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Quantum field in η-ξ spacetime

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A new spacetime, η-ξ spacetime, is constructed. The quantum field in η-ξ spacetime is discussed. It is shown that the vacuum state of quantum field in η-ξ spacetime is a thermal state for an inertial observer in Minkowski spacetime, and the vacuum Green's functions in η-ξ spacetime are just the thermal Green's functions in usual statistical mechanics.

I. INTRODUCTION

In the middle cf the 1970s two important new theories were put forth. One of them is the Hawking radiation theory of a black hole,[1-3] which told us that when quantum effects are considered, pair creation in the gravitational field of a black hole formed by a gravitational collapse leads to a steady emission of particles to infinity; the emitted particles have a thermal spectrum corresponding to a temperature $T_{\mathrm{BH}}=\kappa/2\pi K$, where κ is the surface gravity on the horizon of a black hole. In particular, choosing the suitable boundary conditions, we can find the vacuum in a black-hole spacetime (Hartle-Hawking vacuum) is a thermal state for a static observer in Schwarzschild spacetime. The importance of this Hawking-Unruh effect is that it provides a connection between gravity, quantum mechanics, and thermodynamics.

The other is thermo-field dynamics[4,5] (TFD) proposed by Takahashi and Umezawa, which is based on the idea of augmenting the physical Fock space $\mathcal{F}$ by a fictitious dual space $\tilde{\mathcal{F}}$. We can define a thermal vacuum state $|0(\beta)\rangle$ on the double Fock space $\mathcal{F}\otimes\tilde{\mathcal{F}}$ with the property that the vacuum expectation value of any physical operator agrees with its statistical average for an ensemble in thermal equilibrium. As a dynamical theory, TFD starts from the Hamiltonian $\hat{H}=H-\tilde{H}$, where H is the Hamiltonian in the usual quantum field in Minkowski spacetime, and $\tilde{H}$ is the tilde conjugate of H. The importance of TFD is that it not only provides a field-theory method for calculations of statistical system, but also points out that the unification of all the fundamental interactions on the basis of the concept of a "field" should include thermal physics on the basis of the concept of "quantum fields."

The TFD formalism has been used by Israel[6] in his derivation of Hawking's result. It is interesting that, formally, the Hartle-Hawking vacuum in black-hole theory and the thermal vacuum state $|0(\beta)\rangle$ in TFD have exactly the same expressions, although they have different physical meanings. The Hartle-Hawking vacuum is a thermal state for a static observer outside the black hole who is not an inertial observer. $|0(\beta)\rangle$ in TFD is a thermal state for an inertial observer in Minkowski spacetime. We need to construct a new spacetime to be regarded as a geometrical background directly for TFD.

η-ξ spacetime[7,8] provides a geometrical background for TFD. The vacuum of quantum fields in η-ξ spacetime is just the thermal state for an inertial observer in Minkowski spacetime. The tilde conjugate field in TFD can be regarded as the field distributed in the mirror universe in η-ξ spacetime, which is in principle unmeasurable by an observer in Minkowski spacetime. When any observable is measured, the information about the tilde field will be lost. The loss of the information corresponds to generating entropy, and leads to the change from a pure state in the double Fock space $\mathcal{F}\otimes\tilde{\mathcal{F}}$ to the mixed states. In Ref. 7, only the massless scalar field in two-dimensional η-ξ spacetime was discussed, which may be instructive, but is too special because of its conformal invariant. In this paper, we generalize the discussion to an interacting scalar field in four-dimensional η-ξ spacetime and show the relation between quantum field theory in η-ξ spacetime and equilibrium statistical mechanics.

This paper is organized as follows. η-ξ spacetime and its properties are discussed in Sec. II. In Sec. III, a canonical form of quantum field in η-ξ spacetime is presented. Here the key point is that when choosing a suitable time coordinate, we can get a time-independent Hamiltonian, which directly relates the quantum field in η-ξ spacetime with TFD. Section IV shows that the vacuum Green's function in η-ξ spacetime is equal to a thermal Green's function in usual statistical mechanics by path-integral method. The Euclidean path-integral method used in Sec. IV exposes the character of η-ξ spacetime quite well, but this proof may be somewhat formal. To make the meaning of quantum field in η-ξ spacetime clearer, in Sec. V, we directly discuss the quantum field in Lorentzian section of η-ξ spacetime, and conclude that the vacuum in η-ξ spacetime is just the thermal vacuum state in TFD, i.e., a thermal state for an inertial observer. In Table I, the distinctions between the Rindler space and η-ξ space are listed.

II. η-ξ SPACETIME

η-ξ spacetime is a four-complex-dimensional manifold $\mathbb{C}^4$ (which also may be viewed as an eight-real-dimensional manifold) with complex metric g_{ab} defined by

$$ds^2=\frac{1}{\alpha^2(\xi^2-\eta^2)}(-d\eta^2+d\xi^2)+dy^2+dz^2\ , \qquad (2.1)$$

where α is a constant. It is interesting for us to discuss its Euclidean section and Lorentzian section. By restricting ξ,y,z to be real but η to be pure imaginary, we obtain a four-real-dimensional submanifold with a real Euclidean metric:

$$ds^2=\frac{1}{\alpha^2(\xi^2+\sigma^2)}(d\sigma^2+d\xi^2)+dy^2+dz^2\ , \qquad (2.2)$$

where $\eta=i\sigma$. This submanifold is referred to as a Euclidean section of $(\mathbb{C}^4,g_{ab})$ and is called the Euclidean η-ξ spacetime.

