Electroweak interactions become strong at energy above ~10 TeV

L. McLerran, A. Vainshtein,* and M. Voloshin†
Theoretical Physics Institute, School of Physics and Astronomy, 116 Church St. S.E., University of Minnesota, Minneapolis, Minnesota 55455
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We study baryon-number-changing processes at high energy in electroweak theory. We find that at energies of the order of 10 TeV, some inelastic partial-wave amplitudes saturate the unitarity limit and the cross section for such processes becomes as large as $\sigma \sim \alpha_w^2/M_W^2$. We argue that at high energies, lepton interactions become strong with distributions of particles typical of strong-interaction processes. The multiplicity of particles in the final state is of order $1/\alpha_w$, the transverse momenta are of order M_W, and the types of particles are primarily Higgs and gauge bosons and their decay products.

I. INTRODUCTION

Baryon-number violation is a problem which has been of much interest in cosmology. With a proper understanding of the dynamics of baryon-number violation, it is believed that it should be possible to compute from first principles the observed baryon asymmetry of the Universe. In order to understand the dynamics, one must, of course, have experimental tests.

Until recently it was believed that the only baryon-number-changing processes relevant for cosmology were those whose natural physical scale is of the order of $E \geq 10^{15}$ GeV. Unfortunately, with the failure to observe proton decay, it appears unlikely that it will be possible to experimentally test conjectures concerning the physics at such energy scales.

It has been long known that there are another class of baryon-number-changing processes associated with electroweak theory.[1] The method for computing amplitudes which involve baryon decay uses approximate classical solutions of the electroweak theory, instantons. The problem of instanton-induced scattering amplitudes has long been of interest.[1-3] The first computations of such amplitudes were done in the classic paper of 't Hooft.[1] He considered electroweak theory and employed instantons to determine the rate of baryon- (B-)number- plus lepton- (L-) number-changing decay rates. The amplitudes for such processes can be estimated to be of order $e^{-2\pi/\alpha_w}$ where α_w is the SU(2) weak-interaction coupling constant. This factor is of order 10^{-80}, and is so small that if there is no large factor to compensate for this exponential suppression, then such processes are of no practical phenomenological significance.

There are now a number of results concerning violation of $B+L$ at high temperature which suggest that in some amplitudes the factor of $e^{-2\pi/\alpha_w}$ is compensated by energy-dependent factors.[4-10] This phenomenon was advocated on the basis of qualitative arguments by Arnold and McLerran,[10] who proposed that, in instanton-induced amplitudes containing of order $1/\alpha_w$ external legs of gauge or Higgs bosons, the WKB factor of $e^{-2\pi/\alpha_w}$ is absorbed by enormous preexponential factors. If the energy in a collision becomes sufficient to produce so many bosons, $E \sim M_W/\alpha_w \sim 10$ TeV, then the cross section has no suppression by the WKB factor.

This phenomenon in high-energy two-particle collisions was conjectured in the early paper of Aoyama and Goldberg,[11] and the same behavior is also hinted at in the explicit computations of instanton-induced amplitudes for multiparticle processes recently done by Ringwald.[12] He concludes that at an energy of order 1–10 TeV, the weak-coupling expansion for such amplitudes breaks down. Using his results for energy $E \ll M_W/\alpha_w$, where the cross section is still small, one finds

$$\sigma \sim e^{-4\pi/\alpha_w} e^{\alpha_w A E^2/M_W^2} , \tag{1}$$

where A is some constant of order 1. This result while interesting is not entirely compelling since the computational methods used in its derivation break down entirely when the cross section might become sufficiently large that it is no longer suppressed by the WKB factor.

It is nevertheless useful to consider some of the aspects of his computation in a little more detail. Recall that in electroweak theory, the basic process of $B+L$ nonconservation involves all three generations of quarks. There is a change of one unit of B and one unit of L for each generation. The basic process therefore involves at least 12 fermions as shown in Fig. 1. At high energies we may also include the possible emission of an arbitrary number of Higgs and gauge bosons. We assume there are n such bosons in the final state of a scattering process induced by the collisions of two fermions, each of them being a quark or a lepton. The possible range of values of n is only limited by energy conservation: $n \leq E/M_W$. (We will consider throughout this paper the simple case when $\lambda \sim g^2$ or $M_H \sim M_W$ to avoid the messy complications of two mass scales.)

Ringwald's analysis, which we shall discuss in more detail later, gives the amplitudes for multiparticle production into a state of fixed n (Ref. 12). The sum over the cross sections arising from these amplitudes is

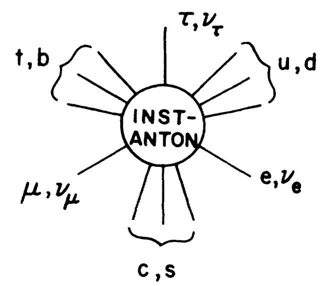

FIG. 1. The basic process involving baryon-number nonconservation in electroweak theory. There are 12 fermions, 3 quarks for each generation and one lepton for each generation.

$$\sigma_{\text{tot}} = \sum_{n}^{n \sim E/M_W} \sigma_n \sim C e^{-4\pi/\alpha_w} e^{\alpha_w A E^2/M_W^2} . \quad (2)$$

A is a constant here and C is a factor which contains an energy dependence which is slower than exponential, and also has some weak dependence on the maximum number of boson legs, and the type of particles are in the initial state. The dominant contribution to this sum comes from σ_n where $n \sim E/M_W$.

