

$$\langle f_i[\alpha^2]|g_i[\alpha^2]\rangle \propto \langle f_j[\alpha]|g_j[\alpha]\rangle \propto \langle f_k|g_k\rangle \tag{4.8}$$

(with $i\neq k$) in such a way that all f_i and g_i in (4.6) are covered. Of course, the cycle length of such non-Hermitian operators is directly related to the possible values for A in (4.7). (Again, however, such a scheme for constructing F will not prove useful for theta-function partition functions.)

Finally, for completeness we also list here our "generalization" from Section III: the Petersson inner product must be real if f_i and g_i have q expansions, and thus purely imaginary functions may be added to F at will. This property can therefore also be incorporated into the general scheme (4.7); i.e., we can also allow situations where $f_i[\alpha]=g_j$ and $g_i[\alpha]=f_j$, etc.

Let us therefore proceed by considering a general class of congruence subgroups Γ' of Γ and determining what normalizers α (not necessarily Hermitian) they may have. Note that we must restrict our attention to *normalizers* of these groups Γ' in order to guarantee that the $[\alpha]$ transform of a modular function with respect to Γ' is itself such a modular function. (Specifically, if $f[\gamma]=f$ for all $\gamma\in\Gamma'$, then requiring $f[\alpha]=f[\alpha\gamma]$ amounts to demanding that $[\alpha\gamma]=[\gamma'\alpha]$, or $\alpha\gamma\propto\gamma'\alpha$ for some $\gamma'\in\Gamma'$. The constant of proportionality must square to 1 since $\det\gamma=\det\gamma'=1$, and can be chosen to be $+1$ by technically restricting ourselves to working with the congruence subgroups $\overline{\Gamma'}\equiv\Gamma'/\{\pm1\}$.) In order to establish our general class of congruence subgroups, we can define[8] the sets of matrices Δ^n:

$$\Delta^n(N,S^\times,M) \equiv \left\{\alpha=\begin{pmatrix} a & b\\ c & d\end{pmatrix} \mid a\in N\mathbf{Z}+S^\times,\ b\in M\mathbf{Z},\ c\in N\mathbf{Z},\ d\in\mathbf{Z},\ \det\alpha=n\right\}\ , \tag{4.9}$$

where n, M, and N are positive integers, and where $S^\times$ is any chosen multiplicative subgroup of $(\mathbf{Z}/N\mathbf{Z})^*$. Note that for $n=1$, the sets Δ^n under matrix multiplication form congruence subgroups of the modular group Γ, and $d \pmod N\in S^\times$ as well. We will therefore take this set of groups $\Delta^1(N,S^\times,M)$ as our general class of congruence subgroups. Note also that the more common congruence subgroups of Γ can indeed be expressed as members of this class:

$$\Gamma_0(N) \overset{N}{=} \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \ : \ \Delta^1(N, S^\times = (\mathbf{Z}/N\mathbf{Z})^*, M = 1) \ ;$$

$$\Gamma_1(N) \overset{N}{=} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \ : \ \Delta^1(N, S^\times = \{1\}, M = 1) \ ;$$

$$\Gamma(N) \overset{N}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ : \ \Delta^1(N, S^\times = \{1\}, M = N) \ ; \tag{4.10}$$

$$\Gamma_0(N,M) \equiv \begin{pmatrix} * & b \overset{M}{=} 0 \\ c \overset{N}{=} 0 & * \end{pmatrix} \ : \ \Delta^1(N, S^\times = (\mathbf{Z}/N\mathbf{Z})^*, M) \ .$$

The above matrix shorthand indicates the groups' defining properties modulo N (or, where appropriate, M); an asterisk indicates the absence of any defining property for the specific matrix element. Still other common congruence subgroups can be expressed as *subgroups* of members of this class, e.g.,

$$\Gamma_1(N,M) \equiv \begin{pmatrix} a \overset{N}{=} 1 & b \overset{M}{=} 0 \\ c \overset{N}{=} 0 & d \overset{M}{=} 1 \end{pmatrix} \subset \Delta^1(N, S^\times = \{1\}, M) \ . \tag{4.11}$$

(The above relation is an equality only when $M|N$.)

We now proceed to determine the matrices α that normalize these groups. Let us first define the following matrices, each of which has four parameters M, N, x, and n:

$$A^\pm(M,N,x,n) \equiv \begin{pmatrix} 0 & gM \\ \pm gN & MNn \end{pmatrix} \ , \ B^\pm(M,N,x,n) \equiv \begin{pmatrix} g & 0 \\ Nn & \pm g \end{pmatrix} \ ,$$

$$C^\pm(M,N,x,n) \equiv \begin{pmatrix} g & Mn \\ 0 & \pm g \end{pmatrix} \ , \ D^\pm(M,N,x,n) \equiv \begin{pmatrix} MNn & gM \\ \pm gN & 0 \end{pmatrix} \ . \tag{4.12}$$

