

## Lorentzian path integral for minisuperspace cosmology

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The path integral for minisuperspace models of cosmology is defined as a sum over Lorentzian geometries, and is a Green function for the Wheeler-DeWitt operator. It is shown to be a symmetric function of the initial and final configurations, and its real part is a solution to the Wheeler-DeWitt equation. The Lorentzian path integral is computed explicitly for the de Sitter minisuperspace model and is convergent. The resulting Green function is then related to the solutions of the Wheeler-DeWitt equation known as the Hartle-Hawking and tunneling wave functions of the Universe. The real part of this Green function is a product of Hartle-Hawking wave functions.

### I. INTRODUCTION

To a large extent, the current wave of interest in quantum cosmology is centered around the belief that the wave function of the Universe can be written as a Euclidean path integral. In particular, the boundary conditions that single out the wave function proposed by Hartle and Hawking<sup>1</sup> are usually expressed as a restriction on the geometries allowed in a Euclidean sum over histories. Unfortunately, a Euclidean path integral for general relativity does not make sense mathematically, even at a somewhat formal level, because the conformal factor integration diverges.<sup>2</sup> Attempts to remedy this problem by rotating the conformal mode integration contour in the complex plane have not been satisfactory;<sup>3</sup> at any rate, such a path integral then would not be a sum over Euclidean geometries, but rather a sum over some other set of complex four-geometries.

Faced with the difficulties inherent in defining a path integral that is in some sense a sum over Euclidean geometries, an obvious alternative is to revert to a definition of the path integral as a sum over Lorentzian geometries.<sup>4,5</sup> In this paper, Lorentzian path integrals are considered for minisuperspace models of cosmology. We further define the sum over histories as including four-geometries with positive spacetime volume only; as a consequence, the path integral is a Green function for the Wheeler-DeWitt operator, rather than a wave function. We show that whenever the system is time reversal invariant in an appropriate sense, the Green function is symmetric under interchange of its initial and final configurations. In addition, the real part of the Green function is a wave function in each argument.

The Lorentzian path integral is computed explicitly for the simplest minisuperspace model of cosmology, the so-called de Sitter model, in which the spatial sections are three-spheres and only the spatial volume  $x$  is retained as a degree of freedom. Denoting the initial three-volume by  $x'$  and the final three-volume by  $x''$ , the resulting Green function is

$$G(x'', x') = \Psi_T(x'')\Psi_H(x')\theta(x'' - x') + \Psi_T(x')\Psi_H(x'')\theta(x' - x''). \quad (1.1)$$

Here,  $\theta$  is the usual step function, and  $\Psi_T(x)$  and  $\Psi_H(x)$  are the solutions to the Wheeler-DeWitt equation known, respectively, as the tunneling<sup>6,7</sup> and Hartle-Hawking<sup>1</sup> wave functions of the Universe. As a function of the final three-volume  $x''$ , and restricted to  $x'' > x'$ , the Lorentzian path integral  $G(x'', x')$  is proportional to the tunneling wave function. This result is consistent with the analysis of Vilenkin,<sup>6</sup> although  $G(x'', x')$  itself is not a wave function. On the other hand, the real part of the Green function is a wave function—specifically, we show that  $\text{Re}G(x'', x')$  is a product  $\Psi_H(x'')\Psi_H(x')$  of Hartle-Hawking wave functions.

There are several reasons why a Euclidean path integral is often promoted over the Lorentzian path integral, despite its mathematical inconsistency and in spite of the fact that a sum over Lorentzian geometries is physically the natural choice. One reason stems from experience with field theory on a fixed background: if the background spacetime is Lorentzian, the integrand of the path integral exhibits oscillatory behavior, while if the background is Euclidean, the integrand exhibits exponentially damped behavior. However, this observation fails to be entirely relevant on two counts. First, the integrand for the Euclidean path integral is not exponentially damped in quantum gravity, because as alluded to previously the conformal mode contribution is exponentially divergent. Second, the fact that the integrand for a Lorentzian path integral is oscillatory does not necessarily imply that the path integral diverges. Indeed, for a Lorentzian functional integral roughly considered to be an infinite product of integrations, the prototype for each integral is the Fresnel integral  $\int dx \exp(ix^2)$ , which is well defined as an improper Riemann integral.<sup>8</sup> While the Euclidean path integral for general relativity is certainly ill defined, the Lorentzian path integral for general relativity might very well be definable.<sup>9</sup>

A second reason for considering a Euclidean path integral in quantum cosmology is based on an analogy with the free relativistic point particle. In this case, the Feynman propagator can be written as a Euclidean path integral,<sup>10</sup> where the process of passing from Lorentzian to Euclidean involves two complex “rotations.” One is the change in the particle’s proper time from timelike to spacelike; in general relativity, the analogue is the change from Lorentzian to Euclidean four-geometries. The other is the change in the particle’s Minkowski time coordinate from real to imaginary, so that the spacetime signature becomes effectively Euclidean; the analogue in general relativity is a complex conformal rotation, so that the signature of superspace becomes effectively positive definite. This procedure works well for the free point particle precisely because the free particle is so simple. In particular, the Minkowski time coordinate only appears in the kinetic term of the particle action. In contrast, the conformal factor in general relativity not only appears in the kinetic term of the action, but also in the gravitational potential-type terms (the scalar three-curvature) and the matter terms. A complex rotation of the conformal factor spoils the simple form and convergence properties of these nonkinetic terms.<sup>3</sup> Similar problems would also arise for a relativistic particle whose mass squared has a nontrivial spacetime dependence.

Another motivation for defining the path integral as a sum over Euclidean geometries is based on an analogy with nonrelativistic mechanics, for which the usual path integral is the transition amplitude for the system to evolve from one configuration at time  $t_1$  to another configuration at time  $t_2$ . It is well known<sup>11,12</sup> that by changing to imaginary time  $\tau = i(t_2 - t_1)$ , then as  $\tau \rightarrow \infty$  the ground-state wave function and energy can be extracted from the path integral. This fact suggests that an “imaginary time” (Euclidean) path integral for cosmology might be identified with the ground-state wave function of the Universe.<sup>1</sup> However, the path integral for quantum gravity is *not* analogous to a quantum-mechanical amplitude because the proper time separation between initial and final three-geometries is integrated, not fixed. The path integral for gravity is instead analogous to the path-integral representation of a Green function for the time-independent Schrödinger equation.<sup>5,13</sup> For such a Green function, the above-mentioned technique for obtaining the ground-state wave function is inapplicable; consequently, this motivation for focusing on the Euclidean path integral appears misguided.

A claim that is often cited as a reason to favor a Euclidean definition of the path integral for general relativity is that topology change can be described within a Euclidean formalism, but not within a Lorentzian formalism. We have no comment on this possibility.<sup>14</sup>

In view of the above observations, we feel that it is more promising to adopt a Lorentzian definition of the path integral for general relativity. In addition, we define the path integral as including a sum over just positive values of the proper time separation between initial and final three-geometries, rather than both positive and negative values. This choice corresponds to Teitelboim’s causality condition,<sup>15</sup> namely, that the histories included

in the functional integral are those for which the final three-surface is located to the future of the initial three-surface. In our minisuperspace models, the proper time separation is measured by spacetime volume, so the causality condition is the natural restriction to positive spacetime volume. As mentioned previously, the path integral so defined is a Green function for the Wheeler-DeWitt operator and its real part is a wave function in each argument. Notice that if one chooses to integrate in the path integral only over negative values of spacetime volume, the overall sign of the gravitational action is changed; so, for example, in the de Sitter model the complex conjugate  $G^*(x'', x')$  of the Green function is obtained. Then because  $G(x'', x') + G^*(x'', x') = 2 \operatorname{Re} G(x'', x') = 2\Psi_H(x'')\Psi_H(x')$ , the product of Hartle-Hawking wave functions can be represented by a path integral in which spacetime volume is integrated over all real values. However, it is important to recognize that such a path integral is not simply a sum over four-geometries. Instead, it is a sum that includes each four-geometry twice, once with positive four-volume and once with negative four-volume, where the negative volume histories are interpreted physically as having their final three-geometry  $x''$  in the past of their initial three-geometry  $x'$ .

The explicit computations in this paper are confined to minisuperspace models. It appears doubtful that a quantized minisuperspace model would be a good approximation to some quantized theory of gravity,<sup>16</sup> in the sense of yielding similar qualitative predictions. On the other hand, minisuperspace models should be helpful in the recognition and development of various formal relationships. Furthermore, it seems unlikely that the technical construction of the path integral can be accomplished for the full field theory of general relativity if it cannot be done properly for a vastly simplified minisuperspace model. For these reasons, we feel that the minisuperspace calculations considered here are worthwhile.