Equation (2) suggests that when $E \geq M_W/\alpha_w$, the WKB instanton suppression factor is compensated. The approximations which Ringwald has used are however only valid if $E \ll M_W/\alpha_w$ and if the number of emitted quanta is $n \ll 1/\alpha_w$. Such limitations are clearly stated in the paper by Ringwald. These approximations are intrinsically tied to the weak-coupling expansion and to go beyond them we will be required to use nonperturbative techniques. It would therefore be premature to conclude that instanton-induced amplitudes are not suppressed at high energy.

In this paper we shall show that the principal limitation in the analysis of Ringwald is that the distortion of the shape of the instanton by the external quanta has not been taken into account. We study here the effect of relaxing this condition and generalize Ringwald's results to $n \geq 1/\alpha_w$ and $E \geq M_W/\alpha_w$. We begin by considering the two-dimensional Abelian Higgs model. In this model the instanton analysis gives a cross section which rises approximately as e^{AE/M_H}, and we find the somewhat surprising result after accounting for this distortion, the cross section continues to rise faster at high energies than the extrapolation of the instanton result would suggest. We therefore conclude that the cross section grows to a size limited by unitarity (in $1+1$ dimensions the probability of the process approaches of order 1), at which point multi-instanton processes become important and unitarize the amplitude.

We then proceed to the more realistic four-dimensional case. We find here that the formula given in Eq. (2) for the cross section is only slightly modified by including the distortion due to the external particles. As was the case in the $(1+1)$-dimensional case, the cross section approaches the unitarity limit, and then becomes unitarized by multi-instanton processes. We argue that at high energies, $E \geq M_W/\alpha_w$, this unitarity limit should be of order

$$\sigma_{\text{tot}} \sim 1/E^2 \sim \alpha_w^2/M_W^2 . \quad (3)$$

We are therefore forced into the somewhat awkward conclusion that leptons interact strongly at high energy. We discuss the qualitative nature of such strong interactions. A picture arises which is in many features similar to hadronic physics. The essential differences are that the final-state particles are Higgs and gauge bosons, the typical transverse momenta per particle are of order M_W, and the rapidity density is of order $1/\alpha_w$. Such physics can clearly have applications for the Superconducting Super Collider (SSC), and possibly proposed e^+e^-, e^-e^-, and ep colliders.

II. THE TWO-DIMENSIONAL ABELIAN HIGGS MODEL

Before turning to the more complicated case of electroweak theory in four dimensions we first study the Abelian Higgs model in two dimensions. This problem is slightly more simple than is the case in four dimensions since when computing instanton-induced amplitudes, there is an instanton solution which is an exact solution which has a fixed size. In four dimensions one must introduce an integration over instanton sizes by first imposing a size scale constraint on the action. Only then is the instanton an exact solution of the equations of motion.

This model has an Abelian gauge field A^μ, a complex scalar field ϕ, and a charge-one Dirac fermion field Ψ. The Euclidean space action is

$$S = \int d^2x \left[\tfrac{1}{4} F_{\mu\nu}^2 + |(\partial_\mu - ieA_\mu)\phi|^2 + V(\phi) + i\bar{\Psi}(\partial_\mu - ieA_\mu)\gamma^\mu \Psi \right] , \quad (4)$$

where the Higgs potential is

$$V(\phi) = \lambda(\phi^*\phi - \eta^2)^2 . \quad (5)$$

This potential naively provides a vacuum expectation for the Higgs field:

$$\langle \phi \rangle = \eta . \quad (6)$$

In the analysis we present here, unlike the four-dimensional case we shall assume that $\lambda \gg e^2$ so that the mass of the Higgs boson $M_H = 2\sqrt{\lambda}\,\eta$ is large compared with that of the gauge boson $M_a = \sqrt{2}\,e\eta$. One can also readily verify that the dimensionless parameter $1/\eta$ plays the same role in perturbation theory in this model as the coupling constant g_w does in the four-dimensional SU(2) theory. We assume throughout this section that $\eta \gg 1$ so that perturbatively the Abelian Higgs model is a weak-coupling theory.