Here $g \equiv \gcd(M,N,x)$, the greatest common divisor of M, N, and x. Note that $C^\pm(M,N,x,n) = [B^\pm(N,M,x,n)]^t$, and that all of the above matrices are independent of x if $M=1$ or $N=1$. Note also that

$$\frac{1}{g} B^+ = -ST^{-Nn/g}S \in \Gamma \ ,$$

$$\frac{1}{g} C^+ = T^{Mn/g} \in \Gamma \ , \tag{4.13}$$

as well as

$$[A^\pm(n)]^{-1} = \frac{\pm 1}{g^2 MN} D^\pm(-n) \ . \tag{4.14}$$

If we permanently adopt the notation $\alpha_y \equiv \left(\begin{smallmatrix} 0 & -1 \\ y & 0 \end{smallmatrix}\right)$, then we may express A^- in terms of the α_y:

$$\begin{aligned} A^-(M,N,x,n) &= S\,\alpha_{gM}\,\alpha_{gN}\,T^{-Mn/g} \\ &= gS\,\alpha_M\,\alpha_N\,T^{-Mn/g} \end{aligned} \tag{4.15}$$

(where we have used the property $\alpha_{gM}\alpha_{gN} = g\alpha_M\alpha_N$, which holds for all g). It then follows from (4.14) and ${\alpha_y}^{-1} = -\alpha_y/y$ that

$$\begin{aligned} D^-(M,N,x,n) &= T^{-Mn/g}\,\alpha_{gN}\,\alpha_{gM}\,S \\ &= gT^{-Mn/g}\,\alpha_N\,\alpha_M\,S \ . \end{aligned} \tag{4.16}$$

Note that if $M=N$, then $(1/gM)A^-$ and $(1/gM)D^- \in \Gamma$.

Next, given our choice of congruence subgroup and corresponding choice for $S^\times$, let us define x to be the largest integer such that for all $s \in S^\times$, we have

$$s \equiv 1 \pmod{x} \ . \tag{4.17}$$

If $S^\times$ has only one element (i.e., if $S^\times$ consists only of the identity), we *define* x to be N, and of course we have $x=1$ if there is more than one element in $S^\times$ but there exists no greater integer x satisfying the above condition. For example, if $N=9$ and $S^\times \equiv \{1,4,7\}$, then $x=3$, whereas for $S^\times \equiv \{1,8\}$ we have $x=7$ and for $S^\times \equiv \{1,2,4,5,7\}$ we have $x=1$. (Note that if N is even, then the smallest such x for any choice of $S^\times$ is $x=2$.) It is then straightforward to show that the group $\Delta^1(N,S^\times,M)$ is normalized by matrices proportional to α, where

$$\alpha \in \left\{ A^\pm(M,N,x,n);\ B^\pm(M,N,x,n);\ C^\pm(M,N,x,n);\ D^\pm(M,N,x,n)\ ,\ \forall n \in \mathbf{Z} \right\} . \tag{4.18}$$

Note that this is not the complete set of normalizers of Δ^1; in particular, the product of two normalizers is also a normalizer, so matrices containing the above matrices as factors will also normalize the chosen group. (However, the above matrices *are* the complete set of normalizers with one vanishing matrix element. We will see shortly that these are sufficient for our purposes.)

We can immediately rule out some of the above normalizers. Recalling the restrictions on (4.2), namely, that

$\alpha \in \mathrm{GL}^+(2,\mathbf{Q})$, we see that we must restrict our attention to those normalizers with positive determinant:

$$\alpha \in \{A^-;\ B^+;\ C^+;\ D^-\}\ . \tag{4.19}$$

[We observe that if τ is in the upper half-plane, then the $\vartheta_i(\alpha\tau)$ will converge only if $\det\alpha > 0$. Thus, if our partition functions are to be built from Jacobi theta functions, this restriction also ensures convergence for the ϑ's.] Similarly, if we had instead simply chosen to have $[\alpha]$ be Hermitian with respect to the Petersson inner product, then for normalizers *not* proportional to **1** we would have required trace $\alpha = 0$. This would have limited us instead to consideration of the normalizers:

$$\alpha \in \left\{ A^\pm(M,N,x,0);\ B^-(M,N,x,n);\right.$$
$$\left. C^-(M,N,x,n);\ D^\pm(M,N,x,0) \right\}. \tag{4.20}$$

Therefore, as an example, if we wish to reproduce strict Atkin-Lehner symmetry directly, we must demand both of these conditions (4.19) and (4.20); hence we can only consider the normalizers $A^-(M,N,x,0)$ and $D^-(M,N,x,0)$. Restricting ourselves to the nonprincipal congruence subgroups $\Gamma_0(N) \equiv \Delta^1(N,S^\times,1)$ for any unspecified $S^\times \in (\mathbf{Z}/N\mathbf{Z})^*$ then yields the normalizers $A^-(1,N,x,0) = D^-(1,N,x,0) = \alpha_N$ (note that $g=1$ regardless of x or $S^\times$). Thus, for the groups $\Gamma_0(N)$, we see that α_N is indeed the only such Hermitian operator that can be considered.