Under coordinate transformation

$$\sigma=\frac{1}{\alpha}e^{\alpha x}\sin\alpha\tau\ ,$$
$$\xi=\frac{1}{\alpha}e^{\alpha x}\cos\alpha\tau\ , \qquad (2.3)$$

the metric (2.2) becomes

$$ds^2=d\tau^2+dx^2+dy^2+dz^2\ ; \qquad (2.4)$$

this is Euclidean flat metric. The coordinate transformation (2.3) means that the τ coordinate is periodically identified such that $\alpha\tau\sim\alpha\tau+2\pi$. In fact, a strip with width of $2\pi/\alpha$ in the direction τ in the Euclidean flat space (2.4) is mapped into a real Euclidean plane $\sigma^2+\xi^2=r^2$; here, $r=(1/\alpha)e^{\alpha x}$. In the σ-ξ plane, $\theta=\alpha\tau$ plays the role of a polar angle with period 2π (Fig. 1).

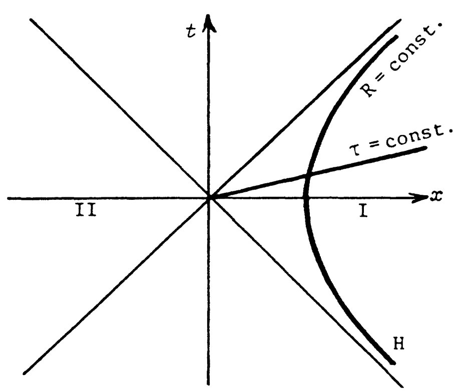
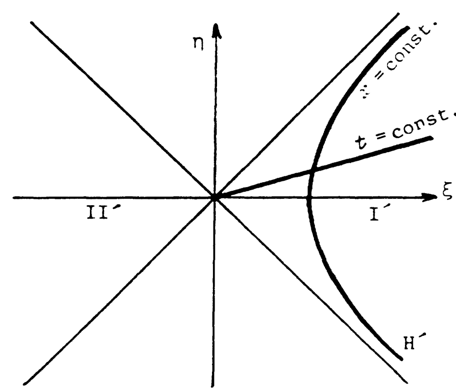
The Euclidean η-ξ spacetime possesses the Killing field

$$K^a=\alpha(\xi\sigma^a-\sigma\xi^a)\ ; \qquad (2.5)$$

the orbits of K^a are simply circles around the origin with radius $r=e^{\alpha x}/\alpha$.

By restricting to real values of η,ξ,y,z, we obtain a Lorentzian section of the complex manifold $(\mathbb{C},g_{ab})$ with the metric

TABLE I. The distinction between Rindler space and η-ξ space.

	Rindler space (τ,R)	η-ξ space $(\eta$-$\xi)$
Transformation and metric	$t=a^{-1}e^{aR}\sinh a\tau$ $x=a^{-1}e^{aR}\cosh a\tau$ $dS^2=e^{2aR}(-d\tau^2+dR^2)$	$\eta=a^{-1}e^{ax}\sinh at$ $\xi=a^{-1}e^{ax}\cosh at$ $dS^2=\frac{1}{a^2(\xi^2-\eta^2)}(-d\eta^2+d\xi^2)$
Spacetime diagram		
Remarks	(1) The coordinates (τ,R) (i.e., Rindler Wedge) cover only a quadrant (region I) of Minkowski space. (2) The hyperbolae H (R =const) correspond to the world line of a uniformly accelerating observer. (3) In particular, the Minkowski vacuum $\lvert 0_M\rangle$ is a thermal state for a uniformly accelerating observer, but the converse is not true; i.e., the Rindler vacuum $\lvert 0_R\rangle$ is not a thermal state for an inertial observer. This is the reason why we try to construct the new space: η-ξ space.	(i) The coordinates (t,x) (i.e., Minkowski space) cover only a quadrant (region I′) of η-ξ space. (ii) The hyperbolae H' (x =const) correspond to the world line of an inertial observer. (iii) The η-ξ vacuum $\lvert 0_{\eta\text{-}\xi}\rangle$ (or $\lvert 0(\beta)\rangle$) is a thermal state for an inertial observer.

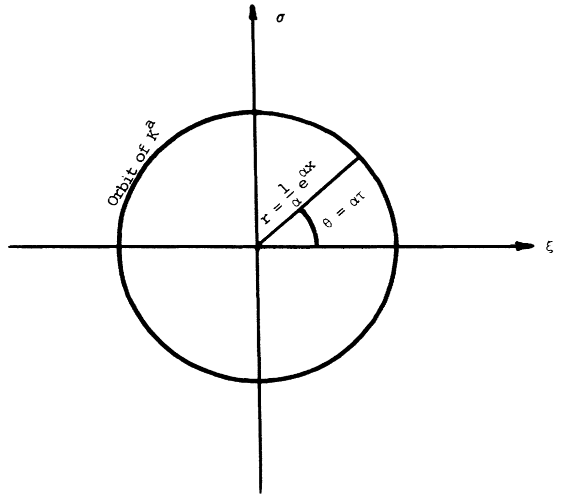


FIG. 1. Euclidean η-ξ spacetime. In the σ-ξ plane, Euclidean time $\alpha\tau$ plays the role of a polar angle with period 2π.

$$ds^2 = \frac{1}{\alpha^2(\xi^2-\eta^2)}(-d\eta^2+d\xi^2)+dy^2+dz^2$$
$$= \frac{dU\,dV}{\alpha^2 UV}+dy^2+dz^2 \; ; \qquad (2.6)$$

here we introduce the null coordinates $U=\eta-\xi, V=\eta+\xi$.