For some choices of initial and final three-volumes  $x'$  and  $x''$  in the de Sitter model, there are no classical Lorentzian four-geometries with these boundary data; consequently there is no single geometry that dominates the evaluation of the Green function  $G(x'', x')$  as a Lorentzian path integral. Instead, the geometries that extremize the action lie off the real Lorentzian “axis” of integration. In these cases, the Green function may be evaluated approximately by employing the method of steepest descents. Specifically, the contour of integration for spacetime volume can be distorted in the complex plane to pass through an appropriate saddle point. In this way the path integral is approximated by the behavior of the system near a complex four-geometry; for some  $x'$  and  $x''$  that four-geometry can be viewed as having Euclidean signature. (For other  $x', x''$ , the signature of the complex four-geometry is Euclidean in some regions and Lorentzian in others.) However this does *not* in any sense mean that the path integral is a sum over Euclidean geometries. The Euclidean geometry plays a role only through the purely mathematical technique used to obtain an approximation to the Lorentzian path integral.

The paper is organized as follows. Section II is devoted

ed to the description of various properties of the Green function for the Wheeler-DeWitt equation in minisuperspace models, and its expression as a Lorentzian path integral. Specifically, it is shown that if the system is time-reversal invariant, then the Green function will be symmetric under interchange of the initial and final configurations, and that the real part of the Green function, as a function of either argument, is a wave function. In Sec. III we concentrate on the de Sitter model of cosmology. The path integral for the Green function is evaluated semiclassically for all positive values of the initial and final three-volumes  $x'$  and  $x''$ . The Hartle-Hawking and tunneling wave functions are then defined as appropriate semiclassical solutions to the Wheeler-DeWitt equation. The Green function is related to these wave functions as expressed in Eq. (1.1), and we argue that this relationship is exact in spite of its semiclassical derivation. Section IV contains a discussion of our results. Here, we also contrast our work with the related work of Halliwell and Louko.<sup>17</sup> The Appendix contains an independent analysis indicating that the Lorentzian path integral for the de Sitter model is well defined, in the sense that it consists of an infinite product of convergent integrals.

## II. MINISUPERSPACE PATH INTEGRAL IN GENERAL

For a finite-dimensional minisuperspace cosmological model of general relativity coupled to nongauge matter fields,<sup>18</sup> the canonical form of the action is

$$S[x, p, N] = \int_{\sigma'}^{\sigma''} d\sigma \left[ p_a \frac{dx^a}{d\sigma} - N \mathcal{H}(x, p) \right], \quad (2.1)$$

where  $x^a(\sigma)$  and  $p_a(\sigma)$  are canonically conjugate variables depending only on the coordinate  $\sigma$  that labels the (closed) spacelike surfaces. The Lagrange multiplier  $N(\sigma)$  in Eq. (2.1) is a metric variable that measures the proper time separation between successive surfaces. For convenience, we will assume that  $N$  is derived through the minisuperspace reduction from a spatial density of weight 1; then  $N$  is a “lapse density” and  $N d\sigma$  is an increment in spacetime volume. Correspondingly, the Hamiltonian constraint  $\mathcal{H}$  comes from a spatial scalar and can be written as

$$\mathcal{H}(x, p) = H(x, p) + \Lambda / \kappa, \quad (2.2)$$

where the cosmological constant  $\Lambda$  appears explicitly as an additive constant. ( $\kappa$  is  $8\pi$  times Newton’s constant.)

The action (2.1) is the appropriate one for a variational principle with fixed values  $x'' \equiv x(\sigma'')$  and  $x' \equiv x(\sigma')$  of the variables  $x^a$  at the initial and final surfaces  $\sigma'$  and  $\sigma''$ . It is moreover analogous to the Jacobi action for nonrelativistic mechanics.<sup>19</sup> In the latter, the physical time required for the system to evolve between initial and final configurations is not fixed; instead the total energy is fixed. Similarly, in (2.1) the spacetime volume  $\int_{\sigma'}^{\sigma''} N d\sigma$ —which measures the separation between initial and final configurations—is not fixed; instead the

cosmological constant is fixed. Because the action (2.1) has the same form as Jacobi’s action, the construction of the path integral for finite-dimensional minisuperspace models is similar to the one for nonrelativistic mechanics at constant energy.<sup>5</sup> For such a system, the invariance of the action under reparametrizations  $\sigma \rightarrow f(\sigma)$  (with suitable restrictions on the function  $f$ ) must be taken into account; histories that are related by a change in parametrization should not be included individually in the sum over histories. The resulting path integral can be obtained from a Becchi-Rouet-Stora-Tyutin (BRST) analysis.<sup>22</sup> It consists of a functional integral over all canonical pairs  $x^a, p_a$  along with an ordinary integral over the spacetime volume between the fixed initial and final configurations:

$$G(x'', x') = \int dT \int \mathcal{D}x \mathcal{D}p \times \exp \left[ \frac{i}{\hbar} \int_0^T dt [p_a \dot{x}^a - \mathcal{H}(x, p)] \right]. \quad (2.3)$$

Here  $dt \equiv N d\sigma$  is an increment in spacetime volume and  $T$  is the total spacetime volume. Also, the overdot denotes differentiation with respect to  $t$  and  $\mathcal{D}x \mathcal{D}p$  is the Liouville measure on phase space.

As discussed in the Introduction, we will choose to integrate in Eq. (2.3) only over positive values of spacetime volume  $T$ . Then expressing the Hamiltonian constraint as in Eq. (2.2), the path integral can be written as the integral transform

$$G(x'', x') = \int_0^\infty dT e^{-i\Lambda T / \kappa \hbar} K(x'', T | x', 0) \quad (2.4a)$$

of a kernel  $K(x'', T | x', 0)$ . The kernel is itself a functional integral given by

$$K(x'', T | x', 0) \equiv \int \mathcal{D}x \mathcal{D}p \times \exp \left[ \frac{i}{\hbar} \int_0^T dt [p_a \dot{x}^a - H(x, p)] \right] \quad (2.4b)$$

and represents the transition amplitude for the system to evolve between configurations  $x'$  and  $x''$  in a fixed total “time”  $T$ . Notice that, at this level of formality, we have not taken into consideration the ranges of integration for the canonical variables  $x^a, p_a$ .<sup>23</sup> Some relevant comments are made in Sec. III in the context of the de Sitter cosmological model.

As in nonrelativistic mechanics, the kernel (2.4b) can be written as a matrix element

$$K(x'', T | x', 0) = \langle x'' | e^{-i\hat{H}T / \hbar} | x' \rangle, \quad (2.5)$$

where  $\hat{H}$  is the quantum Hamiltonian operator corresponding to  $H(x, p)$ , and spacetime volume  $T$  plays the role of time. From the above expressions, it follows that the kernel satisfies the “time-dependent Wheeler-DeWitt equation”

$$\left[ -i\hbar \frac{\partial}{\partial T} + \hat{H} \right] K(x'', T|x', 0) = 0, \quad (2.6)$$

where  $\hat{H}$  is now taken in the coordinate representation acting on  $x''$ . Furthermore, the full path integral (2.4) is a Green function for the time-independent Wheeler-DeWitt operator:

$$(\Lambda/\kappa + \hat{H})G(x'', x') = -i\hbar\delta(x'', x'). \quad (2.7)$$

The particular Green function defined by the path integral is the analogue of the Feynman Green function for a free relativistic particle, and is the analogue of the Fourier transform of the retarded Green function for the time-dependent Schrödinger equation. The result (2.7) is obtained by applying  $\Lambda/\kappa + \hat{H}$  to Eq. (2.4a) and using Eq. (2.6). In this calculation, the term  $\Lambda/\kappa$  is rewritten as a  $T$  derivative acting on the exponential factor, and integration by parts generates an end-point term at  $T=0$  that gives the  $\delta$  function on the right-hand side of Eq. (2.7). For the end-point term at  $T=\infty$ , the kernel is assumed to vanish; for the model considered in the next section, this is indeed the case. (Alternatively the cosmological constant can be given a small negative imaginary part.)

Observe that path integrals involving fixed values of momenta at one or both end points can be constructed in a similar fashion. In these cases, the action functionals differ from (2.1) by the addition of boundary terms  $x^a p_a|_{\sigma'}$  or  $-x^a p_a|_{\sigma''}$  and the resulting path integrals are related to  $G(x'', x')$  by Fourier transforms in  $x'$  or  $x''$ . For example, the path integral with fixed momenta  $p(\sigma')=p'$  and  $p(\sigma'')=p''$  is given by

$$G(p'', p') = \int \prod_a \left[ \frac{dx^a dx^a}{2\pi\hbar} \right] \times e^{-i(x'' \cdot p'' - x' \cdot p')/\hbar} G(x'', x'), \quad (2.8)$$

and is a Green function for the time-independent Wheeler-DeWitt operator:

$$(\Lambda/\kappa + \hat{H})G(p'', p') = -i\hbar\delta(p'', p'). \quad (2.9)$$

Here,  $\hat{H}$  is in the momentum representation acting on  $p''$ .