In the present case, we allow non-Hermitian operators, and hence we can draw normalizers from the full set (4.19). We recall, however, that if $\alpha \in \Gamma$ we obtain supersymmetric total partition functions which yield vanishing cosmological constants by default [see the comments following (2.4)]; we therefore avoid such normalizers (4.13), avoid the case $M = N$, and consider A^- and D^-. Note that this therefore prohibits consideration of the *principal* congruence subgroups $\Gamma(N)$; any suitable Hermitian *or* non-Hermitian normalizer for these groups leads to supersymmetric total partition functions.

V. GENERALIZING ATKIN-LEHNER SYMMETRY: VARIOUS CASES

In order to develop some intuition for the action of these normalizers, let us first consider the cases where $M = 1$ and $N = 2$. This leaves us with the normalizers $-A^- = \alpha_2 T^{-n}$ and $-D^- = T^{-n}\alpha_2$, both of which normalize $\Gamma_0(2)$ and $\Gamma_1(2)$ [as well as a host of other groups within (4.9)]. Let us first focus on $\alpha \equiv -A^- = \alpha_2 T^{-n}$. Choosing to work with $D = 4$ heterotic string partition functions (2.7) and attempting to mimic an explicit construction such as that in (2.10) through (2.13), we observe (see the Appendix)

$$\Delta^{1/2}[\alpha_2 T^{-n}] = \Delta^{1/2}[\alpha_2]\ ,$$
$$\mathcal{M}[\alpha_2 T^{-n}] = (-1)^n\ \mathcal{M}[\alpha_2]\ , \tag{5.1}$$

whereupon we have

$$\left(\frac{\mathcal{M}}{\Delta\overline{\Delta}^{1/2}}\right)[\alpha_2 T^{-n}] = (-1)^n \left(\frac{\mathcal{M}}{\Delta\overline{\Delta}^{1/2}}\right)[\alpha_2] = (-1)^n\,\frac{\mathcal{M}}{\Delta\overline{\Delta}^{1/2}}\ . \tag{5.2}$$

We must therefore proceed to find a suitable F_1 which is to be built from Jacobi theta functions,

$$F_1 = \sum_s \left\{ a_s \prod_{i=2}^{4} \vartheta_i^{m_i^{(s)}}\overline{\vartheta_i}^{\overline{m_i}^{(s)}} \right\}, \tag{5.3}$$

and which should satisfy $F_1[\alpha_2 T^{-n}] = \lambda F_1$ for some λ. Recall that it is not immediately clear what this λ should be; we need not necessarily require net *antisymmetry* under $[\alpha_2 T^{-n}]$ [i.e., we need not produce $A = -1$ in the notation of (4.7)] because this operator is not Hermitian. It is first necessary to determine the order, or "cycle length" of $[\alpha_2 T^{-n}]$. However, when operating on any given term s in (5.3), we have (see the Appendix)

$$[(\alpha_2 T^{-n})^2] = R_{42} \text{ if } n \text{ is even,}$$
$$[(\alpha_2 T^{-n})^4] = R_{32}R_{42} \text{ if } n \text{ is odd,} \tag{5.4}$$

where

$$R_{ij} \equiv \exp\left(-\frac{\pi i}{4}\left[2n(m_i - \overline{m_i}) + n(m_j - \overline{m_j})\right]\right). \tag{5.5}$$

Thus, for even n the cycle length of $[\alpha_2 T^{-n}]$ is 2 (with $A = \lambda = \pm\sqrt{R_{42}}$), while for odd n the cycle length is 4. (If F_1 has $m_3 = m_4$ and $\overline{m_3} = \overline{m_4}$ for all s, then the cycle length of $[\alpha_2 T^{-n}]$ with odd n is also 2, but in this degenerate case the extra two terms involved yield only a net factor of 2 which can be reabsorbed into the coefficients a_s of (5.3).) For odd n, then, we can expect our F_1 to take the general form

$$F_1 = \sum_{i=0}^{3} r^i\ F_2[(\alpha_2 T^{-n})^i] \tag{5.6}$$

where $r \in \mathbf{C}$ is any of the four fourth roots of $(R_{32}R_{42})^{-1}$; this yields four possible eigenvalues $F_1[\alpha_2 T^{-n}] = \lambda F_1$ with $\lambda = r^{-1}$. We then have, for odd n, $A = -\lambda = -r^{-1} = -\sqrt[4]{R_{32}R_{42}}$ for any of these roots. [The relative sign between A and λ is that in (5.2).]

We must also examine the constraints imposed by the assumed invariance of F_1 over the chosen congruence subgroup normalized by $\alpha_2 T^{-n}$. For any value of M, any of these possible subgroups $\Delta^1(N,S^\times,M)$ has T^M as one of its generators. For $M = 1$, therefore, we must demand $F[T] = F$, where we recall the notation of (2.10) with $F_0 \equiv \mathcal{M}\,F_1$. Since $\mathcal{M}/(\Delta\overline{\Delta}^{1/2})$ is odd under $[T]$, we require

$$F_1[T] = -F_1\ . \tag{5.7}$$

Thus, if $F_1[\alpha_2 T^{-n}] = \lambda F_1$, then

$$F_1[\alpha_2 T^{-n}] \ = \ \lambda\ F_1 \ = \ (-1)^n \lambda\ F_1[T^{-n}] \ , \qquad (5.8)$$