The metric (2.6) is singular on the hypersurfaces $\xi^2-\eta^2=0$, and $\xi^2-\eta^2=0$ divides the Lorentzian section into four disjointed parts I, II, III, IV (Fig. 2), each of which is identified with a Minkowski spacetime. To see this, we introduce a coordinate transformation in region I ($\xi > |\eta|$):

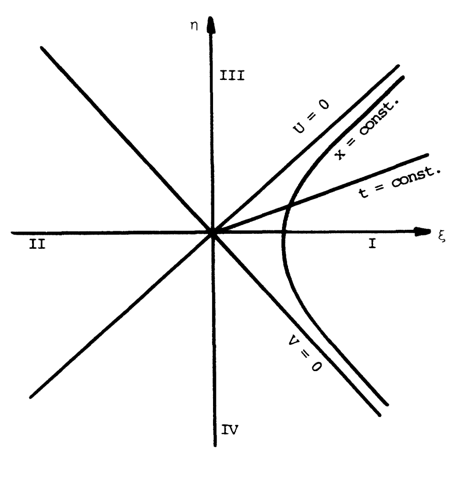


FIG. 2. η-ξ spacetime (Lorentzian section). The hypersurface $UV=\xi^2-\eta^2=0$ divide the Lorentzian section into four disjointed parts I, II, III, IV; each part of them is identified to a Minkowski spacetime. In region I, $t=$const is a straight line through the origin, and the world line ($x=$const) of a static observer in Minkowski spacetime is a hyperbola with null asymptotes $U=0$ and $V=0$.

$$\eta = \frac{1}{\alpha}e^{\alpha x}\sinh\alpha t \; ,$$
$$\xi = \frac{1}{\alpha}e^{\alpha x}\cosh\alpha t \; ; \qquad (2.7)$$

then (2.6) becomes

$$ds^2 = -dt^2+dx^2+dy^2+dz^2 \; ,$$

which is the Minkowski metric. Be careful, the coordinate transformation (2.7) looks like a Rindler coordinate transformation, but, in fact, is not (Table I). In the η-ξ plane, the entire Minkowski spacetime $-\infty < t < \infty$ and $-\infty < x < \infty$ covers only the quarter $\xi > |\eta| > 0$ (region I, our Universe). In region I (and II), the Minkowski time coordinates $t=$const are straight lines through the origin; space coordinates $x=$const are hyperbolae

$$\xi^2-\eta^2 = \frac{1}{\alpha^2}e^{2\alpha x} = \text{const} \qquad (2.8)$$

with null asymptotes $U=0, V=0$.

For other three regions, the transformation takes the following forms:

$$\left.\begin{aligned}\eta &= -\alpha^{-1}e^{\alpha x}\sinh\alpha t\\ \xi &= -\alpha^{-1}e^{\alpha x}\cosh\alpha t\end{aligned}\right\} \text{ region II ("mirror" universe) ,}$$
$$\left.\begin{aligned}\eta &= \alpha^{-1}e^{\alpha x}\cosh\alpha t\\ \xi &= \alpha^{-1}e^{\alpha x}\sinh\alpha t\end{aligned}\right\} \text{ region III ,} \qquad (2.9)$$
$$\left.\begin{aligned}\eta &= -\alpha^{-1}e^{\alpha x}\cosh\alpha t\\ \xi &= -\alpha^{-1}e^{\alpha x}\sinh\alpha t\end{aligned}\right\} \text{ region IV .}$$

Because Minkowski spacetime is complete, the Lorentzian section (2.6) of η-ξ spacetime is geodetically complete; i.e., all geodesics on the Lorentzian section are complete. The singularity of the metric (2.6) at $\xi^2-\eta^2=0$ means that the four parts I, II, III, IV are not connected on the Lorentzian section. If we only stay on the Lorentzian section, it is not easy to find out the connection between the four parts. But if we extend it to a complex manifold, we can find that the four parts of Lorentzian section can be connected via complex path (Fig. 3). If we add a small

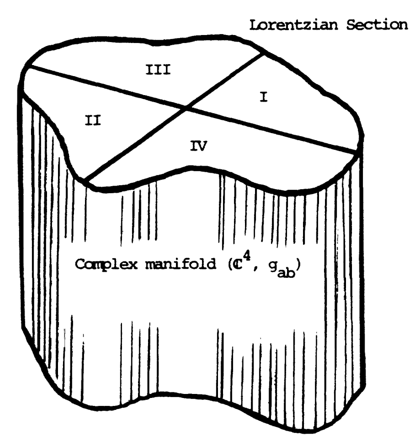


FIG. 3. The Lorentzian section on the complex manifold ($\mathbb{C}^4, g_{ab}$).

imaginary $i\sigma$ to real time η, then metric (2.6) will not be singular again provided $\sigma \neq 0$. Hereafter, we will regard the Lorentzian section given by (2.6) as the limit when we restrict ξ,y,z as real and let the imaginary part $i\sigma$ of time in the complexified metric (2.1) approach zero, and call the Lorentzian section η-ξ spacetime also (although it, in fact, is not a connected spacetime, but rather a piecing together of four Minkowski spacetimes).

