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Strings in strong gravitational fields

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String propagation in exact plane-wave solutions (with nonzero axion and dilaton fields) is analyzed. In these backgrounds, strings can undergo transitions from one state to another. Selection rules are derived which describe allowed and forbidden transitions of the string. It is shown that singular plane waves result in infinitely excited strings. An example is given of a solution whose singular properties are the opposite of an orbifold: it is geodesically complete, but still singular from the standpoint of string theory. Some implications of these results are discussed.

I. INTRODUCTION

One of the key issues that a quantum theory of gravity must address is the nature of the singularities predicted by general relativity. In string theory, the classical equation of motion for the metric can be obtained by demanding conformal invariance of a two-dimensional σ model.[1] In terms of σ-model perturbation theory, this equation takes the form of Einstein's equation with the addition of an infinite number of higher-order terms involving higher powers and derivatives of the curvature. The first step in studying singularities in string theory is to look for solutions to this equation with strong curvature. Recently it has been shown that all solutions to Einstein's vacuum field equation with a covariantly constant null vector are also solutions to string theory.[2–4] This consists of a large class of time-dependent spacetimes which contain arbitrary functions. They are known as plane-fronted waves. Furthermore, one can extend these solutions to include the other massless fields of string theory:[2,4] the dilaton, antisymmetric tensor field (hereafter referred to as the axion field), and even the Yang-Mills field. This yields a large class of solutions to investigate.

Many of these spacetimes are singular in the sense of general relativity; i.e., they are geodesically incomplete. Classical test particles cannot evolve for an infinite time. It is not yet clear what the most appropriate generalization of this definition is for string theory. If one simply considers the motion of classical test strings and asks whether they always have a well-defined evolution for all time, then the answer will be no for any geodesically incomplete spacetime. This is because if one chooses X^μ independent of σ, the equation for a classical string is precisely the (null) geodesic equation. However a definition of singularity in terms of a first-quantized test string would be more satisfactory. The example of orbifolds shows that this can lead to differences with general relativity. Orbifolds are spaces which are locally flat but have conical singularities. Although they are geodesically incomplete, it has been shown[5] that a first-quantized string is well defined on orbifolds. In general, if the expectation value of some physical observable associated with a test string diverges, then the solution will be called singular. Although this is the definition we will use here, it should be noted that it has a limited range of applicability. For a general time-dependent background, the first-quantized description will not be useful due to the string analog of particle creation. This "string creation" must be discussed in the context of a second-quantized theory. Fortunately, for the plane-fronted-wave solutions, it has been shown that there is no particle creation[6] since the covariantly constant null vector leads to a definition of frequency which is conserved. A similar argument shows that there will be no string creation[7] so the first-quantized description will be satisfactory.

Although there is no string creation, there is still "mode creation" arising from the fact that positive frequency is not conserved on the string world sheet. Thus there can be transitions between different modes of the string.[8] We will derive selection rules governing which transitions are allowed and which are forbidden. Of most interest will be how the total excitation of the string behaves at late times when the background fields become large.

One reason that there has not been much attention given to the general definition of a singularity in string theory was the widespread belief that it would not be necessary. Since orbifolds were the first known geodesically incomplete solution to string theory, it was thought (or perhaps hoped) that similar results would hold for other solutions. If all solutions to string theory allowed well-behaved propagation of test strings, then the singularity problem of general relativity would be removed in string theory at the classical level. However we will show that this is not the case. A string which tries to propagate through a class of singular plane-fronted waves becomes infinitely excited. (A summary of some of these results was given in Ref. 4. In a recent paper,[9] de Vega and Sanchez have done a similar calculation for a different background, the Aichelburg-Sexl spacetime,[10] and found that the string excitation remains finite.) One can understand physically why the string becomes infinitely excited in the singular plane waves as follows. A string propagating in a curved spacetime experiences gravitational tidal forces and becomes excited. If the curvature diverges, then in general these tidal forces diverge and the string becomes infinitely excited. From this viewpoint it is clear why orbifold singularities do not cause any problems for the propagation of strings. Since the metric is locally flat, there are simply no tidal forces.

Since the criterion for a singularity in string theory is quite different from general relativity, one may ask if there are spacetimes which are singular in the sense of string theory, but nonsingular in the sense of general relativity, i.e., the opposite of an orbifold. We will show that the answer is yes. One example involves a background axion field which is bounded but discontinuous. Since the curvature is bounded and test particles do not couple to the antisymmetric axion field, this spacetime is geodesically complete. However strings do couple to this field, and we will show that the discontinuity causes enough excitation of the higher modes that the mass squared of the final string state diverges. Although this example may seem unphysical, it illustrates the type of behavior which is possible without some restrictions on classical solutions.

One might expect that the study of a string in a highly curved time-dependent background would be difficult. Even classically, the equations of motion for the string are nonlinear. However, we will see that for plane-fronted waves the analysis simplifies in light-cone gauge. This gauge cannot be imposed for a string in a general curved spacetime. In fact we will show that the standard light-cone gauge is consistent in curved spacetime *only* for plane-fronted waves. For a certain class of plane-fronted waves (called exact plane waves) the unconstrained degrees of freedom satisfy linear equations. Thus, like orbifolds, these backgrounds are simple soluble models of string propagation in nontrivial backgrounds.

This paper is organized as follows. In the next section we review the plane-fronted-wave solutions and discuss classical string propagation in these backgrounds. It is shown that a straightforward imposition of light-cone gauge is possible in curved spacetime only for this class of solutions. In Sec. III we restrict attention to exact plane waves (with nonzero dilaton and axion) and discuss the propagation of a first-quantized string. We first compute the expectation value of the mass squared at late times for a string initially in its ground state. We show that as the background fields diverge, so does the mass of the string. We also discuss the example of a spacetime with bounded curvature such that $\langle M^2\rangle$ diverges at late time. Finally, for nonsingular backgrounds, selection rules are derived governing transitions between initial and final string states. Section IV contains some discussion of these results. This includes implications for the idea that there is a minimum observable length in string theory, and the possible effect on singularities of quantum string corrections to the equations of motion.