The Green function  $G(x'', x')$  for the time-independent Wheeler-DeWitt operator is symmetric in its two arguments provided the system is time-reversal invariant. (Again, the appropriate notion of time is spacetime volume.) To our knowledge, this property has not been previously recognized in the context of quantum cosmology, so we will discuss it in some detail. The result actually stems from a more basic property, namely, that the kernel  $K(x'', T|x', 0)$  is symmetric in  $x'$  and  $x''$  for a system with time-reversal invariance. This can be shown by identifying the kernel with a transition amplitude for a nonrelativistic quantum system as in Eq. (2.5). First introduce the standard antiunitary time-reversal operator  $\hat{\Theta}$ , satisfying<sup>24</sup>

$$(\hat{\Theta}\psi_2, \hat{\Theta}\psi_1)^* = (\psi_2, \psi_1), \quad (2.10)$$

where the parentheses denote the inner product. Next, apply Eq. (2.10) where the state  $\psi_2$  is the position eigen-

state  $|x''\rangle$ ; since a position eigenstate is time-reversal invariant,  $|x''\rangle$  is unchanged by the operator  $\hat{\Theta}$ . For the state  $\psi_1$ , choose  $e^{-i\hat{H}T/\hbar}|x'\rangle$ , and then Eq. (2.10) becomes

$$(\langle x''|\hat{\Theta}e^{-i\hat{H}T/\hbar}|x'\rangle)^* = \langle x''|e^{-i\hat{H}T/\hbar}|x'\rangle. \quad (2.11)$$

Now assume time-reversal invariance for the system, so that  $\hat{\Theta}e^{-i\hat{H}T/\hbar}=e^{+i\hat{H}T/\hbar}\hat{\Theta}$ . This leads to

$$\langle x'|e^{-i\hat{H}T/\hbar}|x''\rangle = \langle x''|e^{-i\hat{H}T/\hbar}|x'\rangle \quad (2.12)$$

and says that the kernel  $K(x'', T|x', 0)$  is symmetric in  $x'$  and  $x''$ .

It is perhaps worthwhile to give a more intuitive argument for this result, based on the path-integral representation (2.4b) for the kernel. For each history  $x(t), p(t)$  of the system beginning at  $x(0)=x'$  and ending at  $x(T)=x''$ , consider the time-reversed history  $\bar{x}(t)=x(T-t)$ ,  $\bar{p}(t)=-p(T-t)$  that begins at  $\bar{x}(0)=x''$  and ends at  $\bar{x}(T)=x'$ . Time-reversal invariance says that the actions for these two histories are the same. Then for each history  $x(t), p(t)$  that enters the sum over histories for  $K(x'', T|x', 0)$ , there is a corresponding history  $\bar{x}(t), \bar{p}(t)$  that makes the same contribution to the sum over histories for  $K(x', T|x'', 0)$ . Therefore the kernel is symmetric:

$$K(x'', T|x', 0) = K(x', T|x'', 0). \quad (2.13)$$

The symmetry of the Green function,

$$G(x'', x') = G(x', x''), \quad (2.14)$$

now follows directly from the symmetry of its integral kernel.

Similar reasoning can be applied to the path integral for quantum gravity in general, independent of any minisuperspace reduction. For example, consider the spacetime manifold to be  $R \times \Sigma$  with  $\Sigma$  a compact orientable three-manifold, although this may be unnecessary. If the action is time-reversal invariant, that is, invariant under a change of time orientation, then the path integral will be symmetric under interchange of the initial and final configurations. It is typical for the action to be independent of time orientation. In particular, any terms in the action that consist of the integral of a coordinate scalar times the natural volume form  $d^4\sigma\sqrt{-g}$  will have this property, since the volume form itself is invariant under changes of spacetime orientation. (The volume  $\int d^4\sigma\sqrt{-g}$  is a positive number, regardless of orientation.) One example of a possible contribution to the action that would not be independent of orientation is the integral of an external four-form field—the overall sign of such a term would depend on the orientation of the four-manifold. An analogous situation arises in relativistic particle mechanics, where coupling to an external magnetic field breaks time-reversal invariance.

Next, we examine the complex conjugate of the Green function, which can be written using Eq. (2.4) as

$$G^*(x'', x') = \int_0^\infty dT e^{+i\Lambda T/\kappa\hbar} K(x', -T|x'', 0). \quad (2.15)$$

(This assumes the cosmological constant is real.) Here,  $K(x', -T|x'', 0)$  is the complex conjugate of the kernel, a

result that follows immediately from the expression of the kernel as a matrix element (2.5). For time-reversal-invariant systems, the configurations  $x'$  and  $x''$  can be interchanged on the right-hand side of (2.15); then changing integration variables by  $T \rightarrow -T$ , the complex-conjugate Green function becomes

$$G^*(x'', x') = \int_{-\infty}^0 dT e^{-i\Lambda T/\kappa\hbar} K(x'', T|x', 0). \quad (2.16)$$

Comparing this result to Eq. (2.4) shows that the path integrals for  $G$  and  $G^*$  differ only in the range of integration chosen for spacetime volume  $T$ .

Applying the Wheeler-DeWitt operator  $\Lambda/\kappa + \hat{H}$  to Eq. (2.16), and assuming that the kernel vanishes as  $T \rightarrow -\infty$ , the conjugate Green function is seen to satisfy

$$(\Lambda/\kappa + \hat{H})G^*(x'', x) = +i\hbar\delta(x'', x'), \quad (2.17)$$

where the operator  $\hat{H}$  acts on  $x''$ . So for a system with time-reversal invariance, the Green function equations (2.7) and (2.17) imply that the real part of the Green function satisfies the Wheeler-DeWitt equation in the argument  $x''$ :

$$(\Lambda/\kappa + \hat{H})\text{Re}G(x'', x') = 0. \quad (2.18)$$

That is,  $\text{Re}G(x'', x')$  is a wave function in  $x''$ . In turn, symmetry implies that  $\text{Re}G(x'', x')$  is a wave function in the argument  $x'$  as well. Also observe that the real part of the Green function, expressed as an integral transform of the kernel, is

$$\text{Re}G(x'', x') = \frac{1}{2} \int_{-\infty}^{\infty} dT e^{-i\Lambda T/\kappa\hbar} K(x'', T|x', 0). \quad (2.19)$$

With the kernel viewed as a path integral (2.4b),  $\text{Re}G(x'', x')$  becomes a functional integral based on the action (2.1), where spacetime volume  $T$  is integrated over both positive and negative values. However, as discussed in the Introduction, we regard the path integral for the full Green function  $G(x'', x')$ , and not the path integral for either  $G^*(x'', x')$  or  $\text{Re}G(x'', x')$ , as the basic expression of a sum over four-geometries (and matter field histories). We therefore base the analysis of the de Sitter model in the following section on a calculation of  $G(x'', x')$  as a path integral.

### III. PATH INTEGRAL FOR THE de SITTER MODEL

For the de Sitter minisuperspace model of cosmology, the metric ansatz is taken to be

$$ds^2 = -\frac{N^2}{x^2} d\sigma^2 + \left[ \frac{x}{2\pi^2} \right]^{2/3} d\Omega_3^2, \quad (3.1)$$

where  $d\Omega_3^2$  is the metric for a unit three-sphere. The spatial three-volume is denoted by

$$x(\sigma) = \int d^3\Omega \sqrt{^3g}, \quad (3.2)$$

and the lapse density is given by

$$N(\sigma) = \int d^3\Omega \sqrt{-^4g}, \quad (3.3)$$

where  $^3g$  and  $^4g$  represent the determinants of the spatial

and spacetime metrics, respectively. Integrating  $N$  along  $\sigma$  yields the total spacetime volume  $T = \int_{\sigma'}^{\sigma''} d\sigma N$  between the initial surface  $\sigma'$  and the final surface  $\sigma''$ .

