Wake fields in a dielectric-lined waveguide

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A cylindrical waveguide of radius a is filled partially with an isotropic dielectric between radii $b<r<a$. A particle travels along the guide, possibly with an offset from the axis, at a velocity $v<c$. The wake fields left behind are calculated. We find that the transverse wake forces do not vanish as $\gamma^{-2}=1-(v/c)^2$ or in any other way when $v\to c$. The longitudinal and transverse wake forces are evaluated as a function of b/a.

I. INTRODUCTION

Wake-field accelerators[1-3] are of great interest because of their potential for providing a very high acceleration gradient for the next generation of accelerators. Best of all, it was shown experimentally[4] that transverse deflections appeared to be small for dielectric-lined waveguides in contrast with the large transverse wake forces measured in structures and plasmas.[5,6] This is important because there will be small beam breakup,[7] which is a traditional source of beam instabilities in linear accelerators. A complete understanding of the transverse wake potential in such a dielectric-lined waveguide is therefore necessary. Recently, there have been some suggestions[8] that the transverse wake forces may vanish as γ^{-2} when the velocity v of the source particle approaches the velocity of light c, $[\gamma^{-2}=1-(v/c)^2]$. The waveguide considered consists of a cylindrical metallic tube of radius a with infinite wall conductivity. The tube is filled partially with an isotropic material with dielectric constant ϵ between radii $b<r<a$. Our derivation of the exact solution of the wake fields shows that the transverse wake forces do not vanish when $v\to c$.

II. SOLUTION

One way to solve for the wake fields is through the introduction of a scalar potential ϕ and a vector potential $\mathbf{A}$. In the Lorentz gauge, Maxwell's equations reduce to wave equations for ϕ and $\mathbf{A}$. The mathematics is rather complicated because the equations in $\mathbf{A}$ are coupled in the cylindrical coordinate.[9] The details are given in the Appendix.

It is well known[10] that the transverse electric fields $\mathbf{E}_t$ and magnetic flux density $\mathbf{B}_t$ in a waveguide can always be expressed in terms of the longitudinal components E_z and B_z. In Gaussian units, these relations are

$$\left[\nabla_z^2-\frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{B}_t=-\frac{i\omega\mu\epsilon}{c}\nabla_t\times\hat{\mathbf{z}}E_z+\nabla_t\nabla_z B_z\ ,$$
$$\left[\nabla_z^2-\frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{E}_t=\frac{i\omega}{c}\nabla_t\times\hat{\mathbf{z}}B_z+\nabla_t\nabla_z E_z\ , \tag{2.1}$$

where μ and ϵ are, respectively, the relative magnetic permeability and dielectric constant of the medium under consideration. In the presence of the dielectric, E_z and B_z are no longer independent. Thus there are two variables E_z and B_z to solve for. The problem is therefore much simpler than working with potentials, where there are four unknowns.

The source particle carrying charge q travels with velocity $v=\beta c$ along the cylindrical waveguide at an offset r_0 from its axis. The Maxwell's equations for longitudinal fields are

$$\left[\nabla^2-\frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\right]E_z=\frac{4\pi}{\epsilon}\frac{\partial\rho}{\partial z}+\frac{4\pi\mu}{c^2}\frac{\partial J_z}{\partial t}\ , \tag{2.2}$$

$$\left[\nabla^2-\frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\right]B_z=0\ . \tag{2.3}$$

The charge density and current density are represented by, respectively,

$$\rho=q\frac{\delta(r-r_0)}{r}\delta(\theta)\delta(z-vt)\ , \tag{2.4}$$

$$J_z=v\rho\ . \tag{2.5}$$

The test particle to be accelerated by the wake of the source particle travels with essentially the same velocity v. Thus, we seek here only those solutions that are functions of $z-vt$.

Note that all the above quantities are functions of (r,θ,z,t). Let us denote the Fourier transforms in the variables $(z-vt)$ and θ by a tilde, i.e.,

$$E_z(r,\theta,z,t)=\sum_{m=-\infty}^{\infty}e^{im\theta}\int_{-\infty}^{\infty}d\omega\, e^{i(z-vt)\omega/v}\tilde{E}_{zm}(r,\omega)\ . \tag{2.6}$$

With the help of

$$\delta(z-vt)=\frac{1}{2\pi v}\int_{-\infty}^{\infty}d\omega\, e^{i(z-vt)\omega/v}\ , \tag{2.7}$$

and

$$\frac{\delta(r-r_0)}{r}\delta(\theta)=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}e^{im\theta}\int_0^{\infty}k\,dk\,J_m(kr)J_m(kr_0)\ , \tag{2.8}$$

where J_m is the Bessel function of order m, Eq. (2.2) can be rewritten as

$$\left[\nabla^2+\frac{\mu\epsilon\omega^2}{c^2}\right]\tilde{E}_{zm}(r,\omega)=\tilde{\phi}_m(r,\omega)\ , \tag{2.9}$$

where

$$\tilde{\phi}_m(r,\omega)=\frac{i4\pi\omega}{v\epsilon}\left[\tilde{\rho}_m-\frac{\mu\epsilon v}{c^2}\tilde{J}_m\right]$$
$$=\frac{iq\omega(1-\mu\epsilon\beta^2)}{\pi v^2\epsilon}\int_0^\infty k\,dk\,J_m(kr)J_m(kr_0)\ . \tag{2.10}$$

The particular solution can then be obtained easily as

$$\tilde{E}_{zm}^{\text{part}}=-\frac{iq\omega}{\pi v^2\gamma^2}\int_0^\infty dk\frac{kJ_m(kr)J_m(kr_0)}{k^2+(\omega/v\gamma)^2},\quad 0<r<b\ , \tag{2.11}$$

where μ and ϵ have been dropped because the particle is traveling in the vacuum sector, and $(1-v^2/c^2)$ has been replaced by γ^{-2}. The integration over k can be done exactly to give

$$\tilde{E}_{zm}^{\text{part}}=-\frac{iq\omega}{\pi v^2\gamma^2}\begin{cases}I_m(\omega r/v\gamma)K_m(\omega r_0/v\gamma), & r<r_0\ ,\\ K_m(\omega r/v\gamma)I_m(\omega r_0/v\gamma), & b>r>r_0\ ,\end{cases} \tag{2.12}$$

where I_m and K_m are, respectively, modified Bessel function and Hankel function of order m.

For the general solution we have

$$\tilde{E}_{zm}^{\text{gen}}=\begin{cases}\mathcal{E}_m I_m(kr_0)I_m(kr), & 0\le r\le b\ ,\\ A_m[J_m(sa)Y_m(sr)-Y_m(sa)J_m(sr)], & b\le r\le a\ ,\end{cases} \tag{2.13}$$

where

$$k=\frac{\omega}{v}\sqrt{1-\beta^2}\quad\text{and}\quad s=\frac{\omega}{v}\sqrt{\mu\epsilon\beta^2-1}\ , \tag{2.14}$$

and Y_m is the Neumann function of order m. We have assumed that the dielectric constant is large enough so that $\mu\epsilon\beta^2>1$. Otherwise, there will not be any Cherenkov radiation produced and as a result there will not be any useful wake potential aside from space charge. In Eq. (2.14) the constants have been so chosen that E_z vanishes on the wall of the guide.

From Eq. (2.3), B_z can also be solved

$$\tilde{B}_{zm}^{\text{gen}}=\begin{cases}\mathcal{B}_m I_m(kr_0)I_m(kr), & 0\le r\le b\ ,\\ C_m[J'_m(sa)Y_m(sr)-Y'_m(sa)J_m(sr)], & b\le r\le a\ ,\end{cases} \tag{2.15}$$

where arrangement has been made so that the radial component of $\tilde{\mathbf{B}}_m$ vanishes at the wall of the guide.