II. PLANE-FRONTED WAVES AND LIGHT-CONE GAUGE

Consider a D-dimensional spacetime with metric of the form

$$ds^2=-dU\,dV+dX^i dX_i+F(U,X^i)dU^2 \tag{2.1}$$

for some function F. The X^i are $D-2$ transverse coordinates. This metric reduces to the standard flat-space form when $F=0$ (with $U=T-Z$, $V=T+Z$). Since the metric is independent of V, there is a null Killing vector field $l_\mu=\partial_\mu U$. One can easily verify that this vector is in fact covariantly constant:

$$\nabla_\mu l^\nu=0\ . \tag{2.2}$$

The Riemann curvature tensor is

$$R_{\mu\nu\rho\sigma}=2l_{[\mu}\partial_{\nu]}\partial_{[\rho}F l_{\sigma]} \tag{2.3}$$

and the Ricci tensor is

$$R_{\mu\nu}=-\tfrac{1}{2}(\partial_T^2 F)l_\mu l_\nu\ , \tag{2.4}$$

where the Laplacian is on the transverse coordinates only, since F is independent of V. The metric (2.1) is thus a solution to the vacuum Einstein field equations if and only if $\partial_T^2 F=0$. The dependence of F on U is completely arbitrary. In four dimensions, (2.1) is the most general solution to Einstein's equation with a covariantly constant null vector.[11] These solutions are known as plane-fronted waves,[12] and in the particular case when F is quadratic in X^i,

$$F(U,X^i)=W_{ij}(U)X^iX^j\ , \tag{2.5}$$

they are called exact plane waves. [A linear contribution $F(U,X^i)=V_i(U)X^i+M(U)$ does not affect the curvature (2.3) and can be removed by a coordinate transformation.]

It has been shown[2,3,4] that plane-fronted waves are also solutions to the classical string equations of motion to all orders in σ-model perturbation theory. This is essentially due to the fact that the curvature is null, so all powers of it vanish.[13] For the case of plane waves (with W_{ij} bounded) it has been shown[3] that they are exact conformal field theories and hence solutions even nonperturbatively in the σ model.

One can also find solutions with the axion and dilaton fields nonzero as well. Consider an axion field strength of the form

$$H_{\mu\nu\rho}=A_{ij}(U)l_{[\mu}\nabla_\nu X^i\nabla_{\rho]}X^j\ , \tag{2.6}$$

where $A_{ij}=A_{[ij]}$ and a dilaton Φ that depends only on U. Then the metric, axion, and dilaton satisfy the string equations to all orders if and only if[2,4]

$$\partial_T^2 F+\tfrac{1}{18}A_{ij}A^{ij}+2\Phi''=0\ . \tag{2.7}$$

One can also extend these solutions to include nontrivial Yang-Mills fields[2] but this will not be considered here.

Before discussing the motion of strings in these backgrounds, we briefly consider the motion of point particles. The equations for a geodesic in a plane-fronted-wave solution are

$$\ddot U=0\ , \tag{2.8a}$$

$$\ddot X_i-\tfrac{1}{2}F_{,i}\dot U^2=0\ , \tag{2.8b}$$

$$\ddot V-F_{,U}\dot U^2-2F_{,i}\dot X^i\dot U=0\ , \tag{2.8c}$$

where an overdot denotes derivative with respect to the affine parameter. These equations can be interpreted as follows. The first shows that $P \equiv \dot{U}$ is conserved. This is simply the inner product between the tangent vector to the geodesic and the null Killing vector l^μ. Equation (2.8b) shows that the transverse components of the geodesic feel an effective force which is $\frac{1}{2}F_{,i}P^2$. The final equation (2.8c) can be viewed as enforcing the constraint $-\dot{U}\dot{V}+\dot{X}^i\dot{X}_i+F\dot{U}^2=\text{const}$. If $P\neq 0$, one can determine V in terms of U and X^i by solving this constraint.

The local tidal forces are, of course, described by the equation for geodesic deviation. The transverse components of this equation are

$$\ddot{\eta}^i - \tfrac{1}{2}P^2\partial^i\partial_j F\eta^j = 0 \ , \tag{2.9}$$

where η^i is the transverse separation between nearby geodesics (with the same value of P). We will see in the next section that there is a qualitative difference in the behavior of a string depending on whether these tidal forces are attractive or repulsive. For a purely gravitational plane wave, the tidal force is attractive in some directions and repulsive in others. For a source with positive energy density such as the axion, $\partial_T^2 F<0$, and the tidal force is attractive in all directions. Note that the tidal forces depend not on F alone, but the combination FP^2. Since one has the freedom to go to a boosted frame $\tilde{U}=\lambda U$, $\tilde{V}=\lambda^{-1}V$, the magnitude of F is coordinate dependent. However under this transformation $\tilde{P}=\lambda P$ and so the combination FP^2 is invariant.

In four dimensions, the plane-wave solutions involve four arbitrary functions of U: the two traceless components of W_{ij} corresponding to the amplitudes for the two polarizations for the gravitational wave, the one component of A_{ij} corresponding to the amplitude for the axion, and the dilaton. Generically, if any of these functions diverge at, say, $U=0$, then by the field equation (2.7), some component of W_{ij} will also diverge. Since general relativity (as it is usually formulated) requires a smooth metric, the spacetime only exists for $U<0$. Every timelike geodesic has $U=P\tau$ with $P\neq 0$, and thus reaches $U=0$ in a finite proper time. The solution is therefore geodesically incomplete and hence singular from the standpoint of general relativity. Since *all* timelike geodesics are incomplete, this spacetime is analogous to one with a cosmological singularity. One can also show that the components of the curvature in a parallel propagated frame diverge as $U\to 0$ along an incomplete geodesic. However, if $A_{ij}A^{ij}+36\Phi''=0$, then W_{ij} can remain bounded (or even zero) while the axion and dilaton both diverge. (This is possible since the dilaton does not satisfy a local energy condition unless the metric is suitably rescaled.) Should this solution still be considered singular in general relativity? The answer is yes. This is because one should consider the motion of test particles. When the dilaton is not constant, test particles no longer follow geodesics. They feel a force due to the dilaton which depends on their composition. If the dilaton diverges, this force will diverge for some test particles and the solution will be singular. We conclude that whenever one of the free functions in the plane-wave solutions diverge, the solution is singular from the standpoint of general relativity.

Of course if the spacetime only exists for $U<0$ it is not possible to ask whether a test string has a well-behaved evolution "through the singularity" to $U>0$. The approach we will adopt is to consider the propagation of strings in nonsingular solutions and then take the limit as the curvature and other background fields diverge. We will investigate whether the propagation of the string remains well behaved in this limit. This approach is possible since the solutions depend on arbitrary functions.