The Einstein action for the metric (3.1) is

$$S[x, N] = \frac{3}{\kappa} \int_{\sigma'}^{\sigma''} d\sigma \left[ -\frac{1}{9N} \left[ \frac{dx}{d\sigma} \right]^2 + N \left[ \frac{2\pi^2}{x} \right]^{2/3} - N \left[ \frac{\Lambda}{3} \right] \right]. \quad (3.4)$$

In Hamiltonian form, the action becomes

$$S[x, p, N] = \int_{\sigma'}^{\sigma''} d\sigma [p(\partial x / \partial \sigma) - N\mathcal{H}], \quad (3.5)$$

where the Hamiltonian constraint is given by

$$\mathcal{H} \equiv - \left[ \frac{3\kappa}{4} p^2 + \frac{3}{\kappa} \left[ \frac{2\pi^2}{x} \right]^{2/3} - \left[ \frac{\Lambda}{\kappa} \right] \right]. \quad (3.6)$$

The momentum  $p = -2(\partial x / \partial \sigma) / (3\kappa N) = -2\dot{x} / (3\kappa)$  conjugate to the spatial volume  $x$  is  $(2/3\kappa)$  times the trace of the extrinsic curvature for the spatial surfaces  $\sigma = \text{const}$ . Notice that the sign of the momentum  $p$  is opposite to the sign of the “velocity”  $\dot{x}$ ; that is, the spatial volume is a negative mode.

As mentioned in the previous section, the action (3.5) is in the form of Jacobi’s action for nonrelativistic mechanics.<sup>13</sup> In Jacobi’s action, the Hamiltonian constraint fixes the kinetic plus potential energy of the system to a constant  $E$ . Then comparing Eq. (3.6) with the Hamiltonian constraint  $\mathcal{H} = p^2/2m + V(x) - E$  for a nonrelativistic particle it becomes apparent that, despite the overall sign difference, the spatial volume  $x$  in the present model can be viewed as the position coordinate for a particle of “mass”  $2/3\kappa$  and total “energy”  $\Lambda/\kappa$  moving in the one-dimensional potential

$$V(x) = \frac{3}{\kappa} \left[ \frac{2\pi^2}{x} \right]^{2/3}. \quad (3.7)$$

This identification will be of considerable use in the remainder of this section.

The Lorentzian path integral for the de Sitter model is given by Eq. (2.4) applied to a single canonical pair  $x, p$ , with

$$H(x, p) = - \left[ \frac{3\kappa}{4} p^2 + \frac{3}{\kappa} \left[ \frac{2\pi^2}{x} \right]^{2/3} \right]. \quad (3.8)$$

In the Appendix, we argue that this path integral is well defined in the sense that it consists of an infinite product of convergent integrals. Here, the path integral will be evaluated approximately using semiclassical methods; this requires an understanding of the classical histories for the system.

#### A. Classical histories

The classical histories extremize the phase in the path integral, which is a functional of  $x(t)$  and  $p(t)$  and a function of spacetime volume  $T$ . Varying the phase with respect to  $x(t)$  and  $p(t)$  gives the canonical equations of

motion associated with the Hamiltonian  $H$  of Eq. (3.8). Varying the phase with respect to  $T$  implies that  $\mathcal{H}|_{t=T}=0$ , where  $\mathcal{H}=H+\Lambda/\kappa$  is the Hamiltonian constraint. But  $H$  is a constant of motion by virtue of the canonical equations, so the Hamiltonian constraint  $\mathcal{H}$  must in fact vanish for all  $t$ .

For a one-dimensional system such as the one considered here, the Hamiltonian constraint completely characterizes the classical dynamics. In terms of the velocity  $\dot{x}$ , the Hamiltonian constraint becomes

$$(\dot{x}^2)/(3\kappa)+V(x)=\Lambda/\kappa, \tag{3.9}$$

where  $V(x)$  is the potential (3.7) shown in Fig. 1. Experience with the nonrelativistic mechanics of a particle moving in a one-dimensional potential reveals the nature of the classical motion  $x(t)$ . For any positive value of the cosmological constant  $\Lambda/\kappa$ , there exists a turning point at  $x_0 \equiv 2\pi^2(3/\Lambda)^{3/2}$ . Then for motion in the classically allowed “over-barrier” region, space contracts from infinite volume to a minimum volume  $x_0$ , reflects off the potential barrier, and expands once again to infinite volume. The classical motion just described is, of course, the de Sitter spacetime solution. Mathematically it is expressed by writing Eq. (3.9) as

$$dt = \pm \frac{dx}{\sqrt{3\kappa[\Lambda/\kappa - V(x)]}} \tag{3.10}$$

and integrating to obtain the spatial volume  $x$  as a function of spacetime volume  $t$ . The plus or minus sign in Eq. (3.10) is determined physically according to whether three-volume  $x$  is expanding or contracting. The geometry of this solution is obtained from Eqs. (3.10) and (3.1), giving the de Sitter metric

$$ds^2 = -\frac{dx^2}{3\kappa x^2[\Lambda/\kappa - V(x)]} + \left[\frac{x}{2\pi^2}\right]^{2/3} d\Omega_3^2. \tag{3.11}$$

[For  $x \geq x_0$ , metric (3.11) covers only the expanding or contracting half of the full de Sitter geometry.]

For the purpose of extremizing the phase in the path integral (2.4), the relevant histories have definite end points  $x'$  and  $x''$ . If the initial and final three-volumes  $x'$

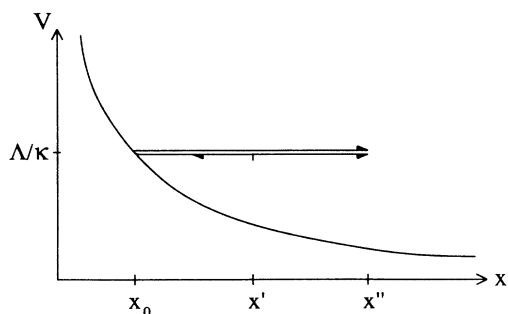


FIG. 1. The potential  $V$  that governs the motion of three-space, as a function of spatial volume  $x$ . When the initial and final three-volumes  $x'$  and  $x''$  are greater than the classical turning point  $x_0$ , there are two classical solutions corresponding to direct and reflected paths.

and  $x''$  are chosen in the over-barrier region as in Fig. 1, there are clearly two classical histories with these initial and final data, each just portions of de Sitter spacetime. For one history, three-space evolves directly from  $x'$  to  $x''$ ; for the other history, three-space first contracts from  $x'$  to the minimum volume  $x_0$ , reflects off the potential barrier, and expands to the final volume  $x''$ . (See Fig. 2.) The total spacetime volume for each of these classical histories is obtained by integrating Eq. (3.10), giving

$$T = \int_{\mathcal{P}} \frac{dx}{k(x)}, \tag{3.12}$$

where we have defined

$$k(x) \equiv \pm \sqrt{3\kappa[\Lambda/\kappa - V(x)]}. \tag{3.13}$$

In Eq. (3.12), the integration path  $\mathcal{P}$  runs directly from  $x'$  to  $x''$  for the “direct” classical history, while for the “reflected” classical history the path runs from  $x'$  to  $x_0$  and then to  $x''$ . By the Hamiltonian constraint (3.9),  $k(x)$  is just the velocity  $\dot{x}$  so the sign in Eq. (3.13) is chosen positive when three-space is expanding and negative when three-space is contracting.

If one of the end points, say  $x'$ , has a value less than that of the turning point  $x_0$ , then there is a single complex history that satisfies the classical equations of motion and thereby extremizes the phase in the path integral. The geometry for this history is described by the metric (3.11) with  $x' \leq x \leq x''$ . It is Euclidean in the under-barrier region  $x < x_0$  and Lorentzian in the over-barrier region  $x > x_0$ . The four-geometry is regular at the interface between the Euclidean and Lorentzian regions. The spacetime volume  $T$  is given by Eq. (3.12) where the path  $\mathcal{P}$  runs directly from  $x'$  to  $x''$ .  $T$  is complex in this case because  $k(x)$  is imaginary for  $x$  smaller than  $x_0$ . Whether  $k(x)$  is taken to be positive or negative imaginary for  $x < x_0$  will be determined later.

If both end points  $x'$  and  $x''$  have values less than  $x_0$ , then there are two complex classical histories. In one, three-space evolves directly from  $x'$  to  $x''$  and in the other, three-space first expands from  $x'$  to a maximum size  $x_0$ , reflects from the potential barrier, then contracts to  $x''$ . The geometries for these histories are portions of the

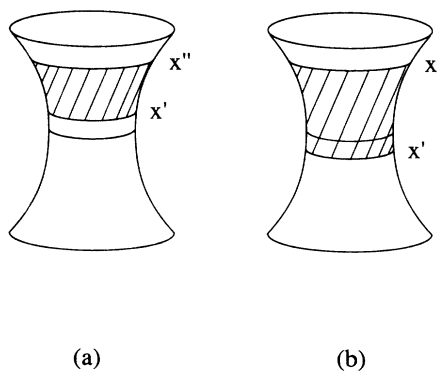


FIG. 2. The direct (a) and reflected (b) classical histories, for the case  $x_0 < x' < x''$ , represented as shaded portions of the de Sitter hyperboloid.