The action of Eq. (4) in the gauge+Higgs-boson sector is identical to the Landau-Ginsburg Hamiltonian. Therefore there are classical solutions corresponding to the well-known Abrikosov vortex. In the polar coordinates (r,θ) the classical solution has the form

$$A_r = 0, \quad A_\theta = \frac{1}{er} f(r), \quad \phi(r,\theta) = \eta e^{i\theta} u(r) , \quad (7)$$

where the functions $f(r)$ and $u(r)$ satisfy the boundary conditions

$$f(0)=0,\quad u(0)=0,\quad f(\infty)=1,\quad u(\infty)=1\ . \tag{8}$$

The rate at which $f(r)$ and $u(r)$ approach their limit at $r\to\infty$ is determined correspondingly by the masses M_a and M_H:

$$1-f(r)\sim e^{-M_a r},\quad 1-u(r)\sim e^{-M_H r}\ . \tag{9}$$

The action of Eq. (4) for the instanton solution of Eq. (7) is given by $S_{\rm inst}=C\eta^2$, where C is a function of the ratio e^2/λ which is just a finite number in the limit $e^2/\lambda\to 0$.

The topological properties in this model are controlled by the current

$$K_\mu=\frac{e}{2\pi}\epsilon_{\mu\nu}A_\nu\ , \tag{10}$$

where $\epsilon_{\mu\nu}$ is the antisymmetric symbol, so that the winding number (the Chern-Simons number) is given by

$$N_{\rm CS}=\int K_0 dx=\frac{e}{2\pi}\int A_1(x)dx\ . \tag{11}$$

The instanton corresponds to tunneling between the field configurations which differ by one unit of $N_{\rm CS}$. This is seen from Eqs. (7) and (8):

$$\int \partial_\mu K_\mu d^2x=\frac{e}{4\pi}\int \epsilon_{\mu\nu}F_{\mu\nu({\rm inst})}d^2x=1\ . \tag{12}$$

The instanton transition is necessarily accompanied by change of chirality of the fermions by two units because of the anomalous divergence of the axial-vector current $j^5_\mu=\overline{\Psi}\gamma_\mu\gamma_5\Psi$:

$$\partial_\mu j^5_\mu=\frac{e}{2\pi}\epsilon_{\mu\nu}F_{\mu\nu}\ . \tag{13}$$

Therefore the instanton-induced amplitudes in this model should always contain at least two external fermionic legs and chiral charge $Q_5=\int j^5_0 dx$ is our two-dimensional analog of the $d=4$ electroweak theory $B+L$ charge.

To formulate scattering amplitudes in the terms used in the perturbation theory one first decomposes the complex field ϕ into the modulus v and the phase χ,

$$\phi=ve^{i\chi}\ , \tag{14}$$

and then expands v around the vacuum expectation value of Eq. (6):

$$v(x)=\eta+\sigma(x)\ . \tag{15}$$

The Nambu-Goldstone degree of freedom associated with the field $\chi(x)$ is absorbed by the gauge field $A(x)$ which becomes massive due to that, while the field $\sigma(x)$ describes the physical Higgs boson H.

We shall study in this section a subset of the instanton-induced vertices: namely, those containing two fermions and an arbitrary number n of the H bosons. According to the Lehman-Symanzik-Zimmermann (LSZ) formula these vertices can be obtained by amputating propagators corresponding to external lines in the $(n+2)$-point Green's function:

$$G(x_1,\ldots,x_n,x_{n+1},x_{n+2})=\langle T[\sigma(x_1)\cdots\sigma(x_n)\overline{\Psi}(x_{n+1})\Psi(x_{n+2})]\rangle\ . \tag{16}$$

In the Euclidean formulation this Green's function can be calculated by adding source terms to the action

$$S[J]=S-\int J(x)\sigma(x)d^2x-\int[\overline{J^f}(x)\Psi(x)+\overline{\Psi}(x)J^f(x)]d^2x \tag{17}$$

with $J(x)$ being the source for the Higgs field and J^f the same for the fermions. The Green's function of Eq. (16) is then found from the path integral $Z[J]$ in which the action S is replaced by $S[J]$:

$$G(x_1,\ldots,x_n,x_{n+1},x_{n+2})=\frac{\delta^{n+2}}{\delta J(x_1)\cdots\delta J(x_n)\delta J^f(x_{n+1})\delta\overline{J^f}(x_{n+2})}\ln(Z[J])\ . \tag{18}$$

In the approximation used by Ringwald[12] the calculation of the instanton contribution to the partition function $Z[J]$ amounts to substitution in the integrals with the source terms in Eq. (17) the bosonic and fermionic fields of the instanton which are unperturbed by the sources (in fact, the fermion field is that of the zero modes):

$$S[J]=S_{\rm inst}-\int J(x)\sigma_0(x)d^2x-\int[\overline{J^f}(x)\Psi_0(x)+\overline{\Psi}_0(x)J^f(x)]d^2x\ . \tag{19}$$