We now consider the motion of strings. The coupling of a string to a general metric, axion, and dilaton background is given by the action

$$S=-\frac{1}{4\pi\alpha'}\int (h^{ab}g_{\mu\nu}\partial_a X^\mu\partial_b X^\nu + B_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \epsilon^{ab} - \tfrac{1}{2}\alpha' R^{(2)}\Phi)\sqrt{h}\, d^2\sigma \ , \tag{2.10}$$

where $X^\mu = X^\mu(\sigma,\tau)$ is the embedding of the world sheet in spacetime, h_{ab} is the two-dimensional world-sheet metric, $R^{(2)}$ is the two-dimensional scalar curvature, and α' is the inverse string tension. The axion field strength is $H_{\mu\nu\rho}=3\nabla_{[\mu}B_{\nu\rho]}$. For the remainder of this section we consider a purely classical string. Since the dilaton term is multiplied by α', it is a quantum correction and does not directly affect the motion of a classical string. [It will have an indirect effect through the field equation (2.7) which relates $g_{\mu\nu}$ and $B_{\mu\nu}$ to Φ.] Thus the equations of motion are obtained by extremizing the first two terms in (2.10) with respect to X^μ and h_{ab}. In the conformal gauge $h_{ab}=e^\phi \eta_{ab}$, these are

$$\partial_a\partial^a X^\mu + \Gamma^\mu_{\lambda\rho}\partial_a X^\lambda \partial^a X^\rho - \tfrac{1}{2}H^\mu{}_{\lambda\rho}\partial_a X^\lambda\partial_b X^\rho\epsilon^{ab}=0 \tag{2.11}$$

and

$$T_{ab}\equiv \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} - \tfrac{1}{2}\eta_{ab}\partial_c X^\mu \partial^c X^\nu g_{\mu\nu}=0 \ . \tag{2.12}$$

Equation (2.12) is the usual reparametrization constraint. If it is satisfied at one time and X^μ satisfies (2.11), then it is satisfied at all times.

In the conformal gauge, there remains the freedom to change σ and τ by solutions to the two-dimensional wave equation. For the trivial flat background (or more generally a product space which includes a two-dimensional Minkowski spacetime) this residual gauge freedom can be fixed by imposing light-cone gauge.[1] We now show that *plane-fronted-waves are the only curved spacetimes for which light-cone gauge can be implemented.*[14] To impose light-cone gauge, one needs a null coordinate U on spacetime which satisfies the two-dimensional wave equation for all classical string solutions. This imposes restrictions on the background fields. To illustrate this, consider first the analogous results for geodesics. Suppose there is a function T such that $d^2T/d\lambda^2=0$ along all geodesics (where λ is the affine parameter). Then

$$0=\frac{d^2T}{d\lambda^2}=\xi^\mu\nabla_\mu(\xi^\nu\nabla_\nu T)=\xi^\mu\xi^\nu\nabla_\mu\nabla_\nu T \ , \tag{2.13}$$

where ξ^μ is the tangent vector to the geodesic and we have used the geodesic equation in the second step. Since this holds for all ξ^μ we conclude $\nabla_\mu\nabla_\nu T=0$. In other words, the spacetime admits a covariantly constant vector $\nabla_\mu T$.

We now derive a similar result for strings. If there is a function U such that $\partial^2 U=0$ for all classical string solutions, then

$$0=\partial^2 U=U_{,\mu}\partial^2 X^\mu+U_{,\mu\nu}\partial_a X^\mu \partial^a X^\nu \ . \tag{2.14}$$

Now using the equation of motion (2.11) we get

$$(\nabla_\mu \nabla_\nu U)\partial_a X^\mu \partial^a X^\nu+\tfrac{1}{2}\nabla_\lambda U H^\lambda{}_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \epsilon^{ab}=0 \ . \tag{2.15}$$

Since the string equation of motion is second order, at any point we can choose the τ and σ derivatives of X^μ, $\dot X^\mu$ and X'^μ, arbitrarily up to the constraints $\dot X^2+X'^2=0$ and $\dot X^\mu X'_\mu=0$. Geometrically these constraints imply that $\partial_a X^\mu \partial^a X^\nu$ is the metric on a timelike two plane and $\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu$ is the volume two-form. Since the timelike plane can be varied *continuously,* the only way (2.15) can be satisfied is if the coefficients vanish: $\nabla_\mu \nabla_\nu U=0$ and $\nabla_\lambda U H^\lambda{}_{\mu\nu}=0$. Therefore light-cone gauge can be imposed in curved spacetime only if there is a covariantly constant null vector $\nabla_\mu U$, i.e., only for the plane-fronted waves.

The action for a string in a plane-fronted-wave background is

$$S=-\int \frac{d\sigma\, d\tau}{2\pi}(-\partial_a U\partial^a V+\partial_a X^i \partial^a X_i+F\partial_a U\partial^a U + \tfrac{1}{3}A_{ij}X^j\partial_a U\partial_b X^i \epsilon^{ab}) \ , \tag{2.16}$$

where we have set $\alpha'=\frac{1}{2}$ and imposed conformal gauge. Extremizing with respect to V yields $\partial^2 U=0$, and we fix the remaining gauge invariance by setting $U=P\tau$. The equations of motion for the transverse X^i are

$$\partial_a \partial^a X_i+\tfrac{1}{2}\partial_i F P^2-\tfrac{1}{3}P A_{ij}X'^j=0 \ . \tag{2.17}$$

The two components of the constraint $T_{ab}=0$ are

$$P\dot V=(\dot X^i)^2+(X'^i)^2+FP^2 \ , \tag{2.18a}$$

$$PV'=2\dot X^i X'^i \ . \tag{2.18b}$$

It is easily verified that the integrability condition $(\dot V)'=(V')^{\cdot}$ is satisfied whenever X^i satisfies its equation of motion (2.17). So provided $P\neq 0$, V is uniquely determined up to an arbitrary zero mode V_0. As for strings in flat spacetime, the X^i are not completely independent. Integrating (2.18b) over σ, one gets the usual constraint $\int \dot X^i X'^i d\sigma=0$ which enforces invariance under shifting σ by a constant.

The light-cone gauge Hamiltonian can be obtained as follows. Setting $U=P\tau$ in the action (2.16) yields

$$S=\int \frac{d\sigma\, d\tau}{2\pi}\left[-P\dot V+\dot X^i \dot X_i-X'^i X'_i +FP^2+\frac{P}{3}A_{ij}X'^i X^j\right] . \tag{2.19}$$

Constructing the canonical Hamiltonian in the usual way we obtain

$$H=\int \frac{d\sigma}{2\pi}\left[\pi^2 P^i P_i+X'^i X'_i-FP^2-\frac{P}{3}A_{ij}X'^i X^j\right] , \tag{2.20}$$

where P_i is the momentum conjugate to X^i. The first two terms are the same as a string moving in flat spacetime. The next two describe the interaction with the spacetime curvature and axion field, respectively. Note that the background fields give rise to *time-dependent* potentials for the X^i.