η-ξ spacetime possesses Killing fields which are timelike in regions I or II. In terms of η-ξ coordinates η,ξ,y,z, the Killing field can be written as

$$b^a = \alpha(\xi\eta^a + \eta\xi^a) , \tag{2.10}$$

where α is a constant. The orbit of b^a coincides with the world line of a static observer in Minkowski spacetime.

Formally, we can call the hypersurfaces $\xi^2 - \eta^2 = 0$ "horizons" only in the sense that an inertial observer in region I cannot accept any signal sent from $U=0$, and cannot send any signal to $V=0$. So the hypersurface $U=0$ or $V=0$ can be formally called a "future horizon" $\mathcal{H}^+$ or "past horizon" $\mathcal{H}^-$ for an inertial observer in region I, although they are not the horizons[9] in the usual sense.

III. CANONICAL FORM OF QUANTUM FIELD IN η-ξ SPACETIME

Let us start by considering a scalar field theory with an action

$$S = \int d^4x \left[\frac{1}{2} \left[\frac{\partial\phi}{\partial t} \right]^2 - \tfrac{1}{2}(\nabla\phi)^2 - V(\phi) \right] \tag{3.1}$$

in Minkowski spacetime with coordinates (x,y,z,t), where $V(\phi)$ is a function of ϕ. Taking transformation (2.7) and extending the domain of integration to whole η-ξ spacetime, we get a new action called the action in η-ξ spacetime:

$$S^{\eta} = \int d\eta\, d\xi\, dx_{\perp} e^{-2\alpha x} \left\{ \frac{1}{2} e^{2\alpha x} \left[\left[\frac{\partial\phi}{\partial\eta} \right]^2 - \left[\frac{\partial\phi}{\partial\xi} \right]^2 \right] - \tfrac{1}{2}(\nabla_{\perp}\phi)^2 - V(\phi) \right\} , \tag{3.2}$$

where $x_{\perp}$ denotes y,z coordinates, $e^{2\alpha x} = \alpha^2(\xi^2 - \eta^2)$.

To get a canonical form of the quantum field in η-ξ spacetime, it is important to choose a suitable time variate. We define a Killing parameter λ by the Killing fields

$$\left[\frac{\partial}{\partial\lambda} \right]^2 = \epsilon\alpha(\xi\eta^a + \eta\xi^a) , \tag{3.3}$$

which are timelike in regions I and II. Here

$$\epsilon = \begin{cases} 1 & \text{in region I, III} \\ -1 & \text{in region II, IV} . \end{cases}$$

[Israel suggests adding the factor ϵ in (3.3) to ensure that the light cones in both regions I and II have the same direction.] It is natural to choose the Killing parameter λ given by (3.3) as the time coordinate. We make the transformation $(\eta,\xi) \to (\lambda, X)$, where X is given by the vector field

$$\left[\frac{\partial}{\partial X} \right]^a = \alpha(\eta\eta^a + \xi\xi^a) . \tag{3.4}$$

The Jacobian of this transformation is

$$\left| \frac{\partial(\eta,\xi)}{\partial(\lambda,X)} \right| = \epsilon\alpha^2(\xi^2 - \eta^2) = \epsilon e^{2\alpha x} .$$

Then the action (3.2) becomes

$$S^{\eta} = \int d\lambda\, dX\, dx_{\perp} \mathcal{L}^{\lambda} , \tag{3.5}$$

where

$$\mathcal{L}^{\lambda} = \epsilon \left\{ \frac{1}{2} e^{2\alpha x} \left[\left[\frac{\partial\phi}{\partial\eta} \right]^2 - \left[\frac{\partial\phi}{\partial\xi} \right]^2 \right] - \tfrac{1}{2}\nabla_{\perp}\phi - V(\phi) \right\} ; \tag{3.6}$$

in (3.6) η and ξ are the functions of λ and X, the integral (3.5) runs over whole η-ξ spacetime.

A Legendre transformation of the action yields the momentum Π^{λ}_{ϕ} canonically conjugate to ϕ in λ-X coordinates,

$$\Pi^{\lambda}_{\phi} = \frac{\partial\mathcal{L}^{\lambda}}{\partial \left[\frac{\partial\varphi}{\partial\lambda} \right]} = \alpha \left[\xi\frac{\partial\phi}{\partial\eta} + \eta\frac{\partial\phi}{\partial\xi} \right] , \tag{3.7}$$

and the Hamiltonian which is independent of the time λ:

$$H^{\lambda} = \int_{\sigma} dX\, dx_{\perp} \epsilon \left\{ \frac{\alpha^2}{2} \left[\left[\xi\frac{\partial\phi}{\partial\eta} + \eta\frac{\partial\phi}{\partial\xi} \right]^2 + \left[\eta\frac{\partial\phi}{\partial\eta} + \xi\frac{\partial\phi}{\partial\xi} \right]^2 \right] + \tfrac{1}{2}(\nabla_{\perp}\phi)^2 + V(\phi) \right\} , \tag{3.8}$$

where σ is a spacelike surface (such as λ=const).