The four constants $\mathcal{E}_m$, $\mathcal{B}_m$, A_m, and C_m will be determined by matching boundary conditions at $r=b$, where the vacuum meets the dielectric. At $r=b$, E_z and B_z are

$$\begin{aligned}&\tilde{E}_{zm}^{v}=\mathcal{E}'_m I_m+\eta'_m K_m,\quad \tilde{E}_{zm}^{d}=A_m p_m\ ,\\ &\tilde{B}_{zm}^{v}=\mathcal{B}'_m I_m,\quad \tilde{B}_{zm}^{d}=C_m r_m\ ,\end{aligned} \tag{2.16}$$

where the superscripts v and d denote vacuum and dielectric, respectively. In the above the following abbreviations have been used:

$$\begin{aligned}&I_m=I_m(kb),\quad K_m=K_m(kb)\ ,\\ &p_m=J_m(sa)Y_m(sb)-Y_m(sa)J_m(sb)\ ,\\ &r_m=J'_m(sa)Y_m(sb)-Y'_m(sa)J_m(sb)\ ,\\ &\eta'_m=\eta\frac{I_m(kr_0)}{\gamma^2},\quad \eta=-\frac{iq\omega}{\pi v^2}\ ,\\ &\mathcal{E}'_m=\mathcal{E}_m I_m(kr_0),\quad \mathcal{B}'_m=\mathcal{B}_m I_m(kr_0)\ .\end{aligned} \tag{2.17}$$

Knowing E_z and B_z, the transverse electric field can now be computed directly using Eq. (2.1). At the boundary $r=b$, these transverse components are

$$\begin{aligned}&\tilde{E}_{\theta m}^{v}=\frac{iv\beta k}{\omega(1-\beta^2)}\mathcal{B}'_m I'_m+\frac{mv}{\omega b(1-\beta^2)}(\mathcal{E}'_m I_m+\eta'_m K_m)\ ,\\ &\tilde{E}_{\theta m}^{d}=-\frac{iv\beta s}{\omega(\mu\epsilon\beta^2-1)}C_m r'_m-\frac{mv}{\omega b(\mu\epsilon\beta^2-1)}A_m p_m\ ;\end{aligned} \tag{2.18}$$

$$\begin{aligned}&\tilde{E}_{rm}^{v}=\frac{mv\beta}{\omega b(1-\beta^2)}\mathcal{B}'_m I_m-\frac{ivk}{\omega(1-\beta^2)}(\mathcal{E}'_m I'_m+\eta'_m K'_m)\ ,\\ &\tilde{E}_{rm}^{d}=-\frac{mv\beta}{\omega b(\mu\epsilon\beta^2-1)}C_m r_m+\frac{ivs}{\omega(\mu\epsilon\beta^2-1)}A_m p'_m\ ;\end{aligned} \tag{2.19}$$

where

$$\begin{aligned}&I'_m=I'_m(kb),\quad K'_m=K'_m(kb)\ ,\\ &p'_m=J_m(sa)Y'_m(sb)-Y_m(sa)J'_m(sb)\ ,\\ &r'_m=J'_m(sa)Y'_m(sb)-Y'_m(sa)J'_m(sb)\ .\end{aligned} \tag{2.20}$$

All polarization charges and currents have been taken care of by the macroscopic magnetic permeability μ and dielectric constant ϵ. Therefore, there should be no surface charge or current at the boundary $r=b$. Thus we expect

$$\begin{aligned}&\tilde{E}_{zm}^{d}=\tilde{E}_{zm}^{v},\quad \tilde{E}_{\theta m}^{d}=\tilde{E}_{\theta m}^{v}\ ,\\ &\epsilon\tilde{E}_{rm}^{d}=\tilde{E}_{rm}^{v},\quad \tilde{B}_{zm}^{d}=\mu\tilde{B}_{zm}^{v}\ .\end{aligned} \tag{2.21}$$

Then, the boundary conditions for B_θ and B_r will be satisfied automatically (since we have only four constants here). The four equations obtained from Eq. (2.21) are

$$A_m p_m=\mathcal{E}'_m I_m+\eta'_m K_m\ , \tag{2.22}$$

$$-\frac{i\beta s}{\mu\epsilon\beta^2-1}C_m r'_m-\frac{m}{b(\mu\epsilon\beta^2-1)}A_m p_m=i\beta k\gamma^2\mathcal{B}'_m I'_m+\frac{m\gamma^2}{b}(\mathcal{E}'_m I_m+\eta'_m K_m)\ , \tag{2.23}$$

$$-\frac{m\beta\epsilon}{b(\mu\epsilon\beta^2-1)}C_m r_m+\frac{i\epsilon s}{\mu\epsilon\beta^2-1}A_m p'_m=\frac{m\beta\gamma^2}{b}\mathcal{B}'_m I_m-ik\gamma^2(\mathcal{E}'_m I'_m+\eta'_m K'_m)\ , \tag{2.24}$$

$$C_m r_m = \mu \mathcal{B}'_m I_m \ . \quad (2.25)$$

We solve for A_m from Eq. (2.22) and C_m from Eq. (2.25). Substituting into Eqs. (2.23) and (2.24) we get two equations for the two unknowns $\mathcal{E}'_m$ and $\mathcal{B}'_m$, the general solutions of the longitudinal electric field and magnetic flux density, respectively,

$$\begin{bmatrix} (\mu\epsilon\beta^2-1)kbI'_m + \frac{\epsilon sb}{\gamma^2}\frac{p'_m I_m}{p_m} & m\beta(\mu\epsilon-1)I_m \\ m\beta(\mu\epsilon-1)I_m & (\mu\epsilon\beta^2-1)kbI'_m + \frac{\mu sb}{\gamma^2}\frac{r'_m I_m}{r_m} \end{bmatrix} \begin{bmatrix} \mathcal{E}'_m \\ i\mathcal{B}'_m \end{bmatrix} = -\eta'_m \begin{bmatrix} (\mu\epsilon\beta^2-1)kbK'_m + \frac{\epsilon sb}{\gamma^2}\frac{p'_m K_m}{p_m} \\ m\beta(\mu\epsilon-1)K_m \end{bmatrix} . \quad (2.26)$$

III. THE MONOPOLE FIELDS

For the monopole case or $m=0$, the square matrix on the left-hand side of Eq. (2.26) is diagonal and the lower element of the right-hand matrix vanishes. We obtain immediately $\mathcal{B}'_0=0$; or there is no longitudinal magnetic field. The longitudinal electric field for the region $r_0 \le r \le b$ is

$$\tilde{E}_{z0} = \mathcal{E}'_0 I_0(kr) + \eta'_0 K_0(kr) \ , \quad (3.1)$$

where

$$\mathcal{E}'_0 = -\eta'_0 \frac{(\mu\epsilon\beta^2-1)kK'_0 + \gamma^{-2}\epsilon s p'_0 K_0/p_0}{(\mu\epsilon\beta^2-1)kI'_0 + \gamma^{-2}\epsilon s p'_0 I_0/p_0} \ . \quad (3.2)$$

For $x \ll 1$,

$$I_0(x) \to 1, \quad xI'_0(x) \to \frac{x^2}{2} \ , \qquad K_0(x) \to -\ln\frac{|x|}{2}, \quad xK'_0(x) \to -1 \ . \quad (3.3)$$

Therefore, when $\gamma \gg \omega b/c$, $\tilde{E}_{z0}$ becomes

$$\tilde{E}_{z0} = \eta I_0(kr_0) \left[\frac{(\mu\epsilon-1)p_0}{p'_0 + (sb/2\epsilon)p_0} \frac{I_0(kr)}{\epsilon sb} - \frac{K_0(kr)}{\gamma^2} \right] , \quad (3.4)$$

where η is given in Eq. (2.17).