III. SINGULARITIES

In this section we restrict our attention to exact plane-wave solutions and consider the propagation of a first-quantized string. As discussed in the previous section we will adopt a light-cone gauge quantization, which has the advantage of being manifestly unitary. It will be convenient to consider spacetimes that are sandwich waves, i.e., no curvature, axion, or dilaton for $U<0$ and $U>T$, so that asymptotically we have the usual free string states. Quantum mechanically, a string which begins in one state at $U<0$ has some amplitude to be in various states after propagating through the curvature. Since we are considering exact plane waves, the transverse modes satisfy a linear equation with time-dependent coefficients. One can thus calculate transition amplitudes by relating the in-oscillator modes of the string to the out-oscillator modes by a Bogoliubov transformation as is done in curved-space quantum-field-theory calculations.[15] One should note, however, that here we are dealing with a two-dimensional quantum-field-theory problem on a flat world sheet with an effective time-dependent potential which arises from the ambient background fields, and not a quantum field theory on curved spacetime.

This section is divided into four parts. In the first, we consider the total excitation at late time of a string moving in a background consisting of a gravitational wave with one polarization. Next we consider the general background with axion and dilaton included. Third, we discuss the example of a solution which is singular in string theory but not general relativity. Finally, we discuss selection rules governing string transitions between low mass states.

A. Constant polarization gravitational wave

Consider first the solution describing a gravitational wave with constant polarization and no axion or dilaton. This is given by

$$F(U,X,Y)=W(U)(X^2-Y^2) \ . \tag{3.1}$$

We will assume that $W=0$ for $U<0$ and $U>T$ for some time T. Let us decompose X and Y into modes:

$$X(\sigma,\tau)=\sum_n X_n(\tau)e^{in\sigma} \ , \tag{3.2a}$$

$$Y(\sigma,\tau)=\sum_n Y_n(\tau)e^{in\sigma} \ , \tag{3.2b}$$

where $X_{-n}=X_n^*$, $Y_{-n}=Y_n^*$. Since the field equation (2.17) is now linear, we obtain

$$\ddot X_n+n^2 X_n-WP^2 X_n=0 \ , \tag{3.3a}$$

$$\ddot Y_n+n^2 Y_n+WP^2 Y_n=0 \ . \tag{3.3b}$$

Note that in this case not only are modes of different n

decoupled, but X_n and Y_n are also decoupled. Coupling between X_n and Y_n will arise when we include both polarizations for the gravitational wave or the axion background. Equations (3.3) are just a collection of harmonic oscillators with time-dependent frequencies.

As mentioned earlier, a linear contribution to $F(U,X,Y)$ is pure gauge and should not affect the physical propagation of a string. Indeed, the addition of a linear term to F will only add a source term to the equation of motion (2.17) which will not affect the mode equations (3.3) for $n\neq 0$. For $n=0$, these equations will change reflecting the description of the center-of-mass motion in the new coordinates. Note that when $n=0$, Eqs. (3.3) are identical to the geodesic equations (2.8b).[16]

Writing each mode in terms of right and left oscillators, we have (for $n>0$)

$$X_n=\frac{i}{2\sqrt{n}}(a_n^x u_n-\tilde{a}_n^{x\dagger}\tilde{u}_n)\ , \tag{3.4}$$

where u_n, $\tilde{u}_n$ are solutions to (3.3a) which have the form

$$u_n=e^{-in\tau},\quad \tilde{u}_n=e^{in\tau} \tag{3.5}$$

for $U<0$, and similarly for Y_n. Imposing the standard canonical commutation relations leads to the usual interpretation of a and $a^\dagger$ as annihilation and creation operators. These are the in oscillators. The string can be similarly decomposed into out modes v and out oscillators b, where the v satisfy (3.3a) but have simple exponential form for $U>T$. Since this equation is linear, the u's and v's are linearly related, and one obtains a linear transformation between the in and out oscillators:

$$b_n^x=A_n a_n^x-B_n^*\tilde{a}_n^{x\dagger}\ ,\quad \tilde{b}_n^x=\tilde{A}_n\tilde{a}_n^x-\tilde{B}_n^* a_n^{x\dagger}\ . \tag{3.6}$$

This is the Bogoliubov transformation. Since (3.3) has real coefficients, we have $A_n=\tilde{A}_n$ and $B_n=\tilde{B}_n$. Since interchanging X and Y is equivalent to $W\mapsto -W$, the coefficients for b_n^y can be obtained from A_n and B_n by replacing W with $-W$.

The mass-squared operator in the region $U<0$ is

$$M_{\text{in}}^2=4\sum_{n=1}^{\infty}n(a_n^{\dagger i}a_n^i+\tilde{a}_n^{\dagger i}\tilde{a}_n^i)-8\ , \tag{3.7}$$

where the -8 is the standard normal-ordering constant. This is usually fixed in light-cone gauge by demanding Lorentz invariance. Since our solutions are flat for $U<0$, the usual arguments can be applied in this region to show that the normal-ordering constant is the same. For $U>T$, the mass squared is given by a similar expression with $a_n^i,\tilde{a}_n^i$ replaced by $b_n^i,\tilde{b}_n^i$. In general, the full spacetime has only a five-parameter group of symmetries. It turns out that the generators of these symmetries do not involve V. Thus the algebra of the quantum operators is identical to that of the classical generators, and there is no anomaly.

From the Bogoliubov transformations, one can find the excitation level in the out region of the nth right and left modes of a string that was initially in the ground state of this mode:

$$\langle 0_{\text{in}}|N_n^{x\,\text{out}}|0_{\text{in}}\rangle=\langle 0_{\text{in}}|b_n^{x\dagger}b_n^x|0_{\text{in}}\rangle=|B_n|^2 \tag{3.8}$$

and similarly

$$\langle 0_{\text{in}}|\tilde{N}_n^x|0_{\text{in}}\rangle=|\tilde{B}_n|^2\ . \tag{3.9}$$

The particular values of A_n and B_n for a given W can be found by solving (3.3a). Note that if we replace X_n by ψ and τ by x, this equation takes the form of a one-dimensional Schrödinger equation for a particle of energy n^2 moving in a potential WP^2. To find the coefficients B_n, we start with a positive-frequency solution $e^{-in\tau}$ at early times and ask for the coefficient of the negative-frequency solution at late times. In the quantum-mechanical analogy, this is equivalent to fixing the amplitude of the transmitted wave to be one and asking for the coefficient for the reflected wave.