from which we conclude (upon operating on both sides with $[T^n]$) that

$$F_1[\alpha_2] \ = \ (-1)^n \lambda\ F_1 \ . \qquad (5.9)$$

However, we know that the Atkin-Lehner operator α_2 has cycle length 2, and can have eigenvalues $\lambda_{\rm AL} = \pm 1$ only. We thus must demand

$$A = (-1)^n \lambda = \lambda_{\rm AL} = \pm 1 \ . \qquad (5.10)$$

From our above results, this means that for even n, R_{42} in (5.5) must equal 1; similarly, for odd n, we must require that $R_{32}R_{42}$ equal 1 as well. This then translates into the following constraints:

$$\text{for even } n: \quad m_2 - \overline{m_2} \ \in \ 2\mathbf{Z} \ ,$$
$$\text{for odd } n: \quad \sum m_i - \sum \overline{m_i} \ \in \ 2\mathbf{Z} \ , \qquad (5.11)$$

which must hold for each term s in (5.3).

It is now clear from (5.10) that if n is even, $A = \lambda_{\rm AL}$. We thus see that in this case not only must we choose $A = -1$, but in so doing we must also have $\lambda_{\rm AL} = -1$. Thus, for even n, our generalized Atkin-Lehner symmetry reduces to strict Atkin-Lehner symmetry (which is, of course, a special case with $n = 0$); any F_1 of the form (5.3) which is odd under $[\alpha_2 T^{-n}]$ and is odd under $[T]$ is necessarily odd under $[\alpha_2]$ as well.

The same situation exists for odd n as well. It might seem from (5.2) that the extra minus sign there would force us to consider, effectively, an F_1 which is *even* under $[\alpha_2 T^{-n}]$ and perhaps under $[\alpha_2]$ as well; this would entail no contradiction in and of itself, since the conclusion to be drawn from *symmetric* strict Atkin-Lehner symmetry is the null result $\Lambda = \Lambda$. However, as (5.10) indicates, any F_1 which is even under $[\alpha_2 T^{-n}]$ is simultaneously odd under $[\alpha_2]$. We thus have again rederived the strict Atkin-Lehner symmetry. Note that in both cases the constraints obtained in (5.11) are therefore the constraints that any strict Atkin-Lehner eigenstate must satisfy; the constraints for *both* even and odd n must be satisfied for $n = 0$ because, as we have shown, both cases reduce to strict Atkin-Lehner symmetry.

An almost identical situation exists for the other of our $M = 1$, $N = 2$ normalizers: $-D^- = T^{-n}\alpha_2$. In particular, we again have (5.2) through (5.4) for the new operator $T^{-n}\alpha_2$, and the argument leading to (5.10) is modified only trivially (but still has the same result). We therefore obtain again the constraints (5.11), and rederive strict Atkin-Lehner symmetry.

Let us therefore consider the next simplest case: $M = 2$, $N = 1$. This has greater potential for yielding something new because the corresponding class of congruence subgroups $\Delta^1(1, S^\times, 2)$ does *not* include $\Gamma_0(2)$ or $\Gamma_1(2)$. [For $S^\times = (\mathbf{Z}/N\mathbf{Z})^*$, this group Δ^1 is usually called $\Gamma^0(2)$.] In fact, the symmetry we will derive here is *not* Atkin-Lehner symmetry, although it is closely related to it. Our normalizers are $A^-(2,1,x,n) = S\alpha_2 S T^{-2n}$ and $D^-(2,1,x,n) = T^{-2n} S\alpha_2 S$; we work here with $\alpha \equiv A^-$. It is then straightforward to show that

$$\left(\frac{\tilde{\mathcal{M}}}{\Delta\overline{\Delta}^{1/2}}\right)[\alpha] \ = \ (-1)^n \ \frac{\tilde{\mathcal{M}}}{\Delta\overline{\Delta}^{1/2}} \ , \qquad (5.12)$$

where

$$\tilde{\mathcal{M}} \ \equiv \ \vartheta_2{}^4 \vartheta_3{}^4 \overline{\vartheta_2}^2 \overline{\vartheta_3}^2 \ = \ -\mathcal{M}[S] \ . \qquad (5.13)$$

Thus, if we assume our corresponding $\tilde{F}_1 \equiv \tilde{F}_0/\tilde{\mathcal{M}}$ to have the form (5.3), then we can similarly show

$$[\alpha^2] \ = \ R_{24} \ \text{ if } n \text{ is even},$$
$$[\alpha^4] \ = \ R_{24}R_{34} \ \text{ if } n \text{ is odd}, \qquad (5.14)$$

where R_{ij} is defined in (5.5). We then must have $\tilde{F}_1[\alpha] = \lambda \tilde{F}_1$ with $\lambda = \pm\sqrt{R_{24}} = A$ for even n, and $\lambda = \sqrt[4]{R_{24}R_{34}} = -A$ for odd n. (Any of these fourth roots are possibilities.)