The transverse forces on the test charge e traveling with velocity v behind the source can be obtained from the Panofsky-Wenzel theorem,[11] and are related to the longitudinal electric field by

$$\tilde{F}_{rm} = \frac{ev}{i\omega}\frac{\partial \tilde{E}_{zm}}{\partial r} \ , \quad (3.5)$$

$$\tilde{F}_{\theta m} = \frac{emv}{i\omega r}\tilde{E}_{zm} \ . \quad (3.6)$$

Although $\tilde{F}_{\theta 0}=0$, it is evident that the radial transverse force $\tilde{F}_{r0}$ is not zero since $\tilde{E}_{z0}$ clearly depends on r through $I_0(kr)$ and $K_0(kr)$. However, since $k=\omega/\gamma v$, this dependence is very small at large γ. In fact, it can be easily shown that when $\gamma \gg \omega r/c$,

$$\tilde{F}_{r0} = \gamma^{-2}(\cdots) \quad (3.7)$$

in the region $r_0 \le r \le b$ and the expression inside the parentheses is γ independent. This reminds us of the behavior of the space-charge forces which also go to zero as γ^{-2}.

However, in the presence of a dielectric lining, the longitudinal electric field tends to a nonzero limit as $\gamma \to \infty$. Using Eqs. (2.6) and (3.4) we obtain

$$E_{z0}(r,z,t) = -\frac{ie\sqrt{\mu\epsilon-1}}{\pi\epsilon bc} \times \int_{-\infty}^{\infty} d\omega\, e^{i\omega(z-ct)/c} \frac{p_0}{p'_0 + (sb/2\epsilon)p_0} \ , \quad (3.8)$$

with $s=\omega\sqrt{\mu\epsilon-1}/c$. We next change the variable of integration to $x=sa$ and integrate in the complex x plane. To satisfy causality, the poles of the integrand are placed slightly below the real x axis. Since p_0 is even and $p'_0+(sb/2\epsilon)p_0$ is odd, we obtain, for $z<ct$,

$$E_{z0}(r,z,t) = -\frac{4q}{\epsilon ab}\sum_\lambda \frac{xp_0}{(d/dx)\mathcal{D}_0(x)} \times \cos\frac{x(z-ct)}{a\sqrt{\mu\epsilon-1}} \Bigg|_{x=x_\lambda} \ , \quad (3.9)$$

where x_λ is the λth *positive* zero of the analytic function

$$\mathcal{D}_0(x) = xp'_0 + \frac{x^2\xi}{2\epsilon}p_0 \ . \quad (3.10)$$

Here $\xi=b/a$ is the ratio of the inner radius to the outer radius of the dielectric. This result is in complete agreement with that of Gai.[8] In the event that $\mathcal{D}_0(x)$ is not analytic, it can be made analytic by the multiplication of x to an appropriate power. Needless to say, we have to multiply the numerator of Eq. (3.9) by the same power of x. The same comment applies also to $\mathcal{D}_m(x)$ in Eq. (4.11) below.

Lots of physics are embedded in Eq. (3.9). Cherenkov radiation is produced inside the dielectric at an angle $\arcsin(1/\sqrt{\mu\epsilon})$ with the axis of the guide. Because the velocity of light in the dielectric $\hat{c}=c/\sqrt{\mu\epsilon}$ is less than the velocity of the source particle, this radiation bounces back and forth inside the dielectric layer and penetrates into the central vacuum region of the guide, creating a wake potential lagging behind. It is this potential that we hope would provide the required acceleration on other particles.

IV. HIGHER-MULTIPOLE FIELDS

For the higher multipole, i.e., $m \neq 0$, the longitudinal magnetic flux density is no longer zero. The square matrix in Eq. (2.26) is not diagonal, contrary to what Gai found.[8] However, both $\mathcal{E}'_m$ and $\mathcal{B}'_m$ can be solved easily. The result will be different from that of Gai.

When $m\neq 0$, Eq. (2.26) can be rewritten as

$$\begin{bmatrix} \dfrac{kbI'_m}{mI_m}+\dfrac{\epsilon k^2 b}{s}\dfrac{p'_m}{mp_m} & \dfrac{\beta(\mu\epsilon-1)}{\mu\epsilon\beta^2-1} \\ \dfrac{\beta(\mu\epsilon-1)}{\mu\epsilon\beta^2-1} & \dfrac{kbI'_m}{mI_m}+\dfrac{\mu k^2 b}{s}\dfrac{r'_m}{mr_m} \end{bmatrix} \begin{bmatrix} \mathcal{E}'_m \\ i\mathcal{B}'_m \end{bmatrix} = -\eta'_m \frac{K_m}{I_m} \begin{bmatrix} \dfrac{kbK'_m}{mK_m}+\dfrac{\epsilon k^2 b}{s}\dfrac{p'_m}{mp_m} \\ \dfrac{\beta(\mu\epsilon-1)}{\mu\epsilon\beta^2-1} \end{bmatrix}, \tag{4.1}$$

where we have used the relation $(s/k)^2=\gamma^2(\mu\epsilon\beta^2-1)$. Employing the small-argument expansions of the modified Bessel functions for $m\neq 0$

$$I_m(x)=\frac{1}{m!}\left[\frac{x}{2}\right]^m\left[1+\frac{1}{m+1}\left[\frac{x}{2}\right]^2\right],$$

$$xI'_m(x)=\frac{1}{(m-1)!}\left[\frac{x}{2}\right]^m\left[1+\frac{m+2}{m(m+1)}\left[\frac{x}{2}\right]^2\right],$$

$$K_m(x)=\frac{(m-1)!}{2}\left[\frac{x}{2}\right]^{-m}\left[1+O\left[\frac{x}{2}\right]^2\right], \tag{4.2}$$

$$xK'_m(x)=-\frac{m!}{2}\left[\frac{x}{2}\right]^{-m}\left[1+O\left[\frac{x}{2}\right]^2\right],$$

we obtain, when $\gamma \gg \omega b/c$,

$$\frac{kbI'_m}{mI_m}=1+O(\gamma^{-2}) \quad\text{and}\quad \frac{kbK'_m}{mK_m}=-1+O(\gamma^{-2}). \tag{4.3}$$

Therefore, Eq. (4.1) for $\mathcal{E}'_m$ and $\mathcal{B}'_m$ becomes

$$\begin{bmatrix} 1+a_1 & 1+a_2 \\ 1+a_2 & 1+a_3 \end{bmatrix}\begin{bmatrix} \mathcal{E}'_m \\ i\mathcal{B}'_m \end{bmatrix} = -\eta'_m\frac{K_m}{I_m}\begin{bmatrix} -1+a_4 \\ 1+a_2 \end{bmatrix}, \tag{4.4}$$

where a_1, a_2, a_3, and a_4 are $O(\gamma^{-2})$. The 1's in the above matrix elements and the signs before them are extremely important. The four 1's in the square matrix lead to a near cancellation of the determinant, leaving behind

$$\det=(a_1-2a_2+a_3)+(a_1a_3-2a_2^2), \tag{4.5}$$

which is $O(\gamma^{-2})$. The -1 and $+1$ in the right-hand matrix, on the other hand, add when solving for $\mathcal{E}'_m$ and $\mathcal{B}'_m$. Keeping the lowest-order contribution we obtain simply

$$\mathcal{E}'_m=\frac{2\eta'_m}{\det}\frac{K_m}{I_m}. \tag{4.6}$$

When $\gamma\to\infty$, the corresponding longitudinal electric field is therefore

$$\tilde{E}_{zm}=\frac{2\eta}{D(s)}\frac{K_m}{I_m}I_m(kr_0)I_m(kr) \to \frac{i2e\omega}{\pi c^2}\frac{1}{mD(s)}\left[\frac{r_0}{b}\right]^m\left[\frac{r}{b}\right]^m, \tag{4.7}$$

where

$$D(s)=\lim_{\gamma\to\infty}(\gamma^2\det) \tag{4.8}$$

is independent of γ. We see clearly that $\tilde{E}_{zm}$ and therefore the transverse forces do not vanish in the limit $\gamma\to\infty$.