We now consider three limiting cases. First suppose W is bounded and consider the limit of large mode numbers $n^2\gg\max|W|P^2$. In this limit the coefficients can be found by treating the last term in (3.3a) as a perturbation. (This is essentially the Born approximation.) In the absence of this term, the solution is simply $X_n=e^{-in\tau}$. Thus expanding to first order about this solution, $X_n=e^{-in\tau}+\delta X_n$, we obtain

$$(\delta\ddot{X}_n)+n^2\delta X_n=WP^2e^{-in\tau}\ . \tag{3.10}$$

For $\tau>T/P$, corresponding to flat spacetime, the coefficient of $e^{in\tau}$ in the solution turns out to be $\hat{W}(2n/P)P/2in$, where $\hat{W}$ is the Fourier transform of W with respect to U. Thus

$$\langle N_n^x\rangle=|B_n|^2=\frac{|\hat{W}(2n/P)|^2P^2}{4n^2}\ . \tag{3.11}$$

The second limit we wish to consider is the case of large repulsive tidal forces. For Eq. (3.3a) this corresponds to $n^2\ll\max(WP^2)$. In the quantum-mechanical analog, this is the case where the energy is much less than the height of the potential. For a wide class of backgrounds W, one can use the standard WKB approximation to show that[17]

$$B_n=\exp\left[\int_0^T\sqrt{W}dU\right]\ . \tag{3.12}$$

Thus $\langle N_n^x\rangle=|B_n|^2$ grows exponentially as W increases. This is exactly what one should expect physically. In this limit, the repulsive tidal forces are much stronger than the string tension. Even a classical string will have the mode X_n increase exponentially, driving up its energy. It is perhaps worth pointing out that the WKB approximation does not require W to be approximately constant, but only slowly varying over one "exponential time" of the solution. This is satisfied, e.g., if $W=U^{-m}$ for $m>2$ and small U.

The third limit is the case of large attractive tidal forces. If $n^2\ll\max(WP^2)$ as above, this is the case for Eq. (3.3b). In the quantum-mechanical analog, this corresponds to a particle in a very deep well. Since the only difference between (3.3a) and (3.3b) is the sign in front of W one can estimate $\langle N_n^y\rangle$ by simply replacing W with $-W$ in (3.12) to obtain $\langle N_n^y\rangle\approx 1$. However, if W includes a transition region where the WKB approximation breaks down, one can use connection formulas to calculate the coefficient in front of this order-one term. For very deep wells, one finds that this coefficient typically grows with the depth of the well. In these cases the excitation of these modes will again diverge as the strength of the gravitational wave increases, although not as fast as the modes with repulsive forces. This is again what one would expect from the behavior of a classical string.

Let us now consider $\langle M^2\rangle = 4\sum n\langle N_n^i\rangle - 8$. The convergence of this sum is determined by its behavior for large n. If W is a C^∞ function of compact support, then its Fourier transform $\hat{W}$ will vanish for a large argument faster than any polynomial. Thus, by (3.11), $\langle M^2\rangle$ will always be finite at late times. However in the limit that $\int_0^T\sqrt{W}dU$ diverges, corresponding to a singular spacetime in general relativity, $\langle N_n^x\rangle$ also diverges for each n. Thus the limiting spacetime is also singular in the sense of string theory.

B. General plane-wave solution

We now consider the case of a general plane wave with all background fields nonzero. Let $W_i(U)$ be the amplitudes for the two polarizations of the gravitational wave, $A(U)=A_{12}(U)$ be the amplitude for the axion, and $\Phi(U)$ be the dilaton. We again assume that these fields vanish for $U<0$ and $U>T$. Since the string equation of motion (2.17) is still linear, the different modes again decouple although the two transverse components are now coupled. Expanding $X^i(\sigma,\tau)$ in modes as in (3.2) we obtain

$$\ddot{X}_n^i + n^2X_n^i + Q_j^iX_n^j = 0\ , \tag{3.13}$$

where the matrix Q_j^i takes the form

$$\begin{bmatrix} \phi_1 & \rho + in\lambda \\ \rho - in\lambda & \phi_2 \end{bmatrix} \tag{3.14}$$

with

$$\phi_1 = P^2\left[-W_1 + \frac{A^2}{36} + \frac{\Phi''}{2}\right]\ , \tag{3.15a}$$

$$\phi_2 = P^2\left[W_1 + \frac{A^2}{36} + \frac{\Phi''}{2}\right]\ , \tag{3.15b}$$

$$\rho = W_2P^2\ , \tag{3.15c}$$

$$\lambda = \tfrac{1}{3}PA\ , \tag{3.15d}$$

and we have used the field equation (2.7). Note that the dilaton only enters the equation for the string evolution through its effect on the metric via this field equation. Since (2.7) involves two derivatives of Φ, it follows that a background dilaton of the form $\Phi = KU$ has no effect on the propagation of a string. This is consistent with earlier discussions of a linear dilaton background in flat spacetimes. It was shown[18] that if $\Phi = V_\mu X^\mu$ for some constant vector V_μ, then the only effect is that the critical dimension and mass levels are both shifted by an amount proportional to $V_\mu V^\mu$. When V^μ is null, there is no effect.

Now an incoming wave $\binom{X}{Y} = \binom{1}{0}e^{-in\tau}$ will at late times be some linear combination of $\binom{1}{0}e^{-in\tau}$, $\binom{0}{1}e^{-in\tau}$, $\binom{1}{0}e^{in\tau}$, and $\binom{0}{1}e^{in\tau}$. The string-mode decomposition becomes (for $n>0$)

$$\mathbf{X}_n = \frac{i}{2\sqrt{n}}(a_n^x\mathbf{u}_n^x + a_n^y\mathbf{u}_n^y - \tilde{a}_n^{x\dagger}\tilde{\mathbf{u}}_n^x - \tilde{a}_n^{y\dagger}\tilde{\mathbf{u}}_n^y)\ , \tag{3.16}$$

where $\mathbf{X} = \binom{X}{Y}$, a_n^i, $\tilde{a}_n^i$ are the right and left in-mode annihilation operators. The $\mathbf{u}_n^i$ are the in-mode solutions to (3.13) which have the following form for $\tau<0$:

$$\mathbf{u}_n^x = \begin{bmatrix}1\\0\end{bmatrix}e^{-in\tau},\quad \mathbf{u}_n^y = \begin{bmatrix}0\\1\end{bmatrix}e^{-in\tau}\ ,$$
$$\tilde{\mathbf{u}}_n^x = \begin{bmatrix}1\\0\end{bmatrix}e^{in\tau},\quad \tilde{\mathbf{u}}_n^y = \begin{bmatrix}0\\1\end{bmatrix}e^{in\tau}\ . \tag{3.17}$$