Euclidean de Sitter space, which is a four-sphere with metric given by Eq. (3.11). The four-volume for each history is purely imaginary, and is given by Eq. (3.12) where the path  $\mathcal{P}$  is either “direct” or “reflected.” [Here again, the sign of  $\text{Im}k(x)$  is to be determined.]

### B. Single-history contribution to $G(x'', x')$

The semiclassical approximation to the path integral for  $G(x'', x')$  is obtained by adding contributions associated with each of the classical histories discussed above. The contributions from these classical histories have a common form—they differ only by overall complex factors associated with reflections from the potential barrier, and by the phase appropriate for  $k(x)$ . We will first construct a “single-history” contribution  $G_{\text{SH}}(x'', x')$  to the Green function that is common to all the classical histories. We will then derive the rules for adding together the single-history terms to obtain the full Green function  $G(x'', x')$ . This will be done by studying a simpler system with the same qualitative features, namely, the system obtained by replacing the potential  $V(x)$  in Eq. (3.7) with a potential that is linear in  $x$ .

The single-history part  $G_{\text{SH}}(x'', x')$  of the Green function is the integral transform of a single-history kernel  $K_{\text{SH}}(x'', T|x', 0)$ . This kernel is the semiclassical contribution to the path integral (2.4b) from a history that extremizes the phase in that path integral. Such a history satisfies the canonical equations of motion generated by the Hamiltonian (3.8). Unlike the classical histories for the Green function, the extremal histories for the kernel have spacetime volume  $T$  fixed while the value of the Hamiltonian is not constrained to equal  $-\Lambda/\kappa$ . Yet the Hamiltonian  $H$  is a constant of motion—denoting its constant value by  $-\tilde{\Lambda}/\kappa$ , the classical histories for the kernel satisfy  $\dot{x} = \tilde{k}(x)$  and

$$T = \int_{\rho} \frac{dx}{\tilde{k}(x)}, \quad (3.14)$$

where

$$\tilde{k}(x) \equiv \pm \sqrt{3\kappa[\tilde{\Lambda}/\kappa - V(x)]}. \quad (3.15)$$

These equations are identical in form to Eqs. (3.12)–(3.13), but in this case they are to be solved in principle for  $\tilde{\Lambda}$  as a function of  $T$  and the end-point values  $x'$  and  $x''$ .

The action associated with the phase of the path integral (2.4b) for the kernel, evaluated along an extremal history, is<sup>5</sup>

$$S(x'', T|x', 0) \equiv -\frac{2}{3\kappa} \int_{\rho} dx \tilde{k}(x) + \tilde{\Lambda}T/\kappa. \quad (3.16)$$

The semiclassical contribution to the kernel from a single classical history can be written as<sup>12</sup>

$$K_{\text{SH}}(x'', T|x', 0) \approx \left[ \frac{1}{2\pi\hbar} \frac{\partial^2 S(x'', T|x', 0)}{\partial x' \partial x''} \right]^{1/2} \times \exp \left[ \frac{i}{\hbar} S(x'', T|x', 0) + \frac{i\pi}{4} \right]. \quad (3.17)$$

The preexponential factor is proportional to the square root of the Van Vleck–Morette determinant, and can be written using Eq. (3.14) as<sup>5</sup>

$$\frac{\partial^2 S(x'', T|x', 0)}{\partial x' \partial x''} = -\frac{1}{\kappa \tilde{k}(x') \tilde{k}(x'')} \frac{\partial \tilde{\Lambda}}{\partial T}. \quad (3.18)$$

The overall factors in Eq. (3.17) are chosen to give the correct result for a “direct” classical history  $x'$  to  $x''$ . In this case, the Van Vleck–Morette determinant is positive and the square root in Eq. (3.17) is defined as the positive root. That the Van Vleck–Morette determinant is positive for a direct history follows from Eq. (3.18) and the facts that the “energy”  $\tilde{\Lambda}$  decreases with increasing “time”  $T$ , and  $\tilde{k}(x)$  must be real for real  $T$ .

Using Eqs. (3.14)–(3.18) for the kernel, the single-history contribution to the Green function becomes

$$G_{\text{SH}}(x'', x') \approx \int_0^{\infty} dT \left[ -\frac{1}{2\pi\hbar\kappa \tilde{k}(x') \tilde{k}(x'')} \frac{\partial \tilde{\Lambda}}{\partial T} \right]^{1/2} \times \exp \left[ \frac{i}{\hbar} \mathcal{S}(T) + \frac{i\pi}{4} \right], \quad (3.19)$$

where the phase  $\mathcal{S}(T)$  is defined by

$$\mathcal{S}(T) \equiv S(x'', T|x', 0) - \Lambda T/\kappa. \quad (3.20)$$

The integral over  $T$  in Eq. (3.19) will be evaluated semiclassically. For this purpose, observe first that

$$\frac{\partial S(x'', T|x', 0)}{\partial T} = \tilde{\Lambda}/\kappa; \quad (3.21)$$

it follows that the phase  $\mathcal{S}(T)$  is extremized for any value  $T_0$  of spacetime volume which satisfies  $\tilde{\Lambda}|_{T_0} = \Lambda$ . The phase can now be expanded as

$$\mathcal{S}(T) = -\frac{2}{3\kappa} \int_{\rho} dx k(x) + \frac{1}{2\kappa} \frac{\partial \tilde{\Lambda}}{\partial T} \Big|_{T_0} (T - T_0)^2 + \dots, \quad (3.22)$$

where  $\tilde{k}(x)|_{T_0} = k(x)$  according to definitions (3.13) and (3.15). Combining these results gives

$$G_{\text{SH}}(x'', x') \approx \left[ -\frac{1}{2\pi\hbar\kappa k(x')k(x'')} \frac{\partial \tilde{\Lambda}}{\partial T} \Big|_{T_0} \right]^{1/2} \exp \left[ \frac{-2i}{3\kappa\hbar} \int_{\rho} dx k(x) + \frac{i\pi}{4} \right] \times \int d(T - T_0) \exp \left[ \frac{i}{2\hbar\kappa} \frac{\partial \tilde{\Lambda}}{\partial T} \Big|_{T_0} (T - T_0)^2 \right]. \quad (3.23)$$

Assuming the history is direct and over barrier,  $k(x')$  and  $k(x'')$  are real and have the same sign, and  $\partial\tilde{\Lambda}/\partial T$  is negative. Then the integral in Eq. (3.23) can be evaluated, leading to the final expression

$$G_{\text{SH}}(x'',x') \approx |k(x')k(x'')|^{-1/2} \times \exp \left[ \frac{-2i}{3\kappa\hbar} \int_{\rho} dx k(x) \right] \quad (3.24)$$

for the semiclassical contribution to the Green function  $G(x'',x')$  associated with a single-classical history.

### C. Phase rules

The next task is to obtain a set of rules that dictate how the single-history Green function obtained above should be modified for classical paths that are not direct and over barrier. For the most part, this simply amounts to determining in Eq. (3.24) the overall phase, and the phase in the argument of the exponential. This is accomplished by considering the  $x^{-2/3}$  potential in Eq. (3.7) to be replaced by a potential that is linear in  $x$ , with negative slope. With either potential, the full Green function (2.4) in the semiclassical approximation receives contributions from direct and reflected classical histories, as described in Sec. III A. But the linear potential is easier to compute, because in that case the kernel (2.4b) is built from a single semiclassical term—that is, there is a unique history with boundary data  $x',x'',T$  that extremizes the phase of the kernel. The same is not true for the  $x^{-2/3}$  potential, which may have one or two histories, or even no histories extremizing the phase of the kernel, depending on the end-point values  $x',x''$ , and  $T$ .

Since there is only one semiclassical contribution to the kernel when the potential is linear, the full kernel is given by the single-history expression (3.14)–(3.18). (Furthermore, this result is exact, a fact that will not be important for what follows.) Equations (3.14) and (3.15) show that for  $T$  smaller than a specific value, the “energy”  $\tilde{\Lambda}/\kappa$  decreases with increasing  $T$ ,  $\partial\tilde{\Lambda}/\partial T < 0$ , and that the path  $\mathcal{P}$  is direct. Then  $\tilde{k}(x)$  is positive if  $x'' > x'$  or negative if  $x'' < x'$ . For  $T$  larger than the specific value, Eqs. (3.14) and (3.15) show that  $\partial\tilde{\Lambda}/\partial T > 0$  and that the path  $\mathcal{P}$  is reflected. The sign of  $\tilde{k}(x)$  depends on whether  $x$  is increasing or decreasing, so that  $\tilde{k}(x')$  is negative and  $\tilde{k}(x'')$  is positive. In each of these cases, the Van Vleck–Morette determinant (3.18) is positive and the preexponential factor in the kernel (3.17) is unambiguously defined using the positive square root. The exponential factor (3.16) is also precisely defined using the known integration path  $\mathcal{P}$  and the appropriate signs for  $\tilde{k}(x)$ . Thus, for a linear potential, Eq. (3.17) is an unambiguous expression for the kernel for all end-point data  $x',x'',T$ .