Now, let us evaluate the determinant. Using the small-argument expansions of the modified Bessel functions in Eq. (4.2) we get

$$\gamma^2 a_1=\frac{1}{\mu\epsilon-1}\left[\frac{s^2b^2}{2m(m+1)}+\frac{\epsilon sbp'_m}{mp_m}\right],$$

$$\gamma^2 a_2=\frac{\mu\epsilon+1}{2(\mu\epsilon-1)}, \tag{4.9}$$

$$\gamma^2 a_3=\frac{1}{\mu\epsilon-1}\left[\frac{s^2b^2}{2m(m+1)}+\frac{\mu sbr'_m}{mr_m}\right].$$

From Eqs. (4.5) and (4.8) we have

$$D(s)=\frac{1}{\mu\epsilon-1}\left[\frac{s^2b^2}{m(m+1)}+\frac{\epsilon sbp'_m}{mp_m}+\frac{\mu sbr'_m}{mr_m}-(\mu\epsilon+1)\right]. \tag{4.10}$$

Substitute the results in Eq. (4.7) and then Eq. (2.6). The integration is then performed in the complex $x=sa$ plane to obtain, for $z<ct$,

$$E_{zm}(r,z,t)=\frac{8q}{a^2}\left[\frac{r_0}{b}\right]^m\left[\frac{r}{b}\right]^m \times\sum_\lambda \frac{xp_m r_m}{(d/dx)\mathcal{D}_m(x)}\cos\frac{x(z-ct)}{a\sqrt{\mu\epsilon-1}}\Bigg|_{x=x_\lambda}, \tag{4.11}$$

where x_λ is the λth *positive* zero of the analytic function

$$\mathcal{D}_m(x)=\left[\frac{x^2\xi^2}{m+1}-m(\mu\epsilon+1)\right]p_m r_m + x\xi(\epsilon p'_m r_m+\mu r'_m p_m). \tag{4.12}$$

Here $\xi=b/a$ is the ratio of the inner radius to the outer radius of the dielectric. Thus, for $m\neq 0$, E_{zm} does not vanish when $\gamma\to\infty$. The transverse wake forces can be obtained readily by Eqs. (3.5) and (3.6), and they also do not vanish as $\gamma\to\infty$.

V. EVALUATION OF WAKE FORCES

Knowing that higher-order transverse wake forces ($m\geq 1$) do not vanish in the limit $\gamma\to\infty$, we would like to determine their sizes relative to the $m=0$ longitudinal wake force. With the aid of Eq. (3.9), the $m=0$ longitudinal force on the test particle carrying charge e at a distance z behind the source particle can be written as

$$F_{z0}(z)=-\frac{eq}{a^2}\sum_{\lambda}\hat{F}_{z0\lambda}(x_{0\lambda})\cos\frac{x_{0\lambda}z}{a\sqrt{\epsilon-1}}\,, \tag{5.1}$$

where

$$\hat{F}_{z0\lambda}=\frac{4}{\epsilon\xi}\frac{x_{0\lambda}p_0(x_{0\lambda})}{\mathcal{D}_0'(x_{0\lambda})}\,, \tag{5.2}$$

and $x_{0\lambda}$ is the λth zero of $\mathcal{D}_0$. Similarly, with the aid of Eqs. (4.11) and (3.5), the $m\geq 1$ transverse force can be written as

$$F_{rm}(r,z;r_0)=\frac{eq}{a^2}\left[\frac{r_0}{a}\right]^m\left[\frac{r}{a}\right]^{m-1}\times\sum_{\lambda}\hat{F}_{rm\lambda}(x_{m\lambda})\sin\frac{x_{m\lambda}z}{a\sqrt{\epsilon-1}}\,, \tag{5.3}$$

where

$$\hat{F}_{rm\lambda}=\frac{8m\sqrt{\epsilon-1}}{\xi^{2m}}\frac{p_m(x_{m\lambda})r_m(x_{m\lambda})}{\mathcal{D}_m'(x_{m\lambda})} \tag{5.4}$$

and $x_{m\lambda}$ is the λth zero of $\mathcal{D}_m$. In the above, the analytic functions $\mathcal{D}_0$ and $\mathcal{D}_m$ have been given by Eqs. (3.10) and (4.12). Also the relative magnetic permeability μ of the dielectric has been put equal to unity. Below, the dimensionless *reduced* wake forces $\hat{F}_{z0\lambda}$ and $\hat{F}_{rm\lambda}$ will be evaluated. The zeros $x_{0\lambda}$ and $x_{m\lambda}$ are dimensionless *reduced* eigenfrequencies of the eigenmodes. The true eigenfrequencies are given by $\omega_{m\lambda}=x_{m\lambda}c/a\sqrt{\epsilon-1}$.

A. Thin dielectric lining

Let $a\delta$ denote the thickness of the dielectric lining or $\delta=1-\xi$. Here we consider the situation of a thin lining, i.e., $\delta\ll 1$. In Eqs. (2.17) and (2.20), p_m, p_m', r_m, and r_m' are defined as functions of $x=sa$ and $x\xi=sb=x-x\delta$. When $x\delta\ll 1$ we Taylor expand them up to δ. With the aid of the Wronskian of J_m and Y_m as well as the Bessel equation we obtain

$$p_m(x)=-\frac{2\delta}{\pi},\quad p_m'(x)=\frac{2(1+\delta)}{\pi x}\,,$$
$$r_m(x)=-\frac{2}{\pi x},\quad r_m'(x)=-\frac{2\delta}{\pi}\left[1-\frac{m^2}{x^2}\right]. \tag{5.5}$$

Thus, retaining only the lowest order of δ,

$$\mathcal{D}_0(x)=\frac{2}{\pi}-\frac{x^2\delta}{\pi\epsilon}\,, \tag{5.6}$$

and, for $m\neq 0$,

$$x\mathcal{D}_m(x)=\frac{4\delta}{\pi^2}\left[\frac{x^2}{m+1}-\frac{\epsilon}{\delta}\right]. \tag{5.7}$$

We see that there is only one positive zero in Eq. (5.6) or (5.7), namely,

$$x_{01}=\left[\frac{2\epsilon}{\delta}\right]^{1/2},$$
$$x_{m1}=\left[\frac{(m+1)\epsilon}{\delta}\right]^{1/2},\quad m\neq 0\,. \tag{5.8}$$

This justifies the approximation used to obtain Eqs. (5.5), i.e., $x\delta\ll 1$ when the dielectric is sufficiently thin. The eigenfrequency happens to be the same for the $m=0$ and $m=1$ modes. One can compute easily the reduced wake forces:

$$\hat{F}_{z01}=4\,, \tag{5.9}$$

$$\hat{F}_{rm1}=4m\left[\frac{(\epsilon-1)(m+1)\delta}{\epsilon}\right]^{1/2}. \tag{5.10}$$

The ratio of the reduced forces is

$$\frac{\hat{F}_{rm1}}{\hat{F}_{z01}}=m\sqrt{m+1}\left[\frac{(\epsilon-1)\delta}{\epsilon}\right]^{1/2}. \tag{5.11}$$

The behavior of the limit $\delta\rightarrow 0$ is not intuitive. As $\delta\rightarrow 0$, one expects the absence of the dielectric lining leaving behind a perfectly conducting pipe wall. The electromagnetic fields generated by the source particle are therefore just the ordinary space-charge fields, which we have omitted after setting $\gamma\rightarrow\infty$. However, the longitudinal wake force as shown by Eq. (5.9) does not vanish as $\delta\rightarrow 0$. Thus, an infinitely thin dielectric lining does not imply no dielectric lining. Our evaluation of the wake forces here is based on Eqs. (5.1) and (5.3) or Eqs. (3.9) and (4.11), where $\gamma\gg\omega a/c$ is assumed. With the substitution of Eq. (5.8) this assumption becomes $\gamma\gg\sqrt{2\epsilon/\delta(\epsilon-1)}$. As a result, our calculation cannot lead to the situation of $\delta=0$, or the removal of the dielectric lining.