Let us now proceed, as before, to constrain these values of λ still further. From $\tilde{F}_1[\alpha] = \lambda \tilde{F}_1$ we obtain (upon operating with $[T^{2n}S]$)

$$\tilde{F}_1[S\alpha_2] \ = \ \lambda\ \tilde{F}_1[T^{2n}S] \ . \qquad (5.15)$$

Furthermore, since $M = 2$, we must demand $[T^2]$ invariance of $\tilde{F} \equiv \tilde{\mathcal{M}}\tilde{F}_1/(\Delta\overline{\Delta}^{1/2})$. Therefore, since $\tilde{\mathcal{M}}[T^2] = -\tilde{\mathcal{M}}$ and $\Delta\overline{\Delta}^{1/2}[T^2] = \Delta\overline{\Delta}^{1/2}$, we require

$$\tilde{F}_1[T^2] \ = \ -\tilde{F}_1 \ . \qquad (5.16)$$

From (5.15) and (5.16) we can conclude

$$\tilde{F}_1[S\alpha_2] \ = \ (-1)^n \lambda\ \tilde{F}_1[S] \ . \qquad (5.17)$$

Let us now consider $\tilde{F}$. From the definition of $\tilde{\mathcal{M}}$ in (5.13) we see $\tilde{\mathcal{M}}[S] = -\mathcal{M}$; this follows because $S^2 = -\mathbf{1}$ and the weight of $\tilde{\mathcal{M}}$ is 6. Consequently, we have

$$\tilde{F}[S] = \frac{\tilde{\mathcal{M}}[S]}{(\Delta\overline{\Delta}^{1/2})[S]}\tilde{F}_1[S] = \frac{\mathcal{M}}{\Delta\overline{\Delta}^{1/2}}\ \tilde{F}_1[S] \ , \qquad (5.18)$$

where we have observed that $\Delta^{1/2}$ is odd under $[S]$. Let us now apply $[\alpha_2]$ to both sides. Since $\mathcal{M}/(\Delta\overline{\Delta}^{1/2})$ is invariant under $[\alpha_2]$ (note that this would *not* have been true with $\tilde{\mathcal{M}}$), we therefore have

$$\tilde{F}[S\alpha_2] \ = \ \frac{\mathcal{M}}{\Delta\overline{\Delta}^{1/2}}\ \tilde{F}_1[S\alpha_2] \ = \ (-1)^n \lambda\ \frac{\mathcal{M}}{\Delta\overline{\Delta}^{1/2}}\ \tilde{F}_1[S] \qquad (5.19)$$

where we have used (5.17).

The implications of this result are significant. It is clear that if we define $F' \equiv \tilde{F}[S]$, then generalized Atkin-Lehner symmetry for $\tilde{F}$ implies some form of strict Atkin-

Lehner symmetry for F'. (We say "some form" to indicate that we have not yet placed further restrictions on λ.) It is therefore essential to understand F'. It is easily demonstrable that for $M \neq 1$, F' is not invariant under transformations in the group normalized by α; hence $F' \neq \tilde{F}$. Thus, our new generalized symmetry, which acts on the invariant $\tilde{F}$, is distinct from the strict Atkin-Lehner symmetry, which acts on F'.

However, although S is not one of the *generators* of our $M \geq 2$ congruence subgroups, it can always be chosen to be one of the effective right coset representatives γ_i of these groups in Γ. (By "effective representatives" we mean matrices $\gamma_i{}'$ satisfying $[\gamma_i{}'] = [\gamma_i]$ when operating on all relevant terms, where γ_i is the corresponding true representative. For example, S^2 is always an effective trivial representative when operating on modular functions of even weight, since $[S^2] = [\mathbf{1}]$ under these conditions. A more substantial example would be T^2, which is also an effective trivial representative when operating on even-weight terms satisfying $m_2 - \overline{m_2} \in 2\mathbf{Z}$.) Thus, any invariant $\tilde{F}$ which has this generalized Atkin-Lehner symmetry will have a coset transform $\tilde{F}[S]$ with strict Atkin-Lehner symmetry. Equivalently (as one can directly show), the strict Atkin-Lehner construction with Hermitian operators α_N and congruence subgroups $\Gamma_0(N)$ will yield a $\Gamma_0(N)$-invariant F' for which one of the coset transforms, $F'[S]$, will have this generalized Atkin-Lehner symmetry.

It is then straightforward to see that our values of λ are likewise constrained. In particular, demanding $F'[\alpha_2] = \lambda_{\rm AL} F'$ with the only values $\lambda_{\rm AL} = \pm 1$ amounts to correspondingly requiring $\lambda = \pm 1$ for even n, and $\lambda = \mp 1$ for odd n. In both cases we therefore have $A = \pm 1$ in the notation of (4.7), and the restriction to generalized Atkin-Lehner *antisymmetry* yields $A = -1$. Constraints analogous to (5.11) are similarly obtained. We note that the other $M = 2$, $N = 1$ operator D^- yields similar results.