λ, X are coincided with t,x and $-t,+x$ in regions I and II, respectively, where t and x are related to η,ξ by (2.7) and (2.9). If using coordinates t,x, then we have

$$H^{\lambda} = H^{\lambda}_{\mathrm{I}} + H^{\lambda}_{\mathrm{II}} = H - \tilde{H} , \tag{3.9}$$

where $H^{\lambda}_{\mathrm{II}}$ is defined on the hypersurface λ=const, and $\tilde{H}$ is defined on the hypersurface t=const (both are in region II). The presence of the sign change of the second term can be regarded as due to the opposite out-normal direction of the hypersurfaces t=const and λ=const. Equation (3.9) directly relates the quantum field in η-ξ spacetime with TFD.

IV. GREEN'S FUNCTION IN η-ξ SPACETIME

In this section we shall show that a thermal Green's function in usual statistical mechanics is just the vacuum Green's function in η-ξ spacetime.

Now we consider the partition function in Minkowski spacetime:

$$Z(\beta)=\mathrm{Tr}(e^{-\beta H}) \tag{4.1}$$

for arbitrary β. The functional-integral form for z is given by[10]

$$Z(\beta)=N_0\int[d\Pi]\int_{\text{periodic}}[d\phi]\exp\left[\int_0^\beta d\tau\int d^3x\left[i\Pi\frac{\partial\phi}{\partial\tau}-\mathcal{H}(\Pi,\phi)\right]\right], \tag{4.2}$$

where $\tau=it$, $\mathcal{H}(\Pi,\phi)=\Pi^2/2+\frac{1}{2}(\nabla\phi)^2+V(\phi)$. The Π integral is Gaussian; then we have

$$Z(\beta)=N_1\int_{\phi(\tau=0)=\phi(\tau=\beta)}[d\phi]\exp\left\{-\int_0^\beta d\tau\int d^3x\left[\frac{1}{2}\left[\frac{\partial\phi}{\partial\tau}\right]^2+\frac{1}{2}(\nabla\phi)^2+V(\phi)\right]\right\}. \tag{4.3}$$

We now perform the change of variables (2.3) from (x,τ) to (ξ,σ). We must emphasize that this is simply a change of integral variables and not to be interpreted as a change of coordinates. If we want the transformation to be single valued, we must have $\beta\alpha\le 2\pi$. For a given β, we can choose a proper α so that $\beta\alpha=2\pi$, then the periodic boundary conditions become consistency condition at $\alpha\tau=2\pi$, and we have

$$\begin{aligned}Z(\beta)&=N_1\int[d\phi]\exp\left[-\int d\sigma\, d\xi\, dx_\perp e^{-2\alpha x}\left\{\frac{1}{2}e^{2\alpha x}\left[\left[\frac{\partial\phi}{\partial\sigma}\right]^2+\left[\frac{\partial\phi}{\partial\xi}\right]^2\right]+\frac{1}{2}(\nabla_\perp\phi)^2+V(\phi)\right\}\right]\\&\equiv N_1\int[d\phi]\exp(-S_E^\eta)=W_E^\eta,\end{aligned} \tag{4.4}$$

where $e^{-2\alpha x}=1/\alpha^2(\xi^2+\sigma^2)$, S_E^η is the action in Euclidean η-ξ spacetime, W_E^η is just the Euclidean generating functional for the theory in η-ξ spacetime.

Now we shall show that the thermal Green's function in Minkowski spacetime is equal to the vacuum Green's function in η-ξ spacetime. We assume that the Green's function defined by the Euclidean generating functional in η-ξ spacetime and analytically continued back to real values of the coordinates gives the correct vacuum Green's function in the Lorentzian section of η-ξ spacetime. The definition of the vacuum in η-ξ spacetime will be discussed in the next section. Consider first

$$K=\mathrm{Tr}[e^{-\beta H}\phi(\mathbf{x}_1,t_1)\cdots\phi(\mathbf{x}_n,t_n)]. \tag{4.5}$$

Following the steps leading to (4.3), for $t_1>t_2>\cdots>t_n$ we find

$$K=N\frac{\delta^n}{\delta J(\mathbf{x}_1,t_1)\cdots\delta J(\mathbf{x}_n,t_n)}\int_{\text{periodic}}[d\phi]\exp\left\{-\int_0^\beta d\tau\int d^3x\left[\frac{1}{2}\left[\frac{\partial\phi}{\partial\tau}\right]^2+\frac{1}{2}(\nabla\phi)^2+V(\phi)-J\phi\right]\right\}\Bigg|_{J=0}. \tag{4.6}$$

We now perform the change of variable (2.3) with $\beta\alpha=2\pi$; then

$$\begin{aligned}K&=N\frac{\delta^n}{\delta J(\mathbf{x}_1,t_1)\cdots\delta J(\mathbf{x}_n,t_n)}\int[d\phi]\exp\left[-\int d\sigma\, d\xi\, dx_\perp e^{-2\alpha x}\right.\\&\qquad\left.\times\left\{\frac{1}{2}e^{2\alpha x}\left[\left[\frac{\partial\phi}{\partial\sigma}\right]^2+\left[\frac{\partial\phi}{\partial\xi}\right]^2\right]+\frac{1}{2}(\nabla_\perp\phi)^2+V(\phi)-J\phi\right\}\right]\Bigg|_{J=0}\\&=\frac{\delta^n}{\delta J(\mathbf{x}_1,t_1)\cdots\delta J(\mathbf{x}_n,t_n)}W_E^\eta[J]\Bigg|_{J=0}.\end{aligned} \tag{4.7}$$