As usual, the Green function  $G(x'',x')$  is computed as the integral transform (2.4a) of the kernel. For the linear potential, the semiclassical evaluation of this integral has been carried out by McLaughlin<sup>25</sup> in the context of non-relativistic particle mechanics. The present situation differs in that the coordinate  $x$  represents a conformal mode, and therefore extra minus signs are introduced into the formulas. We will briefly describe the main

features and results of the calculation.

The integral over  $T$  in Eq. (2.4) is dominated by stationary points  $T_0$ , and just as in the formal analysis of the previous subsection, these values of  $T$  satisfy  $\tilde{\Lambda}|_{T_0} = \Lambda$ . If  $x'$  and  $x''$  are greater than the turning point  $x_0$  of the potential, there are two such stationary points corresponding to direct and reflected classical paths. The direct path makes a semiclassical contribution precisely as in the single-history Green function (3.24). For the reflected path, expression (3.23) is still valid, but now  $k(x')$  and  $\partial\tilde{\Lambda}/\partial T$  have changed signs. Upon integrating over  $T - T_0$ , these sign differences introduce an extra phase factor  $e^{i\pi/2}$  into the single-history contribution to the Green function. So for  $x' > x_0$  and  $x'' > x_0$ , the full Green function is a sum of terms:  $G_{\text{SH}}(x'',x')$  for the direct path and  $iG_{\text{SH}}(x'',x')$  for the reflected path.

Next, consider end points satisfying  $x' < x_0 < x''$ , so that  $x'$  is under the potential barrier and  $x''$  is over the potential barrier, and return to Eq. (3.19) for the single-history contribution to the Green function. In this case the stationary points for the  $T$  integral are complex, satisfying Eq. (3.12) with the path  $\mathcal{P}$  running directly from  $x'$  to  $x''$ . There are four such stationary values, as shown in Fig. 3, corresponding to the distinct ways in which the overall signs can be chosen for  $k(x)$  in the under-barrier and over-barrier regions. Each of these stationary points corresponds to the single complex classical history that was discussed previously. In order to evaluate the  $T$  integral, the contour can be distorted from the positive real axis to pass through appropriate stationary points in the complex plane. Explicit calculation shows that the contour can be distorted as in Fig. 3, to run along a steepest-descent path from the origin to the stationary point in the first quadrant of the complex  $T$  plane, then out parallel to the real axis.<sup>25</sup> The stationary point that dominates this integral corresponds to  $k(x)$  positive real in the over-barrier region and negative imaginary in the under-

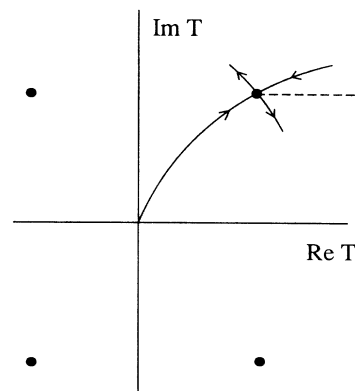


FIG. 3. The complex  $T$  plane for the case  $x' < x_0 < x''$ . The heavy dots are stationary points and the solid lines are curves of constant imaginary part of  $i\mathcal{S}(T)$ . The arrows show the direction of increasing real part of  $i\mathcal{S}(T)$ . The integration contour is distorted first to coincide with the steepest-descent contour between the origin and the stationary point in the first quadrant, and then to run parallel to the real axis along the dashed line.



barrier region. The integral for the Green function is now formally the same as in Eq. (3.23), but  $k(x')$  is negative imaginary and  $\partial\tilde{\Lambda}/\partial T$  is complex; these differences result in an extra phase factor  $e^{i\pi/4}$ . Thus, the full Green function for  $x' < x_0 < x''$  equals  $e^{i\pi/4}G_{\text{SH}}(x'', x')$  where  $k(x)$  is positive real or negative imaginary.

Now consider the situation in which both end points are under the potential barrier,  $x' < x'' < x_0$ . There are four stationary points for the  $T$  integral in Eq. (3.19), all lying along the imaginary axis. The two stationary points nearest the real axis correspond to the direct classical path and the two ways of choosing signs for  $k(x)$ ; the stationary points farthest from the real axis correspond to the reflected classical path. As shown in Fig. 4, the contour of integration for  $T$  can be distorted to pass up the imaginary axis to the second turning point, then out along a path of steepest descent.<sup>25</sup> The contribution to the integral from the part of the contour that runs up the imaginary axis is formally given by Eq. (3.23), and is dominated by the first stationary point. But now  $k(x)$  is negative imaginary, and this leads to an extra overall phase  $e^{i\pi/2}$  in the single-history Green function. The remaining portion of the contour, from the second stationary point to infinity along the steepest-descent path, is dominated by the half Gaussian near that stationary point. It is characterized by  $k(x)$  negative imaginary as  $x$  runs from  $x'$  to  $x_0$ , and positive imaginary as  $x$  runs from  $x_0$  to  $x''$ . So in this case  $k(x')k(x'')$  is real and there are no extra phase factors entering the single-history contribution to the Green function. But because the integral is only a half Gaussian, there is an extra factor of  $\frac{1}{2}$  in the expression for  $G_{\text{SH}}(x'', x')$ . The result is that for  $x' < x'' < x_0$ , the full Green function is a sum of terms,  $iG_{\text{SH}}(x'', x')$  for the direct path and  $\frac{1}{2}G_{\text{SH}}(x'', x')$

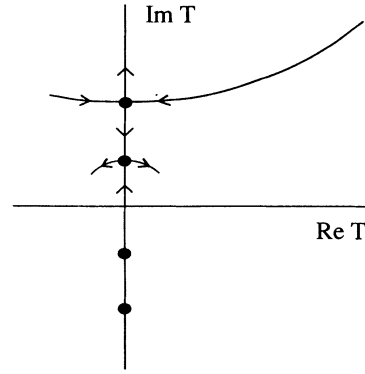


FIG. 4. The complex  $T$  plane for the case  $x' < x'' < x_0$ . The integration contour is distorted to run first up the imaginary axis to the second turning point, then out along a path of steepest descent contour in the first quadrant.

for the reflected path, where the correct signs for  $\text{Im}k(x)$  are indicated above.

#### D. Green functions and wave functions

In the preceding discussions we have indicated how the semiclassical approximation to the Green function for the linear potential is obtained by summing single-history contributions. The same phase rules apply to the  $x^{-2/3}$  potential, since the classical histories that dominate the path integral for the Green function are qualitatively the same in both cases. Using these established rules, the Green function for the de Sitter model of cosmology can be written down directly. For end-point values satisfying  $x_0 < x' < x''$ , the result is [recall definition (3.13)]

$$G(x'', x') \approx \{3\kappa[\Lambda/\kappa - V(x'')]\}^{-1/4} \{3\kappa[\Lambda/\kappa - V(x')]\}^{-1/4} \\ \times \left\{ \exp \left[ \frac{-2i}{3\kappa\hbar} \int_{x'}^{x''} dx \sqrt{3\kappa[\Lambda/\kappa - V(x)]} \right] \right. \\ \left. + i \exp \left[ \frac{-2i}{3\kappa\hbar} \left( \int_{x_0}^{x'} dx \sqrt{3\kappa[\Lambda/\kappa - V(x)]} + \int_{x_0}^{x''} dx \sqrt{3\kappa[\Lambda/\kappa - V(x)]} \right) \right] \right\}. \quad (3.25)$$

For end-point values satisfying  $x' < x_0 < x''$ , the result is

$$G(x'', x') \approx e^{i\pi/4} \{3\kappa[\Lambda/\kappa - V(x'')]\}^{-1/4} \{3\kappa[V(x') - \Lambda/\kappa]\}^{-1/4} \\ \times \exp \left[ \frac{-2i}{3\kappa\hbar} \left( \int_{x_0}^{x''} dx \sqrt{3\kappa[\Lambda/\kappa - V(x)]} - i \int_{x'}^{x_0} dx \sqrt{3\kappa[V(x) - \Lambda/\kappa]} \right) \right]. \quad (3.26)$$

For end-point values satisfying  $x' < x'' < x_0$ , the result is

$$G(x'', x') \approx \{3\kappa[V(x'') - \Lambda/\kappa]\}^{-1/4} \{3\kappa[V(x') - \Lambda/\kappa]\}^{-1/4} \\ \times \left\{ i \exp \left[ \frac{-2}{3\kappa\hbar} \int_{x'}^{x''} dx \sqrt{3\kappa[V(x) - \Lambda/\kappa]} \right] \right. \\ \left. + \frac{1}{2} \exp \left[ \frac{-2}{3\kappa\hbar} \left( \int_{x'}^{x_0} dx \sqrt{3\kappa[V(x) - \Lambda/\kappa]} + \int_{x''}^{x_0} dx \sqrt{3\kappa[V(x) - \Lambda/\kappa]} \right) \right] \right\}. \quad (3.27)$$

All roots in the previous equations are real and by definition positive. Recalling from Sec. II that  $G(x'',x')$  is symmetric in its two arguments, Eqs. (3.25)–(3.27) yield the Green function for all positive values of  $x'$  and  $x''$ . Also observe that the integrals in these expressions can be evaluated explicitly for the potential (3.7).