B. Thick dielectric lining

Here we consider the situation when the inner radius of the dielectric $b=a\xi$ approaches zero. Assuming that $x\xi\ll 1$ we use the small-argument expansions of Bessel functions to obtain

$$p_0(x)=\frac{2}{\pi}\left[\ln\frac{x\xi}{2}+\varepsilon\right]J_0(x)-Y_0(x)\,,$$
$$p_0'(x)=\frac{2}{\pi x\xi}J_0(x)+\frac{x\xi}{2}Y_0(x)\,, \tag{5.12}$$

where $\varepsilon=0.577\,22$ is an Euler number. Retaining only the lowest order in $x\xi$ we get

$$\mathcal{D}_0(x)=\frac{2}{\pi\xi}J_0(x)+\frac{x^2\xi(\epsilon-1)}{2\epsilon}\,. \tag{5.13}$$

Since $x\xi\ll 1$, the λth zero $x_{0\lambda}$ of $\mathcal{D}_0$ should be very close to the λth zero $\bar{x}_{0\lambda}$ of J_0. If we write

$$x_{0\lambda}=\bar{x}_{0\lambda}+\Delta_{0\lambda}\,, \tag{5.14}$$

$$J_0(x_{0\lambda})\approx\Delta_{0\lambda}J_0'(\bar{x}_{0\lambda})=-\Delta_{0\lambda}J_1(\bar{x}_{0\lambda})\,. \tag{5.15}$$

We can then solve from Eq. (5.13) the λth zero of $\mathcal{D}_0$,

$$x_{0\lambda}=\bar{x}_{0\lambda}+\frac{\pi(\epsilon-1)}{\epsilon}\left[\frac{x\xi}{2}\right]^2\frac{Y_0(\bar{x}_{0\lambda})}{J_1(\bar{x}_{0\lambda})} . \tag{5.16}$$

The corresponding $m=0$ reduced longitudinal wake force becomes

$$\hat{F}_{z0\lambda}=\frac{2\pi\bar{x}_{0\lambda}}{\epsilon}\frac{Y_0(\bar{x}_{0\lambda})}{J_1(\bar{x}_{0\lambda})} . \tag{5.17}$$

We see that the eigenfrequencies approach those of the transverse-magnetic ($TM_{0\lambda}$) modes in a cylindrical dielectric-filled waveguide. In fact, this is to be expected because the dielectric fills the whole waveguide when $\xi\to0$.

The higher-order reduced forces can be computed similarly. With $x\xi\ll1$ and $m\neq0$,

$$\begin{aligned}p_m(x)&=-\frac{(m-1)!}{\pi}\left[\frac{2}{x\xi}\right]^m\times\left[J_m(x)+\frac{\pi}{m!(m-1)!}\left[\frac{xt}{2}\right]^{2m}Y_m(x)\right] ,\\ p_m'(x)&=\frac{m!}{\pi x\xi}\left[\frac{2}{x\xi}\right]^m\times\left[J_m(x)-\frac{\pi}{m!(m-1)!}\left[\frac{xt}{2}\right]^{2m}Y_m(x)\right] ,\\ r_m(x)&=-\frac{(m-1)!}{\pi}\left[\frac{2}{x\xi}\right]^m\times\left[J_m'(x)+\frac{\pi}{m!(m-1)!}\left[\frac{xt}{2}\right]^{2m}Y_m'(x)\right] ,\\ r_m'(x)&=\frac{m!}{\pi x\xi}\left[\frac{2}{x\xi}\right]^m\times\left[J_m'(x)-\frac{\pi}{m!(m-1)!}\left[\frac{xt}{2}\right]^{2m}Y_m'(x)\right] .\end{aligned} \tag{5.18}$$

If the terms involving Y_m and Y_m' are neglected as we take the limit $x\xi\to0$, it is easy to see that

$$\mathcal{D}_m(x)\propto J_m(x)J_m'(x) . \tag{5.19}$$

Thus, the eigenmodes are characterized by $\bar{x}_{m\lambda}$, the λth zero of J_m, and $\bar{x}'_{m\lambda}$, the λth zero of J_m'. However, the reduced transverse force $\hat{F}_{rm\lambda}$ which is proportional to $p_m r_m$ as depicted by Eq. (5.4) will vanish identically. As a result we must compute the zeros of $\mathcal{D}_m(x)$ to the next order in $x\xi$. Denote the two series of zeros by

$$x_{m\lambda}=\bar{x}_{m\lambda}+\Delta_{m\lambda} , \tag{5.20}$$

$$x'_{m\lambda}=\bar{x}'_{m\lambda}+\Delta'_{m\lambda} . \tag{5.21}$$

Near the zero $x_{m\lambda}$ we have

$$\mathcal{D}_m(x)=-\frac{2m!(m-1)!}{\pi^2}\left[\frac{2}{x\xi}\right]^{2m}J_m'(x)\times\left[(\epsilon+1)J_m(x)+\frac{\pi}{m!(m-1)!}\left[\frac{x\xi}{2}\right]^{2m}Y_m(x)\right] . \tag{5.22}$$

We expand $J_m(x)$ about $\bar{x}_{m\lambda}$ to obtain

$$J_m(x_{m\lambda})\approx\Delta_{m\lambda}J_m'(\bar{x}_{m\lambda}) , \tag{5.23}$$

and solve Eq. (5.22) to get

$$\Delta_{m\lambda}=-\frac{1}{\epsilon+1}\frac{\pi}{m!(m-1)!}\left[\frac{\bar{x}_{m\lambda}\xi}{2}\right]^{2m}\frac{Y_m(\bar{x}_{m\lambda})}{J_m'(\bar{x}_{m\lambda})} . \tag{5.24}$$

We are now able to compute, up to the next order of $x\xi$,

$$p_m(x_{m\lambda})r_m(x_{m\lambda})=-\frac{\epsilon}{\epsilon+1}\frac{1}{m\pi}J_m'(\bar{x}_{m\lambda})Y_m(\bar{x}_{m\lambda}) , \tag{5.25}$$

and arrive at the corresponding reduced transverse force

$$\hat{F}_{rm\lambda}=\frac{\epsilon\sqrt{\epsilon-1}}{(\epsilon+1)^2}\frac{4\pi}{m!(m-1)!}\left[\frac{\bar{x}_{m\lambda}}{2}\right]^{2m}\frac{Y_m(\bar{x}_{m\lambda})}{J_m'(\bar{x}_{m\lambda})} . \tag{5.26}$$

Near the zero $x'_{m\lambda}$, we have

$$\mathcal{D}_m(x)=-\frac{2m!(m-1)!}{\pi^2}\left[\frac{2}{x\xi}\right]^{2m}J_m(x)\times\left[(\epsilon+1)J_m'(x)+\frac{\epsilon\pi}{m!(m-1)!}\left[\frac{x\xi}{2}\right]^{2m}Y_m'(x)\right] . \tag{5.27}$$