Translate (σ,ξ) to (λ,X) which are defined by

$$\left[\frac{\partial}{\partial\lambda}\right]^a=\alpha(\xi\sigma^a-\sigma\xi^a), \tag{4.8}$$

$$\left[\frac{\partial}{\partial X}\right]^a=\alpha(\sigma\sigma^a+\xi\xi^a), \tag{4.9}$$

where $(\partial/\partial\lambda)^a$ is a Killing field for Euclidean η-ξ spacetime [note that we retain the same symbols λ and X with (3.3) and (3.4)]. Because all points $(\mathbf{x}_1,t),\ldots,(\mathbf{x}_n,t_n)$ are to lie within region I, then we have

$$\frac{\delta^n}{\delta J(\mathbf{x}_1,t_1)\cdots\delta J(\mathbf{x}_n,t_n)}=\frac{\delta^n}{\delta J(\mathbf{X}_1,\lambda_1)\cdots\delta J(\mathbf{X}_n,\lambda_n)}\equiv\frac{\delta^n}{\delta J(\bar{X}_1)\cdots\delta J(\bar{X}_n)},$$

where $\mathbf{X}_i=(X_i,y_i,z_i),\bar{X}_i=(\mathbf{X}_i,\lambda_i)$. So

$$K=N\frac{\delta^n}{\delta J(\bar{X}_1)\cdots\delta J(\bar{X}_n)}\int[d\phi]\exp\left\{-\int d\lambda\,dX\,dx_\perp\left[\frac{1}{2}\alpha^2\left[\xi\frac{\partial\phi}{\partial\sigma}-\sigma\frac{\partial\phi}{\partial\xi}\right]^2+\frac{1}{2}\alpha^2\left[\sigma\frac{\partial\phi}{\partial\sigma}+\xi\frac{\partial\phi}{\partial\xi}\right]^2+\tfrac{1}{2}(\nabla_\perp\phi)^2+V(\phi)-J\phi\right]\right\}\Bigg|_{J=0}$$

$$=\frac{\delta^n}{\delta J(\bar{X}_1)\cdots\delta J(\bar{X}_n)}W_E^\lambda[J]\Bigg|_{J=0}. \tag{4.10}$$

Dividing (4.5) by $\mathrm{Tr}e^{-\beta H}$ and (4.10) by W_E^λ, we conclude that

$$\frac{\mathrm{Tr}\{e^{-\beta H}[\phi(\mathbf{x}_1,t_1)\cdots\phi(\mathbf{x}_n,t_n)]_t\}}{\mathrm{Tr}(e^{-\beta H})}=\frac{1}{W_E^\lambda[J]}\frac{\delta^n}{\delta J(X_1)\cdots\delta J(\bar{X}_n)}W_E^\lambda[J]\Bigg|_{J=0}$$

$$\equiv G^\lambda(\bar{X}_1;\ldots;\bar{X}_n), \tag{4.11}$$

where $(\cdots)_t$ in the left of (4.11) denote t ordering, $G^\lambda(\bar{X}_1,\ldots,\bar{X}_n)$ is the Green's function in η-ξ spacetime. This is just what we want to prove.

It is convenient to use η-ξ coordinates instead of λ-X coordinates in Eq. (4.10). Note that

$$\frac{\delta}{\delta J(\bar{X})}=\left|\frac{\partial(\bar{\xi})}{\partial(\bar{X})}\right|\frac{\delta}{\delta J(\bar{\xi})}=e^{2\alpha x}\frac{\delta}{\delta J(\bar{\xi})}\equiv\frac{\delta}{\delta\tilde{J}(\bar{\xi})},$$

where $\bar{\xi}=(\xi,\eta),\xi=(\xi,y,z),\tilde{J}\equiv e^{-2\alpha x}J$. The points $(\mathbf{x},t)$ are restricted in region I so $(\mathbf{x},t)=(\mathbf{X},\lambda)$, and we have $W_E^\lambda[J]=W_E^\eta[\tilde{J}]$ by comparing (4.7) with (4.10). Then we have

$$\frac{\delta}{\delta J(\bar{X})}W_E^\lambda[J]=\frac{\delta}{\delta\tilde{J}(\bar{\xi})}W_E^\eta[\tilde{J}],$$

and the Green's functions (4.11) in η-ξ spacetime can be expressed as

$$G^\lambda(\bar{X}_1,\ldots,\bar{X}_n)=\frac{1}{w_E^\eta[\tilde{J}]}\frac{\delta^n W_E^\eta[\tilde{J}]}{\delta\tilde{J}(\bar{\xi}_1)\cdots\delta\tilde{J}(\bar{\xi}_n)}\Bigg|_{J=0}$$

$$\equiv G^\eta(\bar{\xi}_1,\ldots,\bar{\xi}_n). \tag{4.12}$$

Here we use η ordering instead of the λ ordering, because both η and λ increase everywhere within the future, null cone of any point, as long as we restrict all points to lie within region I, η ordering and λ ordering will be the same, and the point $(\bar{X}_i)=(\bar{\xi}_i)$.

