Cosmic-string traveling waves

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We find the metric, scalar, and gauge fields for a traveling wave on a strongly gravitating cosmic string. This solution is found by applying a transformation to the corresponding solution for a long straight static string. Limiting cases of the solution are examined and found to agree with previous results.

I. INTRODUCTION

In the cosmic-string scenario the strings are expected to have a small thickness and are usually treated in the zero-thickness approximation.[1] The dynamics of a zero-thickness string is found using the Nambu action, which is simply the area of the string's world sheet. A finite-thickness string is a configuration of self-interacting fields and is thus more complicated than the zero-thickness string. With the exception of Ref. 2 the work on finite-thickness strings has treated only the simplest case: that of a static straight string. In Ref. 2 field-theoretic solutions were found for traveling waves on an infinite string. In this paper we extend the result of Ref. 2 to include the gravitational field of the string. We find solutions of Einstein's equation (as well as the equations for the fields making up the string) that describe a traveling wave on a gravitating string.

In Sec. II we will introduce some notation and present some facts about the results of Ref. 2. Section III contains our method for finding solutions. The string traveling wave solution and some limiting cases are examined in Sec. IV and a discussion of the results is given in Sec. V.

II. NOTATION

We consider strings made of a complex scalar field ϕ and a vector field A_a with the usual Abelian Higgs Lagrangian

$$\mathcal{L}=-\tfrac{1}{2}(D_a\phi)(D^a\phi)^*-\tfrac{1}{4}F^{ab}F_{ab}-\tfrac{1}{8}\lambda(|\phi|^2-\eta^2)^2 . \qquad (2.1)$$

Here $D_a\equiv\nabla_a-ieA_a$ and $F_{ab}\equiv\nabla_a A_b-\nabla_b A_a$. The equations for the fields are

$$D_aD^a\phi=\tfrac{1}{2}\lambda(|\phi|^2-\eta^2)\phi , \qquad (2.2)$$

$$\nabla_aF^{ab}=\frac{ie}{2}(\phi^*\nabla^b\phi-\phi\nabla^b\phi^*) . \qquad (2.3)$$

First consider a nongravitating string, i.e., solutions of Eqs. (2.2) and (2.3) in Minkowski spacetime. Let $(\tilde{\phi},\tilde{A}_a)$ be the fields of a straight static string. Then using the usual Cartesian coordinates (t,x,y,z) the fields have the form

$$\tilde{\phi}=\Phi(x,y) , \qquad (2.4)$$

$$\tilde{A}_a=A_1(x,y)\nabla_ax+A_2(x,y)\nabla_ay . \qquad (2.5)$$

Define the coordinate u by

$$u\equiv z-t \qquad (2.6)$$

and let $f(u)$ and $g(u)$ be arbitrary functions of u. Define the coordinates X and Y by

$$X\equiv x-f(u) , \qquad (2.7)$$

$$Y\equiv y-g(u) . \qquad (2.8)$$

Then the surface $X=Y=0$ is the world sheet of a zero-thickness string traveling wave.[3] Consider the fields ϕ and A_a given by

$$\phi\equiv\Phi(X,Y) , \qquad (2.9)$$

$$A_a\equiv A_1(X,Y)\nabla_aX+A_2(X,Y)\nabla_aY , \qquad (2.10)$$

where the functions Φ, A_1, and A_2 are the same as in Eqs. (2.4) and (2.5) but their arguments are now X and Y rather than x and y. It was shown in Ref. 2 that (ϕ, A_a) is a solution of Eqs. (2.2) and (2.3). Thus (ϕ, A_a) is the solution for the scalar and gauge fields of a cosmic-string traveling wave.