Since

$$J_m'(x'_{m\lambda})\approx\Delta J_m''(\bar{x}'_{m\lambda})=-\Delta_{m\lambda}\left[1-\frac{m^2}{\bar{x}'^2_{m\lambda}}\right]J_m(\bar{x}'_{m\lambda}) , \tag{5.28}$$

we obtain

$$\Delta'_{m\lambda}=\frac{\epsilon}{\epsilon+1}\frac{\pi}{m!(m-1)!}\frac{1}{1-m^2/\bar{x}'^2_{m\lambda}}\left[\frac{\bar{x}'_{m\lambda}\xi}{2}\right]^{2m}\times\frac{Y_m'(\bar{x}'_{m\lambda})}{J_m(\bar{x}'_{m\lambda})} . \tag{5.29}$$

Now we can compute, up to the next order of $x\xi$,

$$p_m(x'_{m\lambda})r_m(x'_{m\lambda})=\frac{1}{\epsilon+1}\frac{1}{m\pi}J_m(\bar{x}'_{m\lambda})Y_m'(\bar{x}'_{m\lambda}) , \tag{5.30}$$

and arrive at the corresponding reduced transverse force:

$$\hat{F}'_{rm\lambda}=\frac{\sqrt{\epsilon-1}}{(\epsilon+1)^2}\frac{4\pi}{m!(m-1)!}\left[\frac{\bar{x}'_{m\lambda}}{2}\right]^{2m}\times\frac{1}{1-m^2/\bar{x}'^2_{m\lambda}}\frac{Y'_m(\bar{x}'_{m\lambda})}{J_m(\bar{x}'_{m\lambda})}\ . \quad (5.31)$$

As expected, when $\xi\rightarrow 0$, the two series of reduced eigenfrequencies $x_{m\lambda}$ and $x'_{m\lambda}$ ($m\neq 0$) correspond to the transverse-magnetic ($TM_{m\lambda}$) and transverse-electric ($TE_{m\lambda}$) modes in a cylindrical dielectric-filled waveguide. In the present dielectric-lined waveguide, the modes are hybrid and are referred to as hybrid-magnetic ($HM_{m\lambda}$) and hybrid-electric ($HE_{m\lambda}$) modes instead. The lowest mode at $\xi=0$ is the HE_{11} mode, with $x'_{11}=1.8411$, which is lower than the lowest longitudinal HM_{01} mode with $x_{01}=2.405$. The lowest transverse HM mode at $\xi=0$ is $x_{11}=3.8171$.

C. Numerical evaluation

The wake forces corresponding to thin and thick dielectric limits have been evaluated analytically. In between, no simple analytic formulas are possible and numerical evaluation is necessary. The zeros of $\mathcal{D}_0$ and $\mathcal{D}_m$ are first located and the summations in Eqs. (5.1) and (5.3) are performed term by term.

Experimentally, the source is not a single particle but a source bunch of total charge q having a rms longitudinal length σ_l. If the center of the bunch travels according to $z=ct$ and the longitudinal charge distribution is Gaussian, the wakes left behind are again given by Eqs. (3.9) and (4.11) with each term in the summand multiplied by

$$\exp\left[-\frac{1}{2}\left[\frac{x_\lambda\sigma_l}{a\sqrt{\mu\epsilon-1}}\right]^2\right]\ . \quad (5.32)$$

Since σ_l is finite, consequently only the first few characteristic waves will contribute significantly. For the sake of clarity we shall restrict ourselves to the lowest mode in the following discussion.

The reduced eigenfrequencies corresponding to the lowest $m=0$ longitudinal mode x_{01} (TM_{01}) and lowest $m=1$ transverse mode x'_{11} (HE_{11}) are shown, respectively, in Figs. 1 and 2 for ξ ranging from 0 to 1, with dielectric constant $\epsilon=1.2$, 2.0, 3.0, and 4.0. The ratio of the two lowest eigenfrequencies is displayed in Fig. 3. In general, larger dielectric constant leads to higher eigenfrequencies. The lowest reduced eigenfrequencies for the monopole and dipole modes start off from, respectively, $x_{01}=2.405$ and $x'_{11}=1.841$ at $\xi=0$, increase rather slowly with ξ when $\xi\lesssim 0.5$, but increase rapidly to infinity according to Eq. (5.8) afterward.

The reduced $m=0$ longitudinal and $m=1$ wake forces of the lowest modes $\hat{F}_{z01}$ and $\hat{F}'_{r11}$ are shown, respectively, in Figs. 4 and 5, and their ratio in Fig. 6. We see that the reduced transverse force as well as the ratio of transverse to longitudinal forces start off almost constant at $\xi\sim 1$ and increase rather slowly with larger ξ. They vanish rapidly only when ξ is sufficiently close to 1; or when the dielectric lining is sufficiently thin. However, $\xi\approx 1$ is not a good region to operate a wake-field accelerator. The wake fields are generated by Cherenkov radiation inside

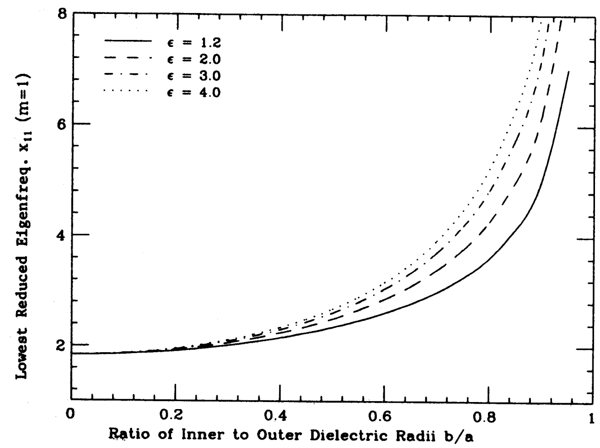

FIG. 2. Lowest reduced eigenfrequency of the $m=1$ transverse mode.

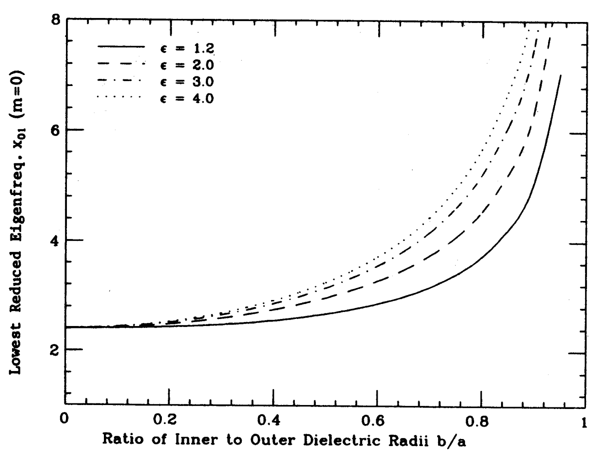

FIG. 1. Lowest reduced eigenfrequency of the $m=0$ longitudinal mode.

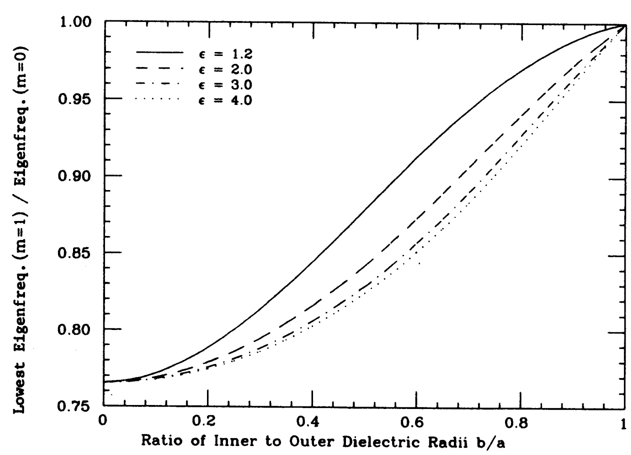

FIG. 3. Ratio of lowest $m=1$ eigenfrequency to lowest $m=0$ eigenfrequency.