Therefore, although we now see that the $M = 2$, $N = 1$ case has not yielded a fundamentally new mechanism for obtaining a vanishing cosmological constant, we have achieved an understanding of the implications of strict Atkin-Lehner symmetry as far as the coset transforms are concerned. Furthermore, by having put this new resultant symmetry on a firm independent footing as arising from *non*-Hermitian operators normalizing more sophisticated congruence subgroups, we have restored a certain parity between the different coset transforms. Atkin-Lehner symmetry, or generalized Atkin-Lehner symmetry, simultaneously appears or fails to appear in all of our coset transforms, and our total partition function (2.4) thus reflects the simultaneous presence or absence of all of these symmetries. We note, as an example, that the expression $F_{\rm AL}{}'$ in (3.22) from Section III does *not* have strict Atkin-Lehner symmetry; rather, $F_{\rm AL}{}'$ is a combination of terms with precisely this form of our *generalized* symmetry. Similarly, our $F^{(k)\prime}$ expressions in (3.22) could have been directly obtained by starting with such *generalized* Atkin-Lehner-symmetric expressions, and subsequently adding purely imaginary functions. In this instance, then, perhaps this form of our generalized Atkin-Lehner symmetry yields a more natural way of obtaining those expressions.

It is now clear that factors of S and T in our normalizers amount to mere coset transformations; together with factors of B^+ and C^+ from (4.12) and (4.13), these normalizers can describe all of the symmetries acting in all of the coset transforms. (Viewed this way, the first case we examined, that with $N = 2$ and $M = 1$, reduced *directly* to strict Atkin-Lehner symmetry for all n because our generalized symmetry acted on the $[T]$-transform of the $[\mathbf{1}]$ coset, which was, in that case, the $[\mathbf{1}]$ coset itself.) The substance of our mechanism therefore lies wholly in the α_y. Strict Atkin-Lehner symmetry is derived from the basic action of one α_y, but from (4.15) and (4.16), we see that A^- and D^- typically have *two* factors of α_y. We therefore must ascertain the conditions under which this might lead to a fundamentally different symmetry. In this regard, we observe that

$$\alpha_M\,\alpha_N \;=\; g'\,\alpha_m\,\alpha_n \;=\; M\,S\,\alpha_{n/m} \;=\; N\,\alpha_{m/n}\,S \tag{5.20}$$

where $g' \equiv \gcd(M,N)$, $m \equiv M/g'$, and $n \equiv N/g'$. We therefore see that it is only when M and N are a common multiple of numbers which are relatively prime (or simply when neither M nor N divides the other) that the action of the normalizers A^- and D^- will indeed be fundamentally different. (Thus our previous cases with either M or N equal to 1 did not exhibit this new behavior; in fact, $\alpha_1 = S$.) Furthermore, we observe that

$$\alpha_M\;S\;\alpha_N \;=\; \alpha_N\;S\;\alpha_M \;=\; -\alpha_{MN}\;, \tag{5.21}$$

so it is only when a factor of S intervenes between two factors of α_y that the resulting symmetry will be a strict Atkin-Lehner symmetry (with a congruence subgroup of higher level MN). Fortunately, we do *not* have such factors of S in our normalizers A^- or D^-, so strict Atkin-Lehner symmetry will not arise in this way. (Of course, there are cases in which this can occur through products with B^+ and C^+.) Thus, we conclude that we have a fundamentally new symmetry in the cases when M and N are chosen as any common multiple of relatively prime numbers.

We have attempted without success to construct expressions which resemble partition functions that might arise in various string theories and which have simple transformation properties under A^- and D^- in the cases where M and N are common multiples of relatively prime numbers. If we generally restrict ourselves to building partition functions composed of theta functions with arbitrary characteristics,[16,17] as string theory would require, then we have two possibilities: we can attempt to find simultaneous eigenfunctions of α_M and α_N, or eigenfunctions simply of $\alpha_M\alpha_N$. Eigenfunctions of α_2 or α_3 separately have been constructed elsewhere,[7,12] yet none of these eigenfunctions is simultaneously an eigenfunction of both α_2 and α_3. In fact, since the α_2 eigenfunctions are constructed with theta functions of half-integer characteristics and the α_3 eigenfunctions are constructed with theta functions of third-integer characteristics, one might get the (mistaken) impression that no identities exist relating $\vartheta(\tau)$ and $\vartheta(h\tau)$ unless ϑ is a theta function with hth-integer characteristics. (See Ref. 16 for general definitions.) This impression is best shown to be false by quoting one such counterexample:[9]

$$\sum_{i=2}^{4} (-1)^i \, \vartheta_i(\tau)\,\vartheta_i(3\tau) = 0 ; \tag{5.22}$$

here the ϑ_i are the usual (*half*-integer characteristic) Jacobi theta functions. Thus, it may well be possible to construct an eigenfunction of $\alpha_2\alpha_3$. Furthermore, as (5.20) indicates, the values of h that we might instead work with in this case are the fractional values $h = 3/2$ or $h = 2/3$. These and other possibilities (corresponding to other suitable values of M and N) are being pursued.