One way to verify that (ϕ, A_a) is a solution is to explicitly evaluate Eqs. (2.2) and (2.3) using the transformations given in Eqs. (2.6)–(2.8). We now present another method which is more easily generalized to the case of gravitating strings. First let $\tilde{\eta}_{ab}$ be the usual Minkowski metric

$$\tilde{\eta}_{ab}=-\nabla_at\nabla_bt+\nabla_az\nabla_bz+\nabla_ax\nabla_bx+\nabla_ay\nabla_by . \qquad (2.11)$$

Define the coordinate v by

$$v\equiv\tfrac{1}{2}(t+z)+\dot{f}X+\dot{g}Y+\tfrac{1}{2}\int(\dot{f}^2+\dot{g}^2)du , \qquad (2.12)$$

where a dot denotes derivative with respect to u. Then it follows that

$$\tilde{\eta}_{ab}=\eta_{ab}-2(\ddot{f}X+\ddot{g}Y)\nabla_a u\nabla_b u \ , \tag{2.13}$$

where

$$\eta_{ab}\equiv 2\nabla_{(a}u\nabla_{b)}v+\nabla_a X\nabla_b X+\nabla_a Y\nabla_b Y \ . \tag{2.14}$$

Note that η_{ab} is also a flat metric: one in which (X,Y) play the role of (x,y) in $\tilde{\eta}_{ab}$. It then follows immediately that (ϕ, A_a) is a solution of the field equations *in the metric* η_{ab}. In fact (ϕ, A_a, η_{ab}) is just the usual static straight-string solution written in unusual coordinates. However, we wish to show that (ϕ, A_a) is a solution in the metric $\tilde{\eta}_{ab}$. From Eq. (2.13) it is straightforward to evaluate the Christoffel symbols relating the derivative operator of $\tilde{\eta}_{ab}$ to that of η_{ab}. Then using the Christoffel symbols and the fact that (ϕ, A_a, η_{ab}) is a solution it is straightforward to show that $(\phi, A_a, \tilde{\eta}_{ab})$ is also a solution to Eqs. (2.2) and (2.3).

The solution (ϕ, A_a, η_{ab}) represents a static string, whereas $(\phi, A_a, \tilde{\eta}_{ab})$ is a string traveling wave. Thus the method of Ref. 2 for finding traveling wave solutions can be viewed as taking the fields of a static string and then finding a new metric in which these same fields describe a cosmic-string traveling wave. This is the method that we will generalize to gravitating strings.

III. GRAVITATING STRINGS

We now consider gravitating strings. Thus there is a metric g_{ab} as well as the fields (ϕ, A_a). In addition to equations (2.2) and (2.3) we need to solve Einstein's equation

$$R_{ab}-\tfrac{1}{2}Rg_{ab}=8\pi T_{ab} \ , \tag{3.1}$$

where T_{ab} is the stress energy tensor of the fields:

$$T_{ab}=(D_{(a}\phi)(D_{b)}\phi)^*+F_{ac}F_b{}^c+\mathcal{L}g_{ab} \ . \tag{3.2}$$

Let (ϕ, A_a, g_{ab}) be the solution of Eqs. (2.2), (2.3), and (3.1) for the straight, static gravitating string. In analogy with the method for nongravitating traveling wave solutions we seek a solution $(\phi, A_a, \tilde{g}_{ab})$ where (ϕ, A_a) are the same fields as for the static string and $\tilde{g}_{ab}$ is given by

$$\tilde{g}_{ab}=g_{ab}+Fk_a k_b \tag{3.3}$$

for some scalar F and vector k_a. In what follows all indices will be raised and lowered with the metric g_{ab}.

We now consider what conditions to impose on k^a and F. In the case of nongravitating strings the vector field $\nabla^a u$ is null, covariantly constant and orthogonal to $\nabla_a\phi$ and A_a. We would like to impose conditions similar to this in the case of gravitating strings. Unfortunately, the static string spacetime does not, in general, possess a covariantly constant vector field. However, it does have a null Killing vector (essentially a linear combination of the time translation and spatial translation Killing vectors). We impose the following conditions on the vector field k^a:

$$k^a k_a=0 \ , \tag{3.4}$$

$$\nabla_{(a}k_{b)}=0 \ , \tag{3.5}$$

$$k_{[a}\nabla_b k_{c]}=0 \ , \tag{3.6}$$

$$k^a\nabla_a\phi=0 \ , \tag{3.7}$$

$$k^a A_a=0 \ , \tag{3.8}$$

$$\mathcal{L}_k A_a=0 \ . \tag{3.9}$$

All these conditions are satisfied in the static string spacetime. The first three equations say that k^a is a null, hypersurface orthogonal, Killing field. Note that these equations imply that k^a is tangent to a shear-free congruence of null geodesics. Therefore the metric $\tilde{g}_{ab}$ is in the class of generalized Kerr-Schild metrics.[4] The last three equations say that k^a Lie derives ϕ and A_a and is orthogonal to A_a. We also impose the condition on F that

$$k^a\nabla_a F=0 \ . \tag{3.10}$$

We now show that (ϕ, A_a) satisfy their field equations in the metric $\tilde{g}_{ab}$. The fields (ϕ, A_a) satisfy Eqs. (2.2) and (2.3) in the metric g_{ab}. The field equations in the metric $\tilde{g}_{ab}$ consist of substituting in Eqs. (2.2) and (2.3) $\tilde{g}_{ab}$ for g_{ab} and $\tilde{\nabla}_a$ for ∇_a where $\tilde{\nabla}_a$ is the derivative operator associated with $\tilde{g}_{ab}$. From the properties of derivative operators it follows that $\tilde{\nabla}_a\phi=\nabla_a\phi$ and $\tilde{\nabla}_{[a}A_{b]}=\nabla_{[a}A_{b]}$. Thus $\nabla_a\phi$ and F_{ab} are unchanged by this substitution. The inverse to $\tilde{g}_{ab}$ is given by

$$(\tilde{g}^{-1})^{ab}=g^{ab}-Fk^a k^b \ . \tag{3.11}$$

It then follows from Eqs. (3.7) and (3.8) that

$$(\tilde{g}^{-1})^{ab}\nabla_b\phi=\nabla^a\phi \ , \tag{3.12}$$

$$(\tilde{g}^{-1})^{ab}A_b=A^a \ . \tag{3.13}$$

From Eqs. (3.8) and (3.9) it follows that $k^a F_{ab}=0$ and therefore that

$$(\tilde{g}^{-1})^{ac}(\tilde{g}^{-1})^{bd}F_{cd}=F^{ab} \ . \tag{3.14}$$

Now consider the derivative operator $\tilde{\nabla}_a$. There is a tensor C^c_{ab} such that

$$\tilde{\nabla}_a\omega_b=\nabla_a\omega_b-C^c_{ab}\omega_c \tag{3.15}$$

for all ω_c. The tensor C^c_{ab} is given by

$$C^c_{ab}=\tfrac{1}{2}(\tilde{g}^{-1})^{cd}(\nabla_a\tilde{g}_{bd}+\nabla_b\tilde{g}_{ad}-\nabla_d\tilde{g}_{ab}) \ . \tag{3.16}$$

From Eqs. (3.5) and (3.6) it follows that there is a scalar A such that

$$\nabla_a k_b=k_{[b}\nabla_{a]}A \tag{3.17}$$

and $k^a\nabla_a A=0$. We then find the following formula for C^c_{ab}:

$$C^c_{ab}=k^c(k_{(a}\nabla_{b)}F+Fk_{(a}\nabla_{b)}A)-\tfrac{1}{2}k_a k_b(\nabla^c F+2F\nabla^c A) . \tag{3.18}$$

From this formula it follows that

$$C^c_{cb}=0 \ . \tag{3.19}$$

It then follows that $\tilde{\nabla}_a v^a=\nabla_a v^a$ for any vector v^a and

$\tilde{\nabla}_a Q^{ab} = \nabla_a Q^{ab}$ for any antisymmetric tensor Q^{ab}. Then using Eqs. (3.11)–(3.14) we find that (ϕ, A_a) continues to satisfy Eqs. (2.2) and (2.3) when $\tilde{g}_{ab}$ and $\tilde{\nabla}_a$ are substituted for g_{ab} and ∇_a, respectively. Therefore (ϕ, A_a) satisfy the field equations in the metric $\tilde{g}_{ab}$.