Let $\mathbf{v}_n^i$ be the out modes, which are solutions to (3.13), that have the simple exponential behavior for $\tau > T/P$. Writing $\mathbf{u}_n^x$ in terms of out modes,

$$\mathbf{u}_n^x = A_n\mathbf{v}_n^x + B_n\mathbf{v}_n^y + C_n\tilde{\mathbf{v}}_n^x + D_n\tilde{\mathbf{v}}_n^y\ , \tag{3.18}$$

where these coefficients depend on ϕ_1, ϕ_2, ρ, and λ. The corresponding expression for $\mathbf{u}_n^y$ can be obtained from Eq. (3.18) by interchanging both the X and Y components, and ϕ_1 and ϕ_2, as well as taking $\lambda\mapsto-\lambda$. Similarly, $\tilde{\mathbf{u}}_n^x$ can be obtained from (3.18) by taking the complex conjugate and replacing λ with $-\lambda$. The Bogoliubov coefficients relating the a and b oscillators can then be found:

$$b_n^x = A_n(\phi_1,\phi_2,\rho,\lambda)a_n^x + B_n(\phi_2,\phi_1,\rho,-\lambda)a_n^y - C_n^*(\phi_1,\phi_2,\rho,-\lambda)\tilde{a}_n^{x\dagger} - D_n^*(\phi_2,\phi_1,\rho,\lambda)\tilde{a}_n^{y\dagger}\ . \tag{3.19}$$

The other coefficients can be found using the same arguments as above. As discussed earlier, from these Bogoliubov coefficients one can calculate the expectation value of the out-number operator in the in vacuum to be

$$\langle N_n^x\rangle = |C_n(\phi_1,\phi_2,\rho,-\lambda)|^2 + |D_n(\phi_2,\phi_1,\rho,\lambda)|^2 \tag{3.20}$$

and similarly for the other modes.

We now consider the same limiting cases we considered earlier. For n^2 much greater than all background fields, we have

$$\mathbf{u}_n^x = \begin{bmatrix}1\\0\end{bmatrix}e^{-in\tau} + \delta\mathbf{X}_n \tag{3.21}$$

and expanding to first order, we obtain

$$\delta\ddot{\mathbf{X}}_n + n^2\delta\mathbf{X}_n = -Q\begin{bmatrix}1\\0\end{bmatrix}e^{-in\tau}\ . \tag{3.22}$$

For $\tau > T/P$, the coefficient of $e^{in\tau}$ in the solution is

$$\frac{i\hat{Q}(2n/P)}{2nP}\begin{bmatrix}1\\0\end{bmatrix}\ , \tag{3.23}$$

where $\hat{Q}$ is the Fourier transform of the matrix Q with respect to U. Thus,

$$C = \frac{i\hat{\phi}_1(2n/P)}{2nP},\quad D = \frac{i\hat{\rho}(2n/P)}{2nP} + \frac{\hat{\lambda}(2n/P)}{2P} \tag{3.24}$$

which implies

$$\langle N_n^x \rangle = \frac{|\hat{\phi}_1|^2+|\hat{\rho}|^2}{4n^2P^2} + \frac{|\hat{\lambda}|^2}{4P^2} + \mathrm{Im}\frac{\hat{\lambda}\hat{\rho}^*}{2nP^2} , \qquad (3.25)$$

where all the Fourier transforms are evaluated at $2n/P$.

We now look at the other limit of n^2 much less than at least one of the background fields. This corresponds to large tidal forces (of either sign). We assume that the WKB approximation is valid (at least away from $\tau=0$ and $\tau=T/P$). Taking as our ansatz

$$\mathbf{X} = \begin{bmatrix} e^{iS_1} \\ e^{iS_2} \end{bmatrix} \qquad (3.26)$$

one obtains, as the fundamental set of four solutions,

$$\begin{bmatrix} \Omega_1^+ \\ \Omega_2^+ \end{bmatrix} e^{\pm iS_+} , \quad \begin{bmatrix} \Omega_1^- \\ \Omega_2^- \end{bmatrix} e^{\pm iS_-} , \qquad (3.27)$$

where

$$\dot{S}_\pm^2 = \frac{\phi_1+\phi_2}{2} \pm \frac{[(\phi_1-\phi_2)^2+4\rho^2+4\lambda^2n^2]^{1/2}}{2} \qquad (3.28)$$

are the eigenvalues of the matrix Q and where Ω_1, Ω_2 can be obtained by expanding (3.13) to next order in the WKB approximation. If WKB is not valid near the transition regions, one can use one of the standard methods to find the connection formulas which link the solutions in $(0,T/P)$ to those outside the region. Independent of the particular form for the connection coefficients, the solution for $\tau > T/P$ will be a linear combination of the $\mathbf{v}_n$'s with coefficients involving

$$\exp\left[\pm i \int_0^{T/P} \sqrt{\dot{S}_+^2}\, d\tau\right] ,$$
$$\exp\left[\pm i \int_0^{T/P} \sqrt{\dot{S}_-^2}\, d\tau\right] . \qquad (3.29)$$

So, as long as $\dot{S}_-^2 < 0$ the excitation of some modes will be exponentially large. Which particular modes of the asymptotic string get excited will generally depend on the behavior near $\tau=0, T/P$ as well. Let us now reexpress $\dot{S}_-^2$ in terms of our original fields and see under what conditions it will be negative. Using (3.15) and (3.28), one obtains

$$\dot{S}_-^2 = P^2\left[\frac{A^2}{36} + \frac{\Phi''}{2}\right] - P^2\left[W_1^2 + W_2^2 + \frac{A^2n^2}{9P^2}\right]^{1/2} . \qquad (3.30)$$

We see that large gravitational-wave amplitudes W_i will cause $\dot{S}_-^2 < 0$ and hence exponential excitation. A large axionic field, however, will keep both $\dot{S}_\pm^2 > 0$. This is a result of the fact that for large axion fields, the tidal forces are all attractive. The excitation in this case will depend on the coefficient in front of the exponential. However as discussed in Sec. III A, in many cases this will grow with the strength of the axion field.