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APPENDIX A: FORMULAS AND DERIVATIONS

In this Appendix we list the important formulae and derivations that are used in the text. We begin with the definitions of the Jacobi theta functions ϑ_i and Dedekind eta function η ($q \equiv e^{2\pi i\tau}$ throughout):

$$\begin{aligned}
\vartheta_2(\tau) &\equiv 2q^{1/8} \prod_{n=1}^{\infty} (1+q^n)^2\,(1-q^n) \\
&= 2\sum_{n=0}^{\infty} q^{(n+1/2)^2/2} , \\
\vartheta_3(\tau) &\equiv \prod_{n=1}^{\infty} (1+q^{n-1/2})^2\,(1-q^n) \\
&= 1 + 2\sum_{n=1}^{\infty} q^{n^2/2} , \\
\vartheta_4(\tau) &\equiv \prod_{n=1}^{\infty} (1-q^{n-1/2})^2\,(1-q^n) \\
&= 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2/2} , \\
\eta(\tau) &\equiv q^{1/24} \prod_{n=1}^{\infty} (1-q^n) \\
&= \sum_{n=-\infty}^{\infty} (-1)^n q^{3(n-1/6)^2/2} .
\end{aligned} \tag{A1}$$

These satisfy the identities

$$\begin{aligned}
\vartheta_3{}^4 &= \vartheta_2{}^4 + \vartheta_4{}^4 , \\
2\,\eta^3 &= \vartheta_2\,\vartheta_3\,\vartheta_4 .
\end{aligned} \tag{A2}$$

Under the transformations $S : \tau \to -1/\tau$ and $T : \tau \to \tau + 1$ generating the modular group, we have

$$\begin{aligned}
&\vartheta_2(S\tau) = \sqrt{-i\tau}\;\vartheta_4(\tau), \quad \vartheta_2(T\tau) = \exp\left(\frac{i\pi}{4}\right)\vartheta_2(\tau) , \\
&\vartheta_3(S\tau) = \sqrt{-i\tau}\;\vartheta_3(\tau), \quad \vartheta_3(T\tau) = \vartheta_4(\tau) , \\
&\vartheta_4(S\tau) = \sqrt{-i\tau}\;\vartheta_2(\tau), \quad \vartheta_4(T\tau) = \vartheta_3(\tau) , \\
&\eta(S\tau) = \sqrt{-i\tau}\;\eta(\tau), \quad \eta(T\tau) = \exp\left(\frac{i\pi}{12}\right)\eta(\tau) ,
\end{aligned} \tag{A3}$$

where we always choose the branch of the square root with non-negative real part.

A modular function f of weight k must by definition satisfy $f(A\tau) = (c\tau+d)^k f(\tau)$ for all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ [where $A\tau \equiv (a\tau+b)/(c\tau+d)$]; hence $\Delta \equiv \eta^{24}$ is a true modular function of weight $k = 12$. (If additionally the modular function f remains finite as $\tau \to i\infty$, then f is a modular *form* with respect to Γ. Δ is such a modular form.) We define the operator $[\alpha] : f \to f[\alpha]$, where

$$(f[\alpha])(\tau) \equiv (\det\alpha)^{k/2}\,(c\tau+d)^{-k}\,f(\alpha\tau) . \tag{A4}$$

Here k is the weight of f, and $\alpha \equiv \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Thus, modular functions and forms must satisfy $f[\gamma] = f$ for all $\gamma \in \Gamma$. (Note that in general $(f[\alpha])[\beta] = f[\alpha\beta]$; in such a product $[\alpha]$ acts *first*.) Defining the notation $(n_2, n_3, n_4) \equiv \vartheta_2{}^{n_2}\vartheta_3{}^{n_3}\vartheta_4{}^{n_4}$, or

$$(n_2,n_3,n_4)\overline{(\overline{n_2},\overline{n_3},\overline{n_4})} \equiv \vartheta_2{}^{n_2}\vartheta_3{}^{n_3}\vartheta_4{}^{n_4}\,\overline{\vartheta_2}^{\overline{n_2}}\,\overline{\vartheta_3}^{\overline{n_3}}\,\overline{\vartheta_4}^{\overline{n_4}} , \tag{A5}$$

we therefore have the general rules

$$\begin{aligned}
&(n_2,n_3,n_4)\overline{(\overline{n_2},\overline{n_3},\overline{n_4})}\;[S] \\
&\quad = \exp\left[\frac{7\pi i}{4}\left(\sum n_i - \sum \overline{n_i}\right)\right](n_4,n_3,n_2)\overline{(\overline{n_4},\overline{n_3},\overline{n_2})} , \\
&(n_2,n_3,n_4)\overline{(\overline{n_2},\overline{n_3},\overline{n_4})}\;[T] \\
&\quad = \exp\left(\frac{\pi i}{4}\,(n_2 - \overline{n_2})\right)(n_2,n_4,n_3)\overline{(\overline{n_2},\overline{n_4},\overline{n_3})} , \\
&\Delta^{1/2}\,[S] = -\Delta^{1/2} , \\
&\Delta^{1/2}\,[T] = -\Delta^{1/2} .
\end{aligned} \tag{A6}$$