We now consider Einstein's equation. Let $\tilde{R}_{ab}$ be the Ricci tensor of $\tilde{g}_{ab}$. Then $\tilde{R}_{ab}$ is given by

$$\tilde{R}_{ab} = R_{ab} - 2\nabla_{[a} C^c_{c]b} + 2C^d_{b[a} C^c_{c]d} \ . \tag{3.20}$$

It then follows from Eq. (3.18) that

$$\begin{aligned}\tilde{R}_{ab} &= R_{ab} + \nabla_c C^c_{ab} \\ &= R_{ab} - \tfrac{1}{2} k_a k_b (\nabla_c \nabla^c F + 2F \nabla_c \nabla^c A \\ &\qquad + 2\nabla^c F \nabla_c A + F \nabla_c A \nabla^c A) \ .\end{aligned} \tag{3.21}$$

Since k^a is a Killing vector it follows that

$$R_{ab} k^b = -\nabla^b \nabla_b k_a = -\tfrac{1}{2} k_a \nabla^b \nabla_b A \ . \tag{3.22}$$

Therefore we find

$$(\tilde{g}^{-1})^{ac} \tilde{R}_{cb} = R^a{}_b - \tfrac{1}{2} k^a k_b e^{-A} \nabla_c \nabla^c (e^A F) \ . \tag{3.23}$$

We now impose the condition on F that

$$\nabla_c \nabla^c (e^A F) = 0 \ . \tag{3.24}$$

It then follows that

$$(\tilde{g}^{-1})^{ac} \tilde{R}_{cb} = R^a{}_b \ . \tag{3.25}$$

Thus the Ricci tensor (with one index up and one down) is unchanged under the substitution of $\tilde{g}_{ab}$ for g_{ab}. Therefore the Einstein tensor $G^a{}_b$ is also unchanged.

Now using Eqs. (3.12)–(3.14) in Eqs. (3.2) and (2.1) it follows that the stress energy tensor $T^a{}_b$ is unchanged when $\tilde{g}_{ab}$ is substituted for g_{ab}. Thus Einstein's equation is still satisfied after the substitution. Therefore (ϕ, A_a, g_{ab}) is a solution of the Einstein-Abelian Higgs equations.

This method of generating solutions should work on a wide class of Lagrangians other than the Abelian Higgs model. However, the method is somewhat limited, since it requires that the background metric g_{ab} possess a null, hypersurface orthogonal Killing vector.

IV. STRING TRAVELING WAVES

We now consider, in more detail, traveling waves on a cosmic string. The metric g_{ab} of the straight static string is

$$g_{ab} = 2e^A \nabla_{(a} u \nabla_{b)} v + S^2 (\nabla_a X \nabla_b X + \nabla_a Y \nabla_b Y) \tag{4.1}$$

and the null Killing vector k^a is

$$k^a = \left[\frac{\partial}{\partial v} \right]^a . \tag{4.2}$$

Here A is the same as in Eq. (3.17) and A and S are functions of X and Y. Since $k^a \nabla_a F = 0$ it follows that F is a function of u, X, and Y. The equation (3.24) for F becomes

$$\frac{\partial}{\partial X} \left[e^A \frac{\partial}{\partial X} (e^A F) \right] + \frac{\partial}{\partial Y} \left[e^A \frac{\partial}{\partial Y} (e^A F) \right] = 0 \ . \tag{4.3}$$