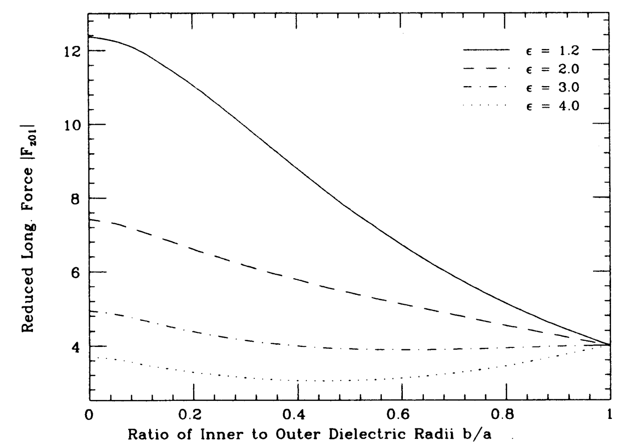

FIG. 4. Reduced $m=0$ longitudinal force $\hat{F}_{z01}$ of the lowest mode.

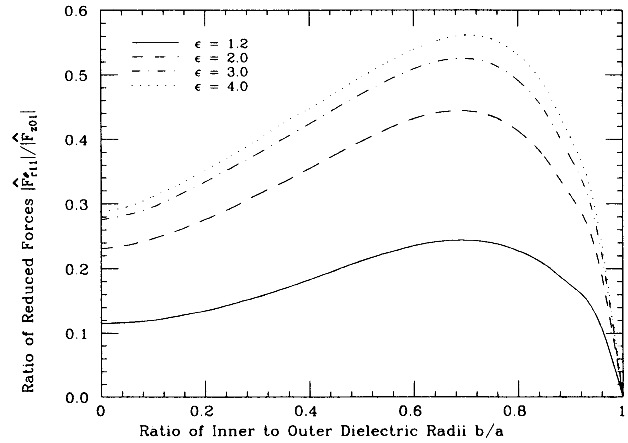

FIG. 6. Ratio of transverse force to longitudinal force, $\hat{F}'_{r11}/\hat{F}_{z01}$. Both forces are of the lowest modes.

the dielectric, and when the dielectric lining is thin, such radiation is minimal. In addition, the wake-field wavelength will be too short to work with. This is seen by examining the cosine factor in Eq. (5.1) with an extremely high reduced eigenfrequency ($\sim\delta^{-1/2}$) as depicted in Eq. (5.8) and Fig. 1. To avoid large transverse forces, the other option appears to be the sector of small ξ. This is the configuration of thick dielectric lining. For this reason the analytic formulas developed for thick dielectric lining are very good approximation in practice. In this region the reduced transverse wake force $\hat{F}'_{r11}$ depends on the dielectric constant mainly through the factor $\sqrt{\epsilon-1}/(\epsilon+1)^2$ and weakly through x'_{11}, as indicated by Eq. (5.31). Consequently we see (also in Fig. 2) $\hat{F}'_{r11}$ reach a maximum at $\epsilon\approx\frac{5}{3}$, and decrease at larger ϵ. However, $\hat{F}_{z01}$ decreases with dielectric constant as $1/\epsilon$ as depicted in Eq. (5.17) and Fig. 1. The result is that the ratio $\hat{F}'_{r11}/\hat{F}_{z01}$ decreases with ϵ (when $\epsilon>1$). Therefore, to reduce $\hat{F}'_{r11}$ as well as $\hat{F}'_{r11}/\hat{F}_{z01}$, a large ϵ is favored.

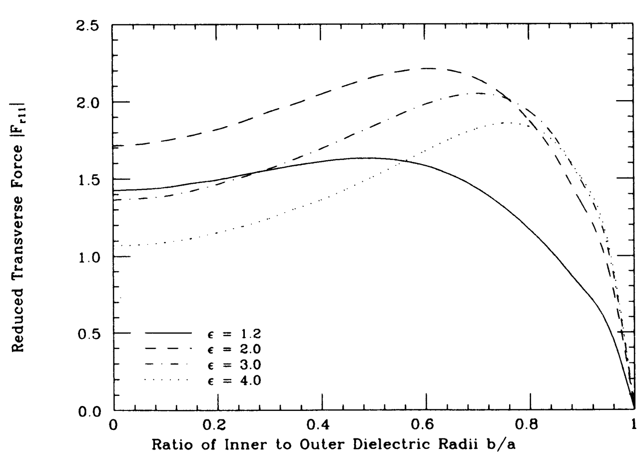

FIG. 5. Reduced $m=1$ transverse force $\hat{F}'_{r11}$ of the lowest mode.

As an illustration let us consider a cylindrical waveguide with outer radius $a=1$ cm lined with a material having a dielectric constant $\epsilon=3$. The thickness of the material is taken as $a-b=0.8$ cm or $\xi=0.2$. The lowest reduced eigenfrequencies are $x_{01}=2.518$ for $m=0$ and $x_{11}=1.954$ for $m=1$. They correspond to frequencies $\omega_{01}/2\pi=8.496$ GHz and $\omega_{11}/2\pi=6.591$ GHz. The reduced wake forces are, respectively, $\hat{F}_{z01}=4.381$ for $m=0$ and $\hat{F}'_{r11}=1.464$ for $m=1$. Using Eq. (5.1) multiplied by $Z_0c/4\pi$, where $Z_0=377\ \Omega$ to convert the force to mks units, we obtain the longitudinal acceleration gradient 3.943×10^{14} eV/m C, or 39.4 MeV/m for a source bunch of 100 nC. Using Eq. (5.2) we obtain a transverse force of $1.32\times10^{16}r_0$ eV/m C, where r_0 is the offset of the source bunch from the axis of the waveguide expressed in m. The accelerating longitudinal force F_{01} (aside from the cosine factor) scales with a^{-2}, while the dipole transverse force F_{11} (aside from the sine factor) scales with a^{-3}. Therefore, increasing the outside radius of the waveguide will lower the transverse force very much. The accelerating force will be decreased also, although not as fast.

As a comparison let us consider an iris-loaded waveguide having inner and outer iris radii $b=5.11$ cm and $a=10$ cm, respectively. Numerical calculation[12] reveals a longitudinal field of frequency 1.15 GHz and a dipole traverse force of $1.2\times10^{14}[r_0\ (\mathrm{m})]$ eV/m C. For a dielectric waveguide of the same frequency we need a guide radius of $a=7.40$ cm provided that we keep $b/a=0.2$ and $\epsilon=3$. The dipole transverse wake force turns out to be $3.25\times10^{13}[r_0\ (\mathrm{m})]$ eV/m C, which is 3.7 times less than that of the iris-loaded waveguide. With suitable choices of parameters, it is possible that the dielectric-lined waveguide can have smaller deflecting wake forces than the iris-loaded waveguide.

VI. DISCUSSIONS

(1) We have solved for the wake forces of a dielectric-lined waveguide and concluded in Sec. IV that the $m\geq1$ transverse wake forces do not vanish in the limit $\gamma\to\infty$.

Simple analytic expressions for the reduced longitudinal and transverse forces have been derived in the limits when the dielectric lining is thin as well as thick. Numerical computation of the lowest modes have also been performed. We learn from these calculations that the reduced dipole transverse wake force has the same order of magnitude as the monopole longitudinal wake force except when the dielectric lining is very thin. This conclusion may not be in contradiction to what was observed experimentally in Ref. 4. This is because the deflecting wake force has never been actually measured there. In that experiment, the inner and outer radii of the three dielectric materials are, respectively, $b=0.63$ cm and $a=1.27$ cm. The three materials have dielectric constants $\epsilon \approx 3.1$, 5.9, and 3.9. According to our calculation the deflecting force should be, respectively, $8.0\times 10^{15}[r_0\ (\mathrm{m})]$ eV/m C, $4.7\times 10^{15}[r_0\ (\mathrm{m})]$ eV/m C, and $6.8\times 10^{15}[r_0\ (\mathrm{m})]$ eV/m C.