The Green function (3.25)–(3.27) was obtained without taking into account the fact that the variable  $x$  represents spatial volume, and should be integrated over only positive values.<sup>23</sup> However, none of the classical histories used to construct  $G(x'',x')$  probe the region  $x \leq 0$ . So the semiclassical contributions associated with these classical histories should be insensitive to whether or not the  $x(t)$  integration is restricted to positive values. Of course, there may be other single-history contributions to  $G(x'',x')$  that we have failed to include, namely, those corresponding to classical histories that reflect off the “boundary” at  $x = 0$ . But such histories tunnel far into the potential barrier, and would make contributions to  $G(x'',x')$  that are exponentially small compared to the single-history contributions already included in Eqs. (3.25)–(3.27). In the de Sitter model, the  $x^{-2/3}$  potential naturally isolates the origin  $x = 0$ , so that in the semiclassical approximation the problem of constructing a path integral on the half-line can be sidestepped.

For  $x'' \neq x'$ , the Green function  $G(x'',x')$  is a solution to the Wheeler-DeWitt equation in each argument. It is therefore possible to identify  $G(x'',x')$  with a wave function in either  $x'$  or  $x''$  when the initial and final three-volumes are restricted by, say,  $x'' > x'$ . In this case,  $G(x'',x')$  can be viewed as a wave function in the larger argument  $x''$ , and it is clear from the semiclassical

analysis above that for  $x''$  in the classically allowed over-barrier region,  $G(x'',x')$  contains only outgoing WKB components. But this is precisely the identifying characteristic of the tunneling wave function<sup>26</sup>  $\Psi_T$  in the context of the de Sitter minisuperspace model. Likewise, viewed as a wave function in the smaller argument  $x'$ ,  $G(x'',x')$  is exponentially increasing with increasing  $x'$  in the classically forbidden under-barrier region. But this is an identifying feature of the Hartle-Hawking wave function  $\Psi_H$  for the de Sitter model.<sup>1,26</sup>

The conclusion of the above observations is that  $G(x'',x')$  behaves like the tunneling wave function in its larger argument, and like the Hartle-Hawking wave function in its smaller argument. Therefore the Green function and wave functions should be related by

$$G(x'',x') = \Psi_T(x'')\Psi_H(x')\theta(x'' - x') + \Psi_T(x')\Psi_H(x'')\theta(x' - x''). \quad (3.28)$$

This prediction can be verified explicitly. Consider the Wheeler-DeWitt equation

$$\left[ -(3\kappa\hbar^2/4)\frac{\partial^2}{\partial x^2} + V(x) - (\Lambda/\kappa) \right] \Psi(x) = 0, \quad (3.29)$$

which follows from the Hamiltonian constraint (3.6) using the “natural” factor ordering  $p^2 \rightarrow \hat{p}^2 = -\hbar^2\partial^2/\partial x^2$ . This has the form of the time-independent Schrödinger equation, so that standard WKB formulas apply. These yield two independent solutions<sup>24</sup> (with overall constants chosen for simplicity):

$$\Psi_H(x) \approx \begin{cases} \frac{1}{2} \{3\kappa[V(x) - \Lambda/\kappa]/4\}^{-1/4} \exp \left[ -\frac{2}{3\kappa\hbar} \int_x^{x_0} dz \sqrt{3\kappa[V(z) - \Lambda/\kappa]} \right], & x < x_0, \\ \{3\kappa[\Lambda/\kappa - V(x)]/4\}^{-1/4} \cos \left[ \frac{2}{3\kappa\hbar} \int_{x_0}^x dz \sqrt{3\kappa[\Lambda/\kappa - V(z)]} - \frac{\pi}{4} \right], & x_0 < x, \end{cases} \quad (3.30)$$

and

$$\bar{\Psi}(x) \approx \begin{cases} -\{3\kappa[V(x) - \Lambda/\kappa]/4\}^{-1/4} \exp \left[ \frac{2}{3\kappa\hbar} \int_x^{x_0} dz \sqrt{3\kappa[V(z) - \Lambda/\kappa]} \right], & x < x_0, \\ \{3\kappa[\Lambda/\kappa - V(x)]/4\}^{-1/4} \sin \left[ \frac{2}{3\kappa\hbar} \int_{x_0}^x dz \sqrt{3\kappa[\Lambda/\kappa - V(z)]} - \frac{\pi}{4} \right], & x_0 < x. \end{cases} \quad (3.31)$$

The WKB solution  $\Psi_H$  is real and exponentially increasing in the region  $x < x_0$ ; it is identified as the semiclassical Hartle-Hawking wave function. The semiclassical tunneling wave function is the linear combination

$$\Psi_T(x) = \Psi_H(x) - i\bar{\Psi}(x), \quad (3.32)$$

which contains only an outgoing wave in the region  $x_0 < x$ . Using these semiclassical wave functions, the right-hand side of Eq. (3.28) can be constructed, yielding

*precisely* expressions (3.25)–(3.27) for the semiclassical Green function derived from a Lorentzian path integral.

The result (3.28) is evidently valid beyond the semiclassical approximation, and is in fact exact in the following sense. The WKB solutions to the Wheeler-DeWitt equation (3.29) are most accurate for large  $x$ , and become exact as  $x \rightarrow \infty$ . Also, exact solutions to the Wheeler-DeWitt equation can be chosen to be real. (This is another consequence of time-reversal invariance.<sup>24</sup>) Then let us *define* the Hartle-Hawking wave function of the Universe

$\Psi_H$  as the real exact solution which approaches the WKB solution (3.30) asymptotically. Likewise, let  $\Psi$  be the real exact solution with asymptotic form (3.31), and define the tunneling wave function of the Universe by Eq. (3.32). It is now straightforward to verify that the right-hand side of Eq. (3.28) is an exact Green function: Applying the Wheeler-DeWitt operator yields

$$(3\kappa\hbar^2/4) \left[ \frac{\partial\Psi_T(x')}{\partial x'} \Psi_H(x') - \frac{\partial\Psi_H(x')}{\partial x'} \Psi_T(x') \right] \delta(x'', x'). \quad (3.33)$$

The term in large parentheses is a constant in  $x'$ , as shown by differentiation; its value is determined to be  $-4i/(3\kappa\hbar)$  from the asymptotic forms for  $\Psi_H$  and  $\Psi_T$ . So the Wheeler-DeWitt operator acting on the right-hand side of Eq. (3.28) gives  $-i\hbar\delta(x'', x')$ , showing that it is indeed a Green function. Furthermore, this Green function is the same Green function as defined by the path integral, since the two agree asymptotically.

Finally, observe that by Eq. (3.32) the real part of the tunneling wave function equals the Hartle-Hawking wave function,  $\text{Re}\Psi_T = \Psi_H$ . Then the real part of the Green function is

$$\text{Re}G(x'', x') = \Psi_H(x'')\Psi_H(x'), \quad (3.34)$$

a product of Hartle-Hawking wave functions.

#### IV. DISCUSSION

The Lorentzian path integral for quantum cosmology does not suffer from any obvious divergences, as does a sum over Euclidean geometries. For this reason, and because the physical Universe is apparently Lorentzian, we have advocated a definition of the path integral as a sum over Lorentzian geometries. A further consequence of defining the path integral as a sum over just four-geometries—that is, histories with only positive proper time separation between initial and final configurations—is that the path integral is a Green function for the Wheeler-DeWitt operator, not a wave function. We have shown in general that for time-reversal-invariant systems, this Green function is symmetric in two arguments, and that its real part is a wave function in each argument. As an example, the Lorentzian path integral for the de Sitter model of cosmology was computed explicitly, and shown to be related to the Hartle-Hawking and tunneling wave functions of the Universe according to Eqs. (3.28) and (3.34). Here, the Hartle-Hawking and tunneling wave functions were defined as solutions to the Wheeler-DeWitt equation with appropriate semiclassical behavior.