Thus in this approximation the Green's function in the Euclidean space is given by

$$G(x_1,\ldots,x_n,x_{n+1},x_{n+2})={\rm const}\times\int d^2y\, e^{-S_{\rm inst}}\sigma_0(x_1-y)\sigma_0(x_2-y)\cdots\sigma_0(x_n-y)\overline{\Psi}_0(x_{n+1}-y)\Psi_0(x_{n+2}-y)\ , \tag{20}$$

where the const which does not depend on either spatial variables or the number n contains the usual preexponential factors in the instanton calculus, $\sigma_0(x)$ is the classical scalar field of the instanton solution, and $\Psi_0(x)$ is the field of the fermionic zero mode. Since the large-x behavior of the solutions $\sigma_0(x)$ and $\Psi_0(x)$ is governed by the free-particle equation the Fourier transform of the Green's function in Eq. (20) displays the on-shell poles of the propagators, corresponding to each external leg:

$$G(p_1,\ldots,p_n,p_{n+1},p_{n+2})={\rm const}\times(2\pi)^2\delta^2\left(\sum_{k=1}^{k=n+2}p_k\right)e^{-S_{\rm inst}}\frac{(\sigma_\mu p^\mu_{n+1})F^f(p^2_{n+1})}{p^2_{n+1}}\frac{(\sigma_\nu p^\nu_{n+2})F^f(p^2_{n+2})}{p^2_{n+2}}\prod_{k=1}^{k=n}\frac{\eta F(p_k^2/M_H^2)}{p_k^2+M_H^2}\ . \tag{21}$$

The functions $F(p^2/M_H^2)$ and $F^f(p^2)$ arise from Fourier transforms of the Higgs-field solution and of the fermion zero mode. These form factors are equal to one at $p^2=0$ and are not singular on the mass shell. The factor η for each external Higgs-boson leg appears here because of the overall normalization of the Higgs field in the instanton solution [cf. Eq. (7)], and we also remind the reader that the fermions are massless in this model. Thus the $(n+2)$-point Green's function corresponds to the instanton-induced $(n+2)$-particle on-shell amplitude of the form

$$A_n = \text{const} \times e^{-S_{\text{inst}}} \bar{\Psi}\Psi (c\eta)^n , \tag{22}$$

where $c = F(-1)$.

With this amplitude one can readily estimate the behavior of the probability (which is the two-dimensional substitute for the cross section) of the fermion-antifermion annihilation into n Higgs bosons at a high energy $E=\sqrt{s} \gg M_H$:

$$\sigma_n \sim e^{-2S_{\text{inst}}} (c\eta)^{2n} \tau_n , \tag{23}$$

where τ_n is the phase space for n identical particles. The latter can be roughly estimated as

$$\tau_n \sim \frac{1}{n!} \left[\frac{1}{2\pi} \int_0^{p(\epsilon)} \frac{dp}{p_0} \right]^n = \frac{1}{n!} \left[\frac{1}{2\pi} \ln \left\{ \frac{\epsilon}{M_H} + \left[\left[\frac{\epsilon}{M_H} \right]^2 - 1 \right]^{1/2} \right\} \right]^n , \tag{24}$$

where $\epsilon \approx E/n$ is the typical energy per particle and $1/n!$ enters because the particles are identical. The integral in Eq. (24) has different behavior for relativistic particles $E \gg nM_H$ when the integral is $\ln[E/(nM_H)]$ and for the nonrelativistic ones, i.e., when $\Delta \equiv E - nM_H \ll nM_H$ and the integral is $\sqrt{2\Delta/(nM_H)}$. In these two cases the τ_n can be estimated as

$$\tau_n \sim \frac{1}{n!} \left[\frac{1}{2\pi} \ln \frac{E}{nM_H} \right]^n \quad \text{if } E \gg nM_H , \tag{25}$$

$$\tau_n \sim \frac{1}{n!} \left[\frac{1}{2\pi} \left[\frac{2\Delta}{nM_H} \right]^{1/2} \right]^n \quad \text{if } \Delta \ll nM_H . \tag{26}$$

We are interested here in the total probability of producing arbitrary number of Higgs bosons at a given energy $E \gg M_H$, i.e., in the sum

$$\sigma_{\text{tot}}(E) = \sum_{n < E/M_H} \sigma_n . \tag{27}$$

Inserting our estimate Eq. (26) for the phase space into Eq. (23), we find in the saddle-point approximation for this sum that if the energy is constrained by the condition $M_H \ll E \ll M_H \eta^2$ the sum receives the dominant contribution from the multiparticle states with $n \approx E/M_H$ so that bosons in the final state are nonrelativistic. [More precisely the maximum of the terms in the sum corresponds to

$$\Delta \sim \frac{E}{2} \frac{1}{\ln(\eta^2 M_H/E)} \tag{28}$$

which is consistent with $\Delta \ll M_H n$ if $E \ll M_H \eta^2$.] We thus find in the exponential approximation

$$\sigma_{\text{tot}}(E) \sim e^{-2C\eta^2} \exp\left[\frac{E}{M_H} \ln \left[\frac{\kappa\eta^2}{(E/M_H)\ln^{1/2}(M_H\eta^2/E)} \right] \right] , \tag{29}$$

where κ is a numerical constant and we have substituted for the unperturbed instanton action $S_{\text{inst}} = C\eta^2$. Notice that this expression is a monotonically increasing function of energy until $E \approx M_H \eta^2$, when the instanton suppression may be canceled. At energies above this, the instanton approximation breaks down, and the result is no longer valid. In addition, our stationary phase evaluation of the cross section within the instanton approximation is invalid, and a proper evaluation of the instanton contribution shows that the cross section continues to rapidly rise.