We have shown (although somewhat formally) that the thermal Green's function in usual statistical mechanics is just equal to the vacuum Green's function in η-ξ spacetime. It is interesting to compare (4.11) and (4.12) with the result[11] by Unruh and Weiss.

V. QUANTUM FIELD IN η-ξ SPACETIME

Now we will discuss the quantum field in the Lorentzian section of η-ξ spacetime. The first question we meet is the following: what is the meaning of the quantum fields in η-ξ spacetime? We have known that the Lorentzian section consists of four disjoint parts. How can we talk about the quantum fields on the Lorentzian section? The answer is that, mathematically, we can find out the solutions of the equations of motion which are analytic on the "horizons" and can be extended to whole η-ξ spacetime.

The scalar wave equation is given by

$$(\Box-\mu^2)\phi=0. \tag{5.1}$$

In region I or II, let $f_\omega(t,x,y,z)\propto\exp(-i\omega t)$ be energy eigenfunctions of Eq. (5.1) as registered by the static observers in Minkowski space. On the "horizons" $\mathcal{H}^\pm$, the asymptotic form of wave packets which are regular there is a superposition of modes

$$f_\omega(t,x,y,z)|_{\mathcal{H}^\pm}\approx\exp(-i\alpha^{-1}\Delta\omega\ln|U^\pm|)P(y,z), \tag{5.2}$$

where $\Delta=\pm1$, $U^\pm=\alpha(\eta\pm\xi)$, $P(y,z)$ is a regular analytic function.

Since regions I and II are causally disjoint, we can associate with any given eigenfunction $f_\omega(t,x,y,z)$ two "Minkowski modes" ${}^{(+)}f_\omega$ and ${}^{(-)}f_\omega$ which are given by

$${}^{(+)}f_\omega=\begin{cases}f_\omega(t,x,y,z) & \text{region I},\\ 0 & \text{region II},\end{cases} \tag{5.3}$$

$${}^{(-)}f_\omega=\begin{cases}0 & \text{region I},\\ f_\omega(t,x,y,z) & \text{region II}.\end{cases} \tag{5.4}$$

Then the two linear combinations

$$\begin{aligned}{}^{(+)}F_\omega&=\cosh\theta_\omega{}^{(+)}f_\omega+\sinh\theta_\omega{}^{(-)}f_\omega,\\ {}^{(-)}F_\omega&=\cosh\theta_\omega{}^{(-)}f_\omega+\sinh\theta_\omega{}^{(+)}f_\omega\end{aligned} \tag{5.5}$$

will be analytic in U and V on the "horizons" provided

$$\tanh\theta_\omega=\exp(-\pi\omega/\alpha). \tag{5.6}$$

Both ${}^{(+)}F_\omega$ and ${}^{(-)}F_\omega$ are regular in the lower halves of the complex U and V planes; their spectra therefore contain only positive frequencies with respect to U and V (called η-ξ modes).

That ${}^{(+)}F_\omega$ and ${}^{(-)}F_\omega$ are analytic in real U and V on the "horizons" $\mathcal{H}^{(\pm)}$ ($U=0$ or $V=0$) does not contradict the fact that the metric (2.6) of the Lorentzian section on the "horizons" $\mathcal{H}^{(\pm)}$ is singular. To see this, we can write (5.5) in η-ξ null coordinates which near the horizons $\mathcal{H}^{(\pm)}$ approaches

$$V^{-i(\omega/\alpha)}P(y,z)$$

and (5.7)

$$U^{i(\omega/\alpha)}P(y,z)\ ,$$

respectively, for all U and V in region $-\infty$ to ∞ (i.e., in both regions I and II); they are clearly analytic across $U=0$ and $V=0$.

A complete set of eigenfunctions $f_{\omega j}$ leads to sets ${}^{(\pm)}f_{\omega j}, {}^{(\pm)}F_{\omega j}$ which are complete over region I∪II and which satisfy orthonormality conditions

$$({}^{(\epsilon)}f_{\omega j}, {}^{(\epsilon')}f_{\omega' j'}) = ({}^{(\epsilon)}F_{\omega j}, {}^{(\epsilon')}F_{\omega' j'}) = \epsilon\delta_{\epsilon\epsilon'}\delta_{jj'}\delta(\omega-\omega') \quad (\omega>0) \tag{5.8}$$

with respect to the Klein-Gordon inner product

$$\langle\phi_1,\phi_2\rangle = \frac{i}{2}\int_\sigma [\phi_1^*\partial_a\phi_2 - (\partial_a\phi_1^*)\phi_2] n^a d\Sigma\ ; \tag{5.9}$$

the integral is to be taken over a complete Cauchy slice with a consistent choice of future-directed normal n^a [the factor ϵ in (5.8) arises from the contraposition of n^a and t^a in region II].