Thus if $W_i \to \infty$, $\langle N_n \rangle$ diverges exponentially for each n and the limiting spacetime is again singular in the sense of string theory. If $A \to \infty$ while W_i remains finite, then it depends on the detailed form of $A(U)$. In many cases $\langle N_n \rangle$ will again diverge. However, it is possible that there exist certain choices of $A(U)$ such that $A \to \infty$ does not result in infinite mass. If all physical observables have finite expectation values, then these spacetimes would be analogous to orbifolds: They would be singular in general relativity but not string theory.

C. Solution which is singular only in string theory

We can now describe our example of a solution which is nonsingular in the sense of general relativity, but nevertheless singular in the sense of string theory. For simplicity we set $W_i=0$ and $\Phi=0$. Let the axion amplitude $A(U)$ be bounded everywhere but discontinuous at some value of U. From Eq. (2.6) its clear that in this case both $H_{\mu\nu\rho}$ and $B_{\mu\nu}$ are discontinuous and not δ functions. The Fourier transform of A will fall off like $1/n$ for large n. From (3.15) and (3.25) we see that $\langle N_n \rangle \sim 1/n^2$ for large n and hence $\langle M^2 \rangle$ diverges. The discontinuity in the axion field results in enough excitation of the high modes of the string to cause the mass to diverge. So this solution is singular in string theory. However, geodesics do not couple to the axion field. From Eqs. (2.5) and (2.7), W_i^i will be discontinuous but the geodesic equations (2.8) still have unique (continuous) solutions for all time. So the spacetime is geodesically complete.

Note that the divergence of $\langle M^2 \rangle$ is a purely quantum effect. For a classical string, if a mode is not oscillating initially, it will not oscillate at any later time (since the modes decouple). Hence the mass will remain finite.

A purely gravitational analog of this solution can be obtained by considering an impulsive gravitational plane wave with constant polarization, i.e., metric (2.1) with $F=\delta(U)(X^2-Y^2)$. Then it has been shown[19] that $\langle N_n \rangle \sim 1/n^2$ so $\langle M^2 \rangle$ again diverges even though the spacetime is geodesically complete. (Although the geodesics are not continuous in this case.) Note that this behavior of $\langle N_n \rangle$ is what one would expect from (3.11) even though strictly speaking the derivation of (3.11) is not valid for a δ function. The key point is that, for plane waves, the coupling of the string to the axion is through the derivative of X^i, whereas the coupling to the metric is proportional to X^i. So a milder axion background will produce the same excitation of the string as a δ function in the metric.

D. Selection rules

We now derive selection rules governing transitions between in- and out-string states for the case of nonsingular plane-wave backgrounds. The light-cone gauge Hamiltonian is given in Eq. (2.20). Reexpressing this in terms of the creation and annihilation operators for the individual modes yields

$$H = H_0 + H_I , \qquad (3.31)$$

where

$$H_0 = \tfrac{1}{4}P_0^iP_{0i} + Q_{ij}X_0^iX_0^j + \sum_{n=1}^{\infty} n(a_n^{i\dagger}a_n^i + \tilde{a}_n^{i\dagger}\tilde{a}_n^i) \qquad (3.32)$$

and

$$H_I=\sum_{n=1}^{\infty}\frac{Q_{ij}}{4n}(a_n^{i\dagger}a_n^j+\tilde{a}_n^{i\dagger}\tilde{a}_n^j-\tilde{a}_n^i a_n^j-a_n^i\tilde{a}_n^j)+\text{H.c.}\ , \tag{3.33}$$

where Q_{ij} is given in terms of the two polarizations for the gravitational wave W_i, the amplitude for the axion A, and the dilation Φ in (3.14) and (3.15).

Using Dyson's formula one can calculate transition amplitudes perturbatively. It is clear from the structure of H_I that transitions which do not preserve $N=\tilde{N}$ are forbidden. For some forms of the background fields, it is possible to show that certain transitions are forbidden to arbitrary order in the interaction Hamiltonian. We now derive these selection rules for transitions from the initial tachyon state to a final massless level state as well as transitions between massless states. The massless states are proportional to

$$P_{ij}a_1^{i\dagger}\tilde{a}_1^{j\dagger}|0\rangle\ , \tag{3.34}$$

where i runs over X and Y. The form of P_{ij} for the dilaton, X^2-Y^2 polarized graviton, XY polarized graviton, and axion are, respectively,

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\quad \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}. \tag{3.35}$$

We note that a transition from the vacuum to a final massless state with polarization P_{ij} will involve arbitrary powers of H_I, i.e., of the matrix Q_{ij}. Upon performing all the contractions of the oscillators, the amplitude will clearly be a sum of terms involving

$$\text{Tr}PQ^k \tag{3.36}$$

as well as traces over some products of Q's as well. Similarly for a transition from a massless state with polarization P to a massless state with polarization P', the amplitude will have terms involving a trace over P, P', and arbitrary powers of Q. Knowing this alone one can derive certain selection rules. We consider the following four classes of plane-wave solutions.

(1) $\Phi\neq 0$, $W_i=A=0$. In this case Q is proportional to the identity. This implies that only transitions to a dilaton from an initial ground state are allowed since only the dilaton polarization has a nonzero trace. Similarly, transitions between distinct massless states are forbidden since the product of any two distinct polarizations is traceless.

(2) $\Phi\neq 0$, $W_1\neq 0$, $W_2=A=0$. In this case Q is diagonal. Since such matrices are closed under multiplication only transitions to a dilaton or X^2-Y^2 graviton are allowed. Only transitions between a dilaton and X^2-Y^2 graviton or between an XY graviton and an axion are allowed.

(3) $\Phi\neq 0$, $W_2\neq 0$, $W_1=A=0$. In this case Q is symmetric with equal diagonal elements. Since such matrices are closed under multiplication, only transitions to a dilaton or XY graviton are allowed. Only transitions between a dilaton and an XY graviton or between an X^2-Y^2 graviton and an axion are allowed.

(4) $\Phi\neq 0$, $A\neq 0$, $W_i=0$. In this case Q has equal diagonal elements and off diagonal elements of opposite sign. Since such matrices are closed under multiplication, only transitions to a dilaton or axion are allowed. Only transitions between a dilaton and an axion or between two gravitons are allowed.

One should note that a massless state can be excited from the ground state even if the corresponding background field is zero. For example, in case (4), transitions to a dilaton are possible even when $\Phi=0$.

IV. DISCUSSION

We have shown the existence of singular solutions to classical string theory. We now discuss several issues which are raised by this investigation.