In Eq. (29) we might guess that the instanton exponent is absorbed when the energy becomes sufficiently large, $E \geq M_H \eta^2$. If this happens, then the instanton-induced processes are unsuppressed.

The appearance of the energy scale $E_0 = M_H \eta^2$ is not a surprise, since this is the height of the energy barrier between sectors of the theory with different values of N_{CS}, the Chern-Simons charge, as shown in Fig. 2. To estimate the height of the barrier which separates degenerate minima of the theory, we first note that there should be a static solution of the classical equations of motion for the effective potential which is classically unstable, and corresponds to sitting on top of the barrier. Such a solution is called the sphaleron.[4] This solution has been constructed for the Abelian Higgs model in two dimensions[8] and is given by the well-known kink solution of $\lambda\phi^4$ theory:[13] theory:[13]

$$\phi(x) = \eta \tanh(\sqrt{\lambda}\, \eta x), \quad A_\mu(x) = 0 . \tag{30}$$

The barrier height energy which corresponds to the energy of this classical solution is

$$E_{\text{sp}} = \int dx \left[\left[\frac{d\phi}{dx} \right]^2 + V(\phi) \right] = \frac{8}{3}\sqrt{\lambda}\, \eta^3 = \frac{4\sqrt{2}}{3} M_H \eta^2 . \tag{31}$$

Equation (29) therefore suggests that when the energy in the collision is larger than the height of the energy bar-

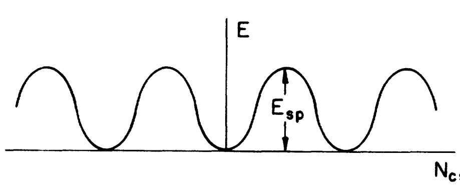

FIG. 2. The energy as a function of Chern-Simons charge. The sphaleron solution corresponds to the top of the barrier between the degenerate minima.

rier, the effect of the barrier penetration factor disappears, and the process with $\Delta N_{\rm CS}=1$ becomes unsuppressed.

The reasoning presented thus far is insufficient to conclude that when $E \geq M_H \eta^2$ the instanton processes are unsuppressed. Recall that we assumed that $E \ll M_H\eta^2$ when we evaluated the sum in Eq. (27). This can be easily corrected, but there is a more serious problem which induces a breakdown of the weak-coupling expansion. This arises because when we compute a multiparticle Green's function, the presence of the external lines distorts the instanton solution. Recall that to compute the multiparticle Green's function in the instanton approximation one does a stationary phase approximation to an integral with an exponential of the classical action and a high-order polynomial of the field. In the naive instanton computation, the stationary phase point is computed ignoring the effect of the high-order polynomial of the field. This will no longer be valid if the polynomial is of sufficiently high order, since at some point the polynomial generates an effect of order 1 for the computation of the stationary phase point of the path integral. If we imagine generating the high-order Green's function by differentiation with respect to an external field, then the distortion of interest is that for the theory in the presence of a strong external field.

Perturbation theory can be seen to be breaking down by considering the corrections to the $n+2$ particle vertex. The corrections due to the exchange of virtual Higgs bosons between the external lines of the vertex have as their expansion parameter n/η^2. Since the sum of Eq. (27) is dominated by $n \sim E/M_H$, the corrections are small only when $E \ll M_H\eta^2$.

We shall soon see that we can compute the behavior of the instanton-induced scattering amplitude for large numbers of external legs by considering the action $S[J]$ in the presence of a strong external field. To simplify the calculation we note that the form factor on each external line $F(p^2/M_H^2)$ in Eq. (21) gives rise to a numerical factor c when evaluated on mass shell. Up to this factor we may, therefore, consider the amplitude evaluated at $p_k^2=0$ instead of on-mass shell at $p_k^2=-M_H^2$. With this simplification, it is sufficient to consider the theory in the presence of a constant external source, $J(x)=J=\text{const}$. The introduction of a constant external source coupled to the Higgs field $\sigma(x)$ gives a Higgs potential of

$$V(v,J)=\lambda[v(x)^2-\eta^2]^2-J[v(x)-\eta] , \qquad (32)$$

where we have used $\phi=ve^{i\chi}$ and $v(x)=\eta+\sigma(x)$.