To quantize a real Klein-Gordon field $\phi(x)$ in terms of the set of Minkowski modes ${}^{(\pm)}f_{\omega j}$ ($\omega>0$), we expand

$$\phi = \begin{cases} \sum_{\omega,j} a_{\omega j}{}^{(+)}f_{\omega j} + a^\dagger_{\omega j}{}^{(+)}f^*_{\omega j} & \text{region I}\ , \\ \sum_{\omega,j} \tilde{a}_{\omega j}{}^{(-)}f^*_{\omega j} + \tilde{a}^\dagger_{\omega j}{}^{(-)}f_{\omega j} & \text{region II}\ , \end{cases} \tag{5.10}$$

where the annihilation and creation operators satisfy the commutation relations

$$[a_{\omega j}, a^\dagger_{\omega' j'}] = [\tilde{a}_{\omega j}, \tilde{a}^\dagger_{\omega' j'}] = \delta_{jj'}\delta(\omega-\omega') \tag{5.11}$$

(other commutators vanish). The summation over ω in (5.10) symbolizes the integral $\int_0^\infty d\omega$. Define

$$a_{\omega j}|0\rangle = 0,\quad \tilde{a}_{\omega j}|\tilde{0}\rangle = 0 \quad \forall\omega,j \tag{5.12}$$

where $|0\rangle$ is the Minkowski vacuum in our Universe and $|\tilde{0}\rangle$ is the Minkowski vacuum in the "mirror" universe.

On the other hand, we obtain an alternative expansion of $\phi(x)$ in terms of η-ξ modes

$$\phi = \sum_{\omega j} (b_{\omega j}{}^{(+)}F_{\omega j} + b^\dagger_{\omega j}{}^{(+)}F^*_{\omega j} + \tilde{b}_{\omega j}{}^{(-)}F^*_{\omega j} + \tilde{b}^\dagger_{\omega j}{}^{(-)}F_{\omega j})\ . \tag{5.13}$$

The operators $b_{\omega j}, \tilde{b}_{\omega j}$ also satisfy the commutation relation (5.11) and are called thermal particle annihilation operators. We can define the η-ξ vacuum (thermal ground state)

$$b_{\omega j}|0(\beta)\rangle = \tilde{b}_{\omega j}|0(\beta)\rangle = 0 \quad \forall\omega,j\ . \tag{5.14}$$

By equating the two expressions (5.10) and (5.13) for ϕ, we obtain in the usual manner the Bogoliubov transformation relating the Minkowski and η-ξ annihilation and creation operators. We find

$$\begin{aligned} b_{\omega j} &= a_{\omega j}\cosh\theta_\omega - \tilde{a}^\dagger_{\omega j}\sinh\theta_\omega\ , \\ \tilde{b}_{\omega j} &= \tilde{a}_{\omega j}\cosh\theta_\omega - a^\dagger_{\omega j}\sinh\theta_\omega\ . \end{aligned} \tag{5.15}$$

The Bogoliubov transformation (5.15) provides the required relation between the thermal ground state (η-ξ vacuum) and the Minkowski vacuum:

$$|0(\beta)\rangle = \prod_{\omega,j}(1-e^{-2\pi\omega/\alpha})^{1/2} \times \sum_{n_{\omega j}} \exp(-n_{\omega j}\pi\omega/\alpha)|n_{\omega j}\rangle|\tilde{n}_{\omega j}\rangle\ , \tag{5.16}$$

where $|n_{\omega j}\rangle$ denotes the state with $n_{\omega j}$ quanta in region I (our Universe), $|\tilde{n}_{\omega j}\rangle$ denotes the state with $n_{\omega j}$ quanta in region II ("mirror" universe). The latter are in principle unobservable by an observer in region I. By "tracing out" over the degrees of freedom associated with region II, we obtain the density matrix ρ for region I given by

$$\rho = \prod_{\omega j}\sum_n \left\{ \frac{e^{-\beta E_n}}{\sum_{m=0}^\infty e^{-\beta E_m}} \right\} |n_{\omega j}\rangle\langle n_{\omega j}|\ , \tag{5.17}$$

with $E_n = n\omega, \beta = 2\pi/\alpha$. This is precisely a thermal density matrix. Thus, we find that the η-ξ vacuum corresponds to a thermal state at temperature $KT=\alpha/2\pi$ for an inertial observer in Minkowski spacetime. When this observer measures an observable A (a functional of $a_{\omega j}, a^\dagger_{\omega j}$ only), the result is

$$\langle 0(\beta)|A|0(\beta)\rangle = \prod_{\omega j}\sum_n \langle n_{\omega j}|A|n_{\omega j}\rangle \frac{e^{-\beta E_n}}{\sum_m e^{-\beta E_m}} = \mathrm{tr}(A\rho)\ ; \tag{5.18}$$

i.e., an expectation value of A in the pure state $|0(\beta)\rangle$ coincides with the statistical average of A in usual statistical mechanics.

Finally, we need to show that the η-ξ vacuum defined by (5.14) is just the vacuum of Green's function G^λ defined by (4.11). From Sec. IV we have known that G^λ is defined in terms of the time λ, so the vacuum in G^λ should be the vacuum defined by positive-frequency modes with respect to the time λ. These can be written as $\exp(-i\omega\lambda)$, where λ is defined by (3.3). If using the null coordinates U, V, we can rewrite (3.3) as

$$\left[\frac{\partial}{\partial\lambda}\right]^a = \epsilon\alpha(\xi\eta^a + \eta\xi^a) = \epsilon\alpha(VV^a - UU^a)\ ,$$

where $V^a = (\partial/\partial V)^a, U^a = (\partial/\partial U)^a$. On the "horizons"

$\mathcal{H}^{\pm}$ (i.e., $U=0$ or $V=0$), the positive-frequency λ modes $\exp(-i\omega\lambda)$ approach $\exp(-i\Delta\alpha^{-1}\omega\ln|U^{\pm}|)$, just according to which we defined the η-ξ vacuum in Sec. V.

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