(2) For a particle of charge e traveling with velocity $v\hat{z}$ in a TE field ($E_z=0$), the transverse force can be easily expressed by using Eq. (2.1):

$$F_r^{\mathrm{TE}}=-eB_\theta^{\mathrm{TE}}(\beta-\beta_p)\ ,$$
$$F_\theta^{\mathrm{TE}}=eB_r^{\mathrm{TE}}(\beta-\beta_p)\ . \tag{6.1}$$

If the electromagnetic field is purely TM ($B_z=0$), the transverse force is

$$\mathbf{F}_t^{\mathrm{TM}}=e\mathbf{E}_t^{\mathrm{TM}}(1-\beta_p\beta)\ . \tag{6.2}$$

In the above, $\beta_p=v_p/c$ and v_p is the phase velocity of the electromagnetic wave in the z direction. There is a theorem[13] which says that an electromagnetic wave can be written as a linear combination of a TM wave, a TE wave, and a TEM wave. Our wake field inside the dielectric waveguide certainly obeys the theorem, but without TEM contribution. Therefore, if the test particle has a velocity v equal to the phase velocity v_p of the electromagnetic field, the TE part of the transverse force on the test particle will vanish according to Eq. (6.1), and the TM part of the transverse force will be suppressed by γ^2 according to Eq. (6.2). But, this suppression is canceled by the fact that $\mathbf{E}_t^{\mathrm{TM}}$ in vacuum is of order γ^2 as illustrated by the part involving $\mathcal{E}'_m$ in Eqs. (2.18) or (2.19).

However, it is difficult to understand why the transverse fields can be of order γ^2. The Lorentz-contracted fields of a particle are of order γ with an opening angle $O(\gamma^{-1})$ so that the total flux is independent of γ. Here, instead of a small opening angle, the transverse fields are distributed longitudinally as a cosine function as illustrated by Eqs. (3.5), (3.6), and (4.11). The total transverse flux will therefore blowup as γ^2. A closer examination of Eqs. (2.18) or (2.19) reveals that $\mathbf{E}_t^{\mathrm{TE}}$, the part involving $\mathcal{B}'_m$, is also of order γ^2. The contribution of the source, the part involving η'_m, is $O(\gamma^0)$. With the solution of $\mathcal{E}'_m$ and $\mathcal{B}'_m$ from Eq. (2.26), we find that the mystery is solved, because the γ^2 parts of $\mathbf{E}_t^{\mathrm{TE}}$ and $\mathbf{E}_t^{\mathrm{TM}}$ cancel exactly, leaving $\mathbf{E}_t$ finite as $\gamma\to\infty$.

Note added. I have recently learned of similar work by other authors. Gluckstern[14] demonstrated the existence of transverse forces in essentially the same way as this paper. Rosing and Gai[15] derived the transverse forces using the vector and scalar potentials $\mathbf{A}$ and ϕ. Jones, Keinings, and Peter[16] solved the problem in the special case of a thin dielectric lining. Also the Advanced Accelerator Test Facility at Argonne National Laboratory has recently reported[17] the experimental observation of transverse wake forces in a dielectric-lined waveguide.

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APPENDIX

We introduce a scalar potential ϕ and a vector potential $\mathbf{A}$, defined by

$$\mathbf{E}=-\nabla\phi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t},\quad \mathbf{B}=\nabla\times\mathbf{A}\ . \tag{A1}$$

In order to transform Maxwell's equations into independent inhomogeneous equations in ϕ for $\mathbf{A}$,

$$\nabla^2\phi-\frac{\mu\epsilon}{c^2}\frac{\partial^2\phi}{\partial t^2}=-\frac{4\pi\rho}{\epsilon}\ , \tag{A2}$$

$$\nabla^2\mathbf{A}-\frac{\mu\epsilon}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\frac{4\pi\mu\mathbf{J}}{c}\ , \tag{A3}$$

with $\mathbf{J}=\rho\mathbf{v}$, use is made of the Lorentz condition

$$\nabla\cdot\mathbf{A}+\frac{\mu\epsilon}{c}\frac{\partial\phi}{\partial t}=0\ . \tag{A4}$$

Note that, in cylindrical coordinate,

$$\nabla^2\mathbf{A}=\hat{\mathbf{r}}\left[\nabla^2 A_r-\frac{A_r}{r^2}-\frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta}\right]+\hat{\boldsymbol{\theta}}\left[\nabla^2 A_\theta-\frac{A_\theta}{r^2}+\frac{2}{r^2}\frac{\partial A_r}{\partial\theta}\right]+\hat{\mathbf{z}}\nabla^2 A_z\ . \tag{A5}$$

In other words, the equations for A_r and A_θ are coupled. Because of the symmetry of Eqs. (A2) and (A1), one is tempted to assign

$$\mathbf{A}=\frac{\mathbf{v}}{c}\mu\epsilon\phi\ . \tag{A6}$$

This will simplify the problem tremendously because it leaves behind only one equation in one variable. However, Eq. (A6) may not be correct. In fact, the relation between ϕ and $\mathbf{A}$ has been given explicitly by Eq. (A4). Any additional constraint can arise only from the speciality of the problem. For example, Eq. (A6) can be correct if (1) $(\mathbf{A},\phi)$ rotates as a four-vector in the Minskowski space and (2) there is no longitudinal magnetic field (pure TM modes). But, in the presence of a dielectric, space-time does not constitute a Minskowski space and the electromagnetic fields do not separate into pure TE or TM modes.

[1]R. K. Keinigs, M. E. Jones, and W. Gai, Part. Accel. **24**, 223 (1989).
[2]C. Callen *et al.*, JASON Report No. JSR-86-109, 1987 (unpublished).
[3]A. Piwinski, DESY Report No. 72/72 (unpublished).
[4]W. Gai *et al.*, Phys. Rev. Lett. **24**, 2756 (1988).
[5]H. Figueroa *et al.*, Phys. Rev. Lett. **60**, 2144 (1988).
[6]J. B. Rosenzweig *et al.*, Phys. Rev. Lett. **61**, 98 (1988).
[7]Y. Y. Lau, Phys. Rev. Lett. **63**, 1141 (1989).
[8]W. Gai, Argonne Lab Report No. WF-124, 1989 (unpublished); W. Gai, in *Advanced Accelerator Concepts,* proceedings of the 1989 Workshop, Lake Arrowhead, California, edited by Chan Joshi (AIP Conf. Proc. No. 193) (AIP, New York, 1989), p. 111.
[9]M. Rosing, Argonne Lab Report No. WF-131, 1989 (unpublished).
[10]See, for example, J. D. Jackson, *Classical Electrodynamics,* 1st ed. (Wiley, New York, 1962), p. 240.
[11]W. K. H. Panofsky and W. A. Wenzel, Rev. Sci. Instrum. **27**, 967 (1965).
[12]J. D. Simpson, Argonne National Lab Report No. ANL-HEP-TR-89-81, 1989 (unpublished).
[13]C. G. Montgomery, R. H. Dicke, and E. M. Purcell, *Principles of Microwave Circuits* (McGraw-Hill, New York, 1948).
[14]R. L. Gluckstern, LANL Report No. AT-6:ATN-90-1, 1990 (unpublished).
[15]M. Rosing and W. Gai, Argonne Lab Report No. WF-132, 1990 (unpublished).
[16]M. E. Jones, R. Keinigs, and W. Peter, LANL Report No. LA-UR-89-4234 (unpublished).
[17]E. Chojancki *et al.*, Argonne National Lab report (unpublished).