Let us consider the behavior of the action $S_{\rm inst}[J]$ for large positive J. The main effect of the presence of J is a shift in the vacuum expectation value of the field $v(x)$, which is determined by solving

$$\frac{dV(v,J)}{dv}=4\lambda v^3-4\lambda v\eta^2-J=0, \quad v \geq 0 . \qquad (33)$$

For large J we can neglect the term linear in v and get the solution

$$\langle v \rangle \approx \left[\frac{J}{4\lambda}\right]^{1/3} . \qquad (34)$$

This is the limit as $x \to \infty$ of the $v(x)$ field in the distorted instanton solution. The characteristic size of the solution is $(\sqrt{\lambda}\langle v \rangle)^{-1} \sim J^{-1/3}$, so that the action becomes

$$\lim_{J\to\infty} S_{\rm inst}[J] \sim \bar{c}\langle v \rangle^2 \sim \left[\frac{J}{\lambda}\right]^{2/3} . \qquad (35)$$

It should be noted that this asymptotic behavior of $S_{\rm inst}[J]$ is not affected by perturbative corrections since these have $1/\langle v \rangle^2 \sim J^{-2/3}$ as an expansion parameter.

As a consequence of Eq. (35), $S_{\rm inst}[J]$ has a branch cut in the complex J plane. The nature of this cut is clear upon looking at the solution for v for negative J. The local minimum of the potential $V(v,J)$ for positive v disappears when

$$J=J_{c2}=-\frac{8}{3\sqrt{3}}\lambda\eta^3 . \qquad (36)$$

In fact $J<J_{c2}$ corresponds to the situation when there is no stable or metastable state with $\langle v \rangle \geq 0$ at all. The local minimum of $V(v,J)$ starts to be metastable [i.e., higher than $V(0,J)$] already for

$$J \leq J_{c1} \approx -1.089\lambda\eta^3 \qquad (37)$$

which means that $\ln(Z[J])$ starts to develop an imaginary part at $J \leq J_{c1}$ which is however exponentially small in the region $J_{c2} \leq J \leq J_{c1}$. At $J \leq J_{c2}$ the theory becomes a strongly coupled theory since the dimensionless expansion parameter $1/\langle v \rangle^2$ is no longer small. At this branch point the characteristic size of the instanton $(\sqrt{\lambda}\langle v \rangle)^{-1}$ also becomes large which invalidates the validity of our approximation that $J=\text{const}$. Therefore we cannot examine the behavior of $S_{\rm inst}[J]$ near the branch point at $J=J_{c2}$.

The existence of a branch cut for J varying at a scale $\sim 1/(\sqrt{\lambda}\,\eta)$ is also ensured by our previous consideration of the limit $J\to\infty$, since in this analysis the characteristic size of the instanton Higgs-boson core $(\sqrt{\lambda}\langle v \rangle)^{-1} \sim J^{-1/3}$ is much smaller than $(\sqrt{\lambda}\,\eta)^{-1}$. In our future analysis we shall use for the position of the branch cut

$$J_c=-\bar{c}\lambda\eta^3 \qquad (38)$$

with $\bar{c}$ as some undetermined numerical coefficient which is of order 1.

One might think that we implicity assume here that the exact amplitude has the separable form analogous to that given by Eq. (21), i.e., that it depends only on the virtuality of individual external legs and does not depend on invariants like p_ip_j which correlate the momenta of emitted particles. This however is not the case because due to shrinking of the instanton at large positive J the behavior described by Eq. (35) at $J\to+\infty$ is also valid for the source varying at a fixed scale $\sim M_H^{-1}$, which generates the on-shell amplitudes for production of soft (nonrelativistic) Higgs bosons, i.e., of the configuration which gives the dominant contribution to $\sigma_{\rm tot}$ at $E > M_H\eta^2$.

The branch cut for $S_{\rm inst}[J]$ obviously produces a cut for $\ln(Z_{\rm inst}[J]) \sim e^{-S_{\rm inst}[J]}$. According to Abel's theorem, the existence of this cut implies that when $Z_{\rm inst}[J]$ is expanded around $J=0$ in a Taylor series

$$\ln(Z[J]) = \sum_{n=0}^{\infty} a_n J^n \tag{39}$$

the radius of convergence of the series is equal to $|J_c|$ and that the coefficients of the Taylor-series expansion have the limiting behavior

$$\lim_{n\to\infty} |a_n|^{1/n} = 1/|J_c| \ . \tag{40}$$

We have therefore that the n particle connected Green's function behaves as

$$\lim_{n\to\infty} \left[\frac{\delta}{\delta J}\right]^n \ln(Z[J]) = \zeta(n) n! \left[\frac{1}{|J_c|}\right]^n , \tag{41}$$

where $\zeta(n)$ is an arbitrary function of n which has a strength weaker than an exponential.

According to Eq. (18), this corresponds to the behavior in the $n\to\infty$ for the $(n+2)$-particle amplitudes

$$A_n \sim n! \left[\frac{cM_H^2}{|J_c|}\right]^n \bar{\Psi}\Psi \sim n!(\bar{c}/\eta)^n \bar{\Psi}\Psi \ , \tag{42}$$

where the factor of $(cM_H^2)^n$ arises from multiplying the appropriate multiparticle Green's function by the inverse propagator for each Higgs particle, and allowing for the form factor effect of $F(-1)$, and $\bar{c}$ is some undetermined constant of order 1.