Far from the string the static string metric approaches Minkowski spacetime minus a wedge whose angular size we denote $\Delta\phi$. The asymptotic form of A and S is $A \to a_0$, $S \to s_0 (X^2 + Y^2)^{-\Delta\phi/4\pi}$ where a_0 and s_0 are constants. Thus, far from the string F approaches a solution of the two-dimensional Laplace equation. A solution to Eq. (4.3) is determined by the asymptotic values of F. We choose F so that its asymptotic form is

$$F \to -2(\ddot{f} X + \ddot{g} Y) \ , \tag{4.4}$$

where f and g are arbitrary functions of u. This form is chosen to agree with the nongravitating string in the limit of weak gravity. Taking the limit as the strength of the string's gravitational field vanishes the metric becomes flat and the fields (ϕ, A_a) approach the values of Ref. 2. One can choose a different solution of Eq. (4.3) for F, one where F grows faster than linearly with X and Y at large X and Y. Linear growth corresponds to a string traveling wave. Faster growth corresponds to an additional gravitational wave traveling along with the cosmic-string traveling wave.

The asymptotic form of the traveling wave metric agrees with the metric of Ref. 5 for a traveling wave on a zero-thickness cosmic string. Furthermore, taking the zero-thickness limit of our solution produces the solution of Ref. 5. Thus the metric of Ref. 5 is also the exterior (i.e., outside the core region) metric of a finite-thickness cosmic-string traveling wave.

There is one case where the equation for F can be solved in closed form. As shown by Linet[6] when the Abelian Higgs coupling constants satisfy the relation $e^2 = \lambda$ the scalar A vanishes. In this case F is simply a solution of the two-dimensional Laplace equation and has the form given in Eq. (4.4) everywhere.

V. DISCUSSION

We now consider some possible extensions of the results of this paper. In Ref. 2 traveling waves on global strings and domain walls were treated in addition to those on gauge strings. Can our method be used to treat self-gravitating global strings and domain walls? The static straight global string has a null Killing vector; so our method can certainly produce solutions for gravitating traveling waves on such strings. However, the asymptotic properties of the static global string metric are far more pathological than those of the gauge string. Therefore it is not clear what boundary conditions to place on the function F; so there may not be a particular string traveling wave solution that is preferred. The self-gravitating domain wall does not have a null Killing vector; so our method cannot be used to find gravitating domain wall traveling waves. It may be that self-gravity prevents domain walls from having traveling waves. Alternatively, a different method might be used to find traveling waves on domain walls.

Though in our solution the traveling wave does not dissipate, there remains the possibility that cosmic-string traveling waves might dissipate through quantum creation of particles. However, it has been shown that there is no particle production in the plane-wave solutions of Einstein's equation.[7,8] It is likely that this result can be extended to a large class of spacetimes with a null Killing vector, probably including ours. Therefore we expect that cosmic-string traveling waves produce no particles.

The static cosmic string has two null Killing vectors corresponding to the two directions of translation along the string. Thus our method can produce solutions corresponding to left-moving traveling waves or right-moving traveling waves. The method can even be used to find two oppositely directed traveling waves. Consider a metric of the form

$$\bar{g}_{ab}=g_{ab}+Fk_a k_b+Gl_a l_b \ , \tag{5.1}$$

where g_{ab} and k^a are given by Eqs. (4.1) and (4.2), respectively, and where $l^a=(\partial/\partial u)^a$. Let F and G satisfy Eq. (4.3) and $k^a\nabla_a F=l^a\nabla_a G=0$. Furthermore let F have compact support in u and G have compact support in v and restrict attention to the region of the spacetime where the supports of F and G do not overlap. Then $(\phi, A_a, \bar{g}_{ab})$ is a solution of the Einstein-Abelian Higgs equations representing two oppositely directed traveling waves on a cosmic string. Since we consider only that region where the supports of F and G do not intersect, we have a solution for the spacetime before the two traveling waves collide. After the waves collide the spacetime will, in general, have no symmetries and we do not expect to be able to find a solution in this region. The oppositely directed traveling waves can be constructed so that they carry a large amount of energy in a small volume. If the energy is sufficiently large and the waves are sufficiently well aligned, one might expect that their collision will form a black hole. It would be interesting to see whether this and other properties of colliding traveling waves could be deduced from an examination of the solution describing the waves before collision.

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