It has been shown that plane-fronted waves are solutions to all orders in σ-model perturbation theory. However, this series is often assumed to be valid only when $(\alpha' R)\ll 1$, where R is a measure of the spacetime curvature. Since this inequality is clearly violated for the singular plane waves, is it justified to continue to use this expansion? The answer is yes. The above inequality is usually assumed in order to ensure some form of convergence of the perturbation series. However for plane-fronted waves, this series always converges since all the higher-order terms vanish. So no restriction is necessary.

We have considered test strings in the singular plane-wave backgrounds. Since we find that $\langle M^2\rangle$ diverges, one might ask whether it is consistent to ignore the back reaction. The analysis certainly is consistent in the sense that the backgrounds have been shown to be classical solutions to all orders in σ-model perturbation theory. The first-quantized strings are introduced simply to test whether these solutions are singular, and we find that they are.

However since test strings can be viewed as small perturbations, one can ask whether the fact that $\langle M^2\rangle$ diverges means that the solutions are unstable. They are clearly stable under a class of perturbations which maintain the plane-wave form. However for more general perturbations this is a difficult question to answer. Even if it were known that generic perturbations became large near the singularity it would not be clear whether or not nearby solutions were singular. In general relativity we know that singularities are stable precisely because of the singularity theorems.[20] Unfortunately, there is no analog of these powerful results for string theory. However in our view, stability is not a crucial issue. These solutions are not of direct physical interest due to the existence of the null translational symmetry. The singular solutions are singular for all time and not the result of evolution from nonsingular initial conditions. Nevertheless they are of interest since they are the first examples of singular solutions in string theory.

In Sec. III we only considered strings in plane-wave solutions. More general plane-fronted waves contain singularities (in the sense of general relativity) which are more analogous to gravitational collapse than a cosmological singularity. Some timelike geodesics are complete and others are not. String propagation in these backgrounds can also be analyzed in the light-cone gauge. However now the equations for the transverse modes are nonlinear and more difficult to analyze near the singularity. Since the tidal forces again diverge, one expects that strings will again become infinitely excited showing that these solutions are also singular in the sense of string theory.

Over the past few years a number of arguments have been given to support the idea that there is a minimum observable length in classical string theory. These were often viewed as giving further evidence (in addition to orbifolds) that string theory should not have singularities. In light of the results presented here, the range of validity of these arguments needs to be reexamined. One argument applies to backgrounds of the form $M \times K_r$, where M is Minkowski spacetime and K_r is a compact internal space with characteristic radius r. For a variety of different choices for K_r, it has been shown[21] that string propagation on this background is equivalent to string propagation on $M \times K_{r^{-1}}$. So a small internal space is indistinguishable from a large one. This is referred to as "spacetime duality." A second argument comes from high-energy (fixed-angle) string scattering in flat spacetime. As one increases the center-of-mass energy s, one probes shorter distances until one reaches the Planck scale. At higher energies it has been shown[22] that the dominant contribution to tree-level scattering comes from world sheets which grow in size like $\sqrt{s}$. Thus at higher energies one does not probe arbitrarily short distances.

Neither of these arguments apply directly here, and indeed the results we find are qualitatively different. When the curvature becomes large, the strings do not behave as if the curvature is becoming small. Thus one must be careful in applying spacetime duality in areas where it has not been explicitly demonstrated. In particular, there seems to be little justification for assuming that some form of spacetime duality holds in the early Universe in string cosmology.

With regard to the second argument, although the interaction of a string with a singular gravitational wave is in some sense a "high-energy" collision, the string is strongly affected by the singularity. It is difficult to directly compare the calculations here with the flat-space scattering analysis, but the following comments can be made. A nonsingular gravitational wave can presumably be viewed as a collection of gravitons. The conserved component of the momentum P can be viewed as a measure of the speed at which the string is approaching the gravitational wave. Thus the limit as $P \to \infty$ should be analogous to a high-energy collision between the string and the gravitons. In Sec. III A it was shown that for a gravitational wave with amplitude $W(U)$ and constant polarization, and for modes satisfying $n^2 \ll \max(WP^2)$, the excitation at late times is given by

$$\langle N_n \rangle = \exp\left[\int \sqrt{W}\, dU\right] . \tag{4.1}$$

Two points should be noted about this formula. First, the string responds to an integral of the gravitational-wave amplitude, which is consistent with the idea that strings are not probing arbitrarily short distances. Second, $\langle N_n \rangle$ is independent of P.[23] Thus as P gets large, the excitation of each mode remains finite, but the number of excited modes increases with P. This implies that $\langle M^2 \rangle$ increases with P. This should be compared with the fact that the flat-space (fixed-angle) elastic scattering amplitudes vanish exponentially fast with center-of-mass energy. The implication is *not* that string interactions become weak at high energy, but rather that the result of high-energy collisions is likely to be highly excited strings. Since the number of states of a string increases exponentially with mass, one might expect an exponential decrease in the (fixed-angle) elastic scattering amplitudes. Finally we remark that a different high-energy limit of flat-space scattering amplitudes (with t fixed) has been shown[24,19] to be related to the Aichelburg-Sexl geometry[10] which is just a special case of the plane-fronted waves, with the metric function $F(U,X^i)=\delta(U)\ln(X^iX_i)$.

We have considered backgrounds which satisfy the classical equations of motion for string theory. In Ref. 4 two arguments were given for why these backgrounds are likely to remain solutions to the field equations which include perturbative quantum corrections. The first was that any local correction term constructed from the background fields and their derivatives will vanish. The second was the fact that these spacetimes have some unbroken supersymmetry, so perhaps a nonrenormalization theorem could be established.

However even if there is a nonzero quantum correction, one can still argue that some singular solutions should exist in the quantum theory as follows. In attempts to quantize general relativity directly, the effective dimensionless coupling constant is GR, where G is Newton's constant and R is a measure of the curvature. Thus strong curvature automatically corresponds to strong coupling and one expects quantum effects will be important. In string theory the situation is different. The dimensionless coupling constant is the dilaton. The field equation for the dilaton takes the form of a wave equation with sources which are powers of the curvature and other background fields. One expects that in regions of large curvature, the dilaton will become large and quantum effects will again be important. However plane-fronted waves have the property that the source term for the dilaton vanishes. Thus $\Phi = \text{const}$ is an exact solution even when the curvature diverges. So one can consider a singular plane wave with constant dilaton. By setting the dilaton to correspond to arbitrarily small coupling, one can reduce the size of perturbative quantum corrections. If one makes the reasonable assumption that nonperturbative quantum effects also go to zero as the coupling tends to zero, one is led to the possibility that there exists a consistent weak-coupling regime in the full string theory which contains spacetime singularities.

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