We can now compute the total probability using the vertex given by Eq. (42). In this case we shall evaluate it in the limit where $E \gg M_H \eta^2$:

$$\sigma_{\rm tot} \sim \sum_{n \le E/M_H} (n!)^2 \left[\frac{\bar{c}}{\eta}\right]^{2n} \tau_n \ . \tag{43}$$

From Eq. (24) for the τ_n we find for asymptotically large E that the dominant contribution to this sum comes from the values of n where $n \approx E/M_H$ such that the particles are nonrelativistic:

$$\frac{E}{M_H n} - 1 \approx \frac{1}{2\ln(E/M_H\eta^2)} \ . \tag{44}$$

The contribution to the total probability arising from one instanton process behaves therefore as

$$\sigma_{\rm tot}(E) \sim \left[\frac{AE}{M_H \eta^2 \ln^{1/2}[E/(M_H\eta^2)]}\right]^{E/M_H} . \tag{45}$$

This contribution therefore grows more rapidly than the naive extrapolation of instanton amplitudes computed without the correction due to the distortion arising from external particles. This means that accounting for the distortion does not provide a cutoff for instanton-induced amplitudes.

The estimate of Eq. (45) cannot be taken literally for the total probability since it violates the unitarity condition that the probability of scattering in $1+1$ dimensions cannot exceed 1. However, we have just shown that there is no unitarization of the amplitude arising from the one-instanton contribution. Unitarization can only be achieved by including multi-instanton–anti-instanton effects. This however implies that the total probability becomes of order unity once the multi-instanton effects begin to become of the order of the single-instanton effects. This happens at the energy $E \sim E_{\rm sp} \sim \sqrt{\lambda}\,\eta^3$.

We therefore have found that in the two-dimensional Abelian Higgs model, the probability of chirality-violating processes becomes of order unity at an energy $E \sim \sqrt{\lambda}\,\eta^3$, and that the dominant final states are those with $n \sim E/M_H \sim \eta^2$ Higgs bosons.

III. FOUR-DIMENSIONAL SU(2) GAUGE THEORY

We now proceed to analyze the realistic case of four-dimensional electroweak theory. The only simplification we here adopt is that we consider a pure $SU(2)_L$ gauge theory and disregard the U(1) interaction. This simplification corresponds to setting the Weinberg angle $\Theta_W = 0$.

The Euclidean action for $SU(2)_L$ gauge theory is

$$S = \int d^4x \, [\tfrac{1}{4} W_{\mu\nu}^2 + |(\partial_\mu - igW_\mu \cdot \tau)\phi|^2 + \lambda(\phi^\dagger\phi - \eta^2)^2 + S_F] \ , \tag{46}$$

where S_F is the action for the fermions and $W_{\mu\nu}$ is the gauge field strength:

$$W_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\nu^a + g\epsilon^{abc} W_\mu^b W_\nu^c \ . \tag{47}$$

The matrices $\tau^a = \sigma^a/2$ are generators in the fundamental representation of SU(2). The field ϕ is an SU(2) Higgs-boson doublet which we parametrize as

$$\phi(x) = v(x)\exp[i\pi(x)\cdot\tau] \begin{bmatrix} 0 \\ 1 \end{bmatrix} . \tag{48}$$

The fields $\pi^a(x)$ are Goldstone fields absorbed by the gauge field W_μ^a which thus acquires a mass $M_W = g\eta/\sqrt{2}$. The excitations of the field $v(x)$ around the mean value

$$\langle v(x) \rangle = \eta \tag{49}$$

describe the physical Higgs boson with a mass $M_H = 2\sqrt{\lambda}\,\eta$.

The instanton has one zero mode for each fermion doublet so that the minimal number of fermionic legs in an instanton-induced vertex is equal to $4N_f$ where N_f is the number of quark-lepton families. This fact can also be deduced from studying the anomalous divergence of the baryon-lepton number current:

$$\Delta B = \Delta L = \Delta N_{\rm CS} = N_f \frac{g^2}{32\pi^2} \int d^4x \, W_{\mu\nu} W_{\mu\nu}^d = N_f \ , \tag{50}$$

where $W_{\mu\nu}^d = \epsilon_{\mu\nu\lambda\sigma} W^{\lambda\sigma}$. Unlike the two-dimensional case, for SU(2) gauge theory in the presence of spontaneous symmetry breaking, the instanton with finite-scale size ρ is not a solution of the classical equations of motion. It is an approximate solution only when $\rho \ll 1/\langle v \rangle$. The technical way to deal with this problem has been known since the work of 't Hooft[1] and Affleck.[14] One can introduce a constraint into the path integral which constrains the size of instantons, and then in the end the size parameter is integrated out. For our purposes it is sufficient to know that the instanton action $S_{\rm inst}$ receives an additional contribution of the form $\Delta S_{\rm inst}=2\pi^2\rho^2\langle v\rangle^2$ from the Higgs expectation value $\langle v\rangle$, so that

$$S_{\rm inst}=8\pi^2/g^2+2\pi^2\rho^2\langle v\rangle^2 \; . \tag{51}$$

The asymptotic long-distance behavior of the gauge, Higgs, and fermion fields is governed by e^{-Mr} where M is the mass of the corresponding particle in the physical broken-symmetry vacuum.