The de Sitter minisuperspace model of cosmology has been previously studied by Halliwell and Louko.<sup>17</sup> There are numerous technical differences between our analysis and theirs. For example, Halliwell and Louko choose a metric ansatz that differs from our Eq. (3.1), and this leads to a different factor ordering in the Wheeler-DeWitt operator and corresponding differences in the wave functions and Green functions. Also, they obtain solutions and Green functions that are exact to within multiplica-

tive constants, while our explicit calculations are restricted to a semiclassical approximation. Although exact results are always nice, the semiclassical approach does have certain conceptual advantages, and has allowed us to recognize the exact relationships (3.28) and (3.34).

The most important difference between the present work and the work of Halliwell and Louko is the treatment of integration contours. Following a suggestion made by Hartle,<sup>27</sup> Halliwell and Louko propose to define the path integral as a sum over complex metrics (and complex matter fields), where the contour is taken along a steepest-descent path. Although their motivation for choosing a complex contour is to obtain a convergent path integral, such an approach is perhaps unnecessary—we find no mathematical difficulty or inconsistency in defining the path integral for the de Sitter model as a sum over Lorentzian four-geometries. Another consequence of using complex contours is that every physical system would have its own individual set of steepest-descent paths, so there could be no general definition of the path integral as a sum over some specific class of geometries. In contrast, we believe a reasonable approach is to work towards a more general definition of the path integral, preferably as a sum over Lorentzian geometries (and real matter fields). The advantage is that integration contours are uniquely and physically determined.

A practical difficulty with using steepest-descent contours to define the path integral is that, thus far, the mathematical basis for this idea is not understood, not even at a formal level. In the de Sitter model, Halliwell and Louko consider complex contours only for the single integral over the lapse (in our notation, the integral over spacetime volume), but do not consider complex contours for the functional integral over canonical variables (in our language, the functional integral for the kernel). As a result, they must reply on scaling arguments that fix the form of the functional integral only to within an overall constant. Another feature of using steepest-descent contours is that there may be no simple relationship between the path integral with coordinates fixed at the end points and the path integral with momenta fixed at the end points.

At this point, it is worthwhile recalling that in our calculations, the geometries that extremize the phase in the path integral are Euclidean whenever the initial and final three-volumes are smaller than the minimum de Sitter space three-volume  $x_0 = 2\pi^2(3/\Lambda)^{3/2}$ . But a Euclidean four-geometry has imaginary four-volume. So in order to evaluate the Lorentzian path integral semiclassically, the integration contour for spacetime volume can be distorted from the real line into the complex plane to pass through the imaginary stationary points. This is mathematics, not physics: the complex contour is used here only as a device to obtain an approximation to an integral over real values of spacetime volume. There is no physical content in this technique—we do not use steepest-descent contours to define the path integral.

In this paper we have discussed the definition of the path integral for quantum cosmology, and evaluated the path integral for a simple model. But there are crucial

questions that we have not addressed: What is the meaning of the path integral? How is it used? As a Green function for the Wheeler-DeWitt equation, the path integral is analogous to a Green function for the time-independent Schrödinger equation. For unbounded potentials in nonrelativistic quantum mechanics, that Green function seems to have little use beyond perturbation calculations in scattering theory.<sup>24</sup> The real part of the path integral, which solves the Wheeler-DeWitt equation, may be the more useful object, assuming it plays a role analogous to the (time-dependent or time-independent) wave functions of nonrelativistic quantum mechanics. However, in both cases, we are faced with problems of interpretation.

As a final observation, it is possible that in quantum cosmology the kernel is physically fundamental and is a more useful object than the Green function. As mentioned in Sec. II, the kernel is given by a path integral in which spacetime volume is fixed, but not the cosmological constant. Indeed, in minisuperspace models there is a motivation for focusing on the kernel rather than the Green function: the kernel is mathematically like the transition amplitude of nonrelativistic quantum mechanics, and unlike the Green function it obeys the usual composition law. Unfortunately, this attractive property is an artifact of the minisuperspace reduction, and will not hold in a full field theory of quantum cosmology.

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#### APPENDIX: CONVERGENCE

For the de Sitter minisuperspace model of cosmology, the following simple arguments suggest that the Lorentzian path integral is well defined. The kernel (2.4b) can be evaluated in principle as the infinite limit of a finite number of integrations through a time-slicing procedure.<sup>28</sup> Then the integrals over  $p$  at each time  $t$  are well defined as Fresnel (Gaussian) integrals, and the resulting exponent is  $(i/\hbar)$  times the discretized version of the Lagrangian action. In turn, the integrations over  $x$  for each time are well defined: At large  $x$ , the  $x^{-2/3}$  potential is suppressed in the action, and each  $x(t)$  integral is seen to converge as  $x \rightarrow \infty$  as a Fresnel integral. At small  $x$ , the potential term dominates the action and a change of variables from  $x$  to  $x^{-1/3}$  shows that the integrals converge as  $x^{-1/3} \rightarrow \infty$  ( $x \rightarrow 0$ ) as well. Therefore, assuming the limit of infinitely small time steps is convergent, the kernel is well defined. By the same criteria outlined above, the kernel for a sum over Euclidean histories diverges.

The full path integral (2.4) includes an integration over spacetime volume  $T$ . The convergence of this integral depends on the behavior of the kernel for small and large  $T$ . First consider small values of  $T$ . In this case, the kernel

can be evaluated approximately using the semiclassical methods of Sec. III. The result is a sum, with appropriate phases, of two single-history contributions as in Eqs. (3.14)–(3.17)—one contribution from a direct classical path and the other from a reflected classical path. For the direct path, the “energy”  $\bar{\Lambda}/\kappa$  is large and the potential is unimportant. Then the kernel (3.17) assumes the form, for a “free” universe,<sup>5</sup>

$$K_{\text{free}}(x'', T|x', 0) \sim T^{-1/2} \exp[-i(x'' - x')^2 / (3\kappa\hbar T)], \quad (\text{A1})$$

in the limit  $T \rightarrow 0$ . Likewise, direct calculation shows that as  $T \rightarrow 0$  the single-history kernel for the reflected classical path has this same form, but with  $x'' - x'$  replaced by  $x'' + x'$ . For each of these single-history terms, the  $T$  integral in Eq. (2.4a) for the Green function converges at  $T = 0$ ; this may be seen by changing variables from  $T$  to  $T^{-1/2}$  and comparing the result with a Fresnel integral.

For large  $T$ , notice first that the “free” kernel (A1) tends to zero like  $T^{-1/2}$ . This can be understood physically by viewing the kernel as a wave function. Using the language of nonrelativistic quantum mechanics, the kernel is a time- ( $T$ -) dependent wave function in  $x''$  for a particle that is initially localized at  $x'$  at  $T = 0$ . In time, this wave packet spreads: as  $T$  increases the magnitude of the wave function at  $x'' \neq x'$  initially increases, but at late times it falls to zero as the wave function spreads to plus and minus infinity. Evidently, this behavior will also occur for the kernel (2.4b), which describes a “particle” in a potential  $V(x)$ . In this case, the kernel may not tend to zero as quickly as does the “free” kernel, because the wave packet can only spread to  $x = +\infty$  and not to  $x = -\infty$ . However, that difference should only change the asymptotic behavior of the kernel by a factor of 2. [On the other hand, the presence of the potential (3.7) will expedite the vanishing of the kernel by “pushing” the wave packet to infinity.] The conclusion is that the kernel (2.4b) tends to zero at large  $T$  at least as quickly as  $T^{-1/2}$ . Then the integral over  $T$  for the Green function is seen to converge at the upper limit of integration. Consequently, the Lorentzian path integral for the de Sitter model converges.

As a final comment, it is important to recognize that for  $d$ -dimensional minisuperspace models, the kernel will consist of sums of “free” kernels as  $T$  approaches zero. In this case, the contribution to the sum corresponding to, say, a direct classical path will have the form of Eq. (A1), but where  $(x'' - x')^2$  is a sum of squares and the factor  $T^{-1/2}$  is replaced by  $T^{-d/2}$ . For  $d \geq 4$ , the integral over  $T$  near zero converges only when the kernel is viewed as the boundary value of a function that is analytic in the lower half complex  $(x'' - x')^2$  plane.<sup>29</sup> This same situation arises both for the proper time representation of the Feynman Green function for the relativistic point particle, and also for the Green function (2.4a) for nonrelativistic mechanical systems, whenever the dimension is four or more. We hope to explore this issue in a future publication.

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