Similarly, under $\alpha_2 \equiv \binom{0\;-1}{2\;\;\;0}$, we have

$$\begin{aligned}
\vartheta_2{}^2\,[\alpha_2] &= \frac{-i}{\sqrt{2}}\,(\,2\vartheta_3\,\vartheta_4\,)\,,\\
\vartheta_3{}^2\,[\alpha_2] &= \frac{-i}{\sqrt{2}}\,[\,\vartheta_3{}^2+\vartheta_4{}^2\,]\,,\\
\vartheta_4{}^2\,[\alpha_2] &= \frac{-i}{\sqrt{2}}\,[\,\vartheta_3{}^2-\vartheta_4{}^2\,]\,,\\
(\vartheta_3\vartheta_4)\,[\alpha_2] &= \frac{-i}{\sqrt{2}}\,\vartheta_2{}^2\,,\\
\Delta^{1/2}\,[\alpha_2] &= -\frac{\vartheta_2{}^4}{2\vartheta_3{}^2\vartheta_4{}^2}\,\Delta^{1/2}\,.
\end{aligned}\tag{A7}$$

In order to obtain these results, it is necessary to use the doubling formulas[8,9] for $\theta_i(2\tau)$ in terms of $\theta_i(\tau)$, as well as the results for $[S]$ given above. Note that in this last equation for $\Delta^{1/2}$, an extra factor is obtained under $[\alpha_2]$ in which the numerator $-\frac{1}{2}\vartheta_2{}^4$ is the $[\alpha_2]$ transform of the denominator $(\vartheta_3\vartheta_4)^2$. This is the origin of the reason for the $\mathcal{M}$-decomposition discussed in the text. Since this last equation is derived from the relation

$$2^8\,\Delta^{1/2}(2\tau) \;=\; \vartheta_2{}^8\,\vartheta_3{}^2\,\vartheta_4{}^2 \tag{A8}$$

(where the argument is τ if no argument is listed), we easily obtain

$$\frac{\Delta^{1/2}}{(\vartheta_3\vartheta_4)^2} \;=\; [\eta\,\eta(2\tau)]^4 \;=\; \tfrac{1}{2}(\eta\,\eta[\alpha_2])^4\,, \tag{A9}$$

thus leading to (2.12) in the text.

We can proceed to use these formulas to derive the results quoted in the text; as an illustrative example, we derive (5.4). We seek the behavior of the terms in (5.3) under $[\alpha_2T^{-n}\alpha_2T^{-n}]$. Rather than use the unwieldy formulas in (A7) above, we observe that $\alpha_2T^{-n} = -ST^{-2n}S\alpha_2$, whereupon it follows from $\alpha_2{}^2 = -2$ that $(\alpha_2T^{-n})^2 = 2ST^{-2n}ST^{-n}$ and

$$[(\alpha_2T^{-n})^2] \;=\; [ST^{-2n}ST^{-n}]\,. \tag{A10}$$

Thus, operating on some term $(m_2,m_3,m_4)\overline{(m_2,m_3,m_4)}$, the operator $[S]$ first yields the result quoted above in (A6). Operating subsequently on this with $[T^{-2n}]$ yields

$$\exp\left[\frac{7\pi i}{4}\,\left(\sum m_i-\sum\overline{m_i}\right)\right]\,\exp\left(-\frac{\pi i}{2}\,n(m_4-\overline{m_4})\right) \times(m_4,m_3,m_2)\overline{(m_4,m_3,m_2)}\,, \tag{A11}$$

and then operating with $[S]$ yields

$$\exp\left[\frac{7\pi i}{2}\,\left(\sum m_i-\sum\overline{m_i}\right)\right]\,\exp\left(-\frac{\pi i}{2}\,n(m_4-\overline{m_4})\right) \times(m_2,m_3,m_4)\overline{(m_2,m_3,m_4)}\,. \tag{A12}$$

Finally operating with $[T^{-n}]$, we find that our theta-function powers become

$$\begin{aligned}
&\text{for even } n:\quad (m_2,m_3,m_4)\overline{(m_2,m_3,m_4)}\,,\\
&\text{for odd } n:\quad (m_2,m_4,m_3)\overline{(m_2,m_4,m_3)}\,,
\end{aligned}\tag{A13}$$

and our overall phase picks up another factor

$$\exp\left(-\frac{\pi i}{4}\,n(m_2-\overline{m_2})\right)\,. \tag{A14}$$

For such terms in (5.3), we have

$$\sum m_i-\sum\overline{m_i} \;=\; (18-D)\;-\;(10-D)\;=\;8 \tag{A15}$$

in all spacetime dimensions D, and hence the phase $\exp[7\pi i(\sum m_i-\sum\overline{m_i})/2]$ from the $[S]$ transformations is always 1. For even n, therefore, we obtain the first equation in (5.4), where R_{42} is the remaining phase. For odd n, on the other hand, our powers in (A13) are not as they started; hence [except for certain circumstances listed after (5.5)] we see that $[(\alpha_2T^{-n})^2]$ is not proportional to the identity. Applying $[(\alpha_2T^{-n})^2]$ a second time then yields the result quoted in (5.4) for odd n.

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