In analogy to the analysis of the two-dimensional Abelian Higgs model we consider the behavior of the instanton action in the presence of a source J of the Higgs field:

$$S[J]=S-\int d^4x\, J(x)\sigma(x) \; . \tag{52}$$

In the approximation where the distortion of instantons due to the presence of J is ignored, this action generates amplitudes for processes involving 12 fermions and n Higgs particles:[12]

$$A_n \sim n!(c/\eta)^n e^{-2\pi/\alpha_w} \; , \tag{53}$$

where c is a constant order 1.

Using the above amplitudes one can estimate the total cross section for quark-quark scattering plus baryon-number change. This is given by the sum over cross sections for the production of n particles as

$$\sigma_{\rm tot}=\sum_n \sigma_n \sim e^{AE^2/\eta^2} e^{-4\pi/\alpha_w} \; . \tag{54}$$

This expression is derived in the limit where the energy is in the range $M_H \ll E \ll \eta^2/M_H$. The sum is dominated by $n\sim E^2/\eta^2$, and thus in this region it is consistent to treat the produced Higgs bosons as relativistic since in this case $E/n\sim\eta^2/E \gg M_H$. In this limit, then n particle phase space may be approximated as

$$\tau_n \sim \frac{(E^2/16\pi^2)^{n-2}}{n!^3} \; . \tag{55}$$

As discussed by Ringwald, the derivation of the amplitude of Eq. (53) fails to be correct when $n\sim 1/\alpha_w \sim 1/\lambda$. (In the four-dimensional case we are always working in the limit where $\lambda\sim g^2$ so as to avoid the complications of two scales.) The reason for this is that the parameter for the perturbation theory radiative corrections is $n\alpha_w$. It is therefore impossible to conclude from Eq. (54) that the cross section becomes of order unity when the energy becomes $E\sim\eta/g\sim M_W/\alpha_w$, an energy which corresponds to the barrier between different topological sectors of electroweak theory.

As was the case in our analysis of the two-dimensional Abelian Higgs model we can overcome this failure by considering the distortion of the instanton field in the presence of a strong external current J for the Higgs field. As we saw before, the main effect of such a source is to shift the expectation value of the Higgs field. The gauge and Higgs part of the instanton action is given by Eq. (51) with $\langle v\rangle$ being a function of J as

$$S[J]=\frac{8\pi^2}{g^2}+2\pi^2\rho^2(\langle v\rangle_J)^2 \; . \tag{56}$$

It is straightforward to perform the integration over ρ to obtain the following result for the one-instanton contribution to the generating function $Z[J]$:

$$\left[\frac{\delta}{\delta J^f}\right]^{12} \ln Z_{\rm inst}[J] = \frac{1}{(\langle v\rangle_J)^{14}}\int d^4y \left[\prod_{f=1}^{12}\Psi_0^f(x_f-y)\right]\left[\frac{\Lambda_W}{\langle v\rangle_J}\right]^b \; , \tag{57}$$

where $(\delta/\delta J^f)^{12}$ symbolically represents differentiating over the 12 fermionic sources of different flavor f and $\Psi_0^f(x)$ is the field of the fermion zero mode for flavor f, Λ_W is the infrared energy scale of the $SU(2)_L$ theory at which it would become infrared unstable if there were no symmetry breaking,

$$\begin{aligned}\Lambda_W &= {\rm const}\times M \exp\left[-\frac{2\pi}{b\alpha_w(M)}\right] \\ &= {\rm const}\times M_W \exp\left[-\frac{2\pi}{b\alpha_w(M_W)}\right] \\ &\sim 10^{-15}\ {\rm eV} \; ,\end{aligned} \tag{58}$$

where M is the ultraviolet cutoff, and finally b is the coefficient of the β function in the theory:

$$b=7-\tfrac{4}{3}N_f \; . \tag{59}$$

In fact the result of Eq. (57) is obvious on dimensional grounds since the power b with which Λ_W enters the vertex is fixed by the bare instanton action $2\pi/\alpha_w$ and renormalization-group arguments.

We may now repeat the same reasoning as was used in the two-dimensional Abelian Higgs model. The large J asymptotics of $\langle v\rangle_J$ is given by $\langle v\rangle_J=(J/4\lambda)^{1/3}$, and we again come to the conclusion that $\ln Z[J]$ has a branch cut at negative J starting at $J\sim J_c\sim -\lambda\eta^3$. Applying Abel's theorem we therefore find that the $n\to\infty$ asymptotics of the $(n+12)$-particle instanton amplitudes are

$$\begin{aligned}A_n &\sim \left[\prod_{f=1}^{12}\Psi^f\right]\zeta(n)n!\left[\frac{cM_H^2}{\lambda\eta^3}\right]^n \\ &= \left[\prod_{f=1}^{12}\Psi^f\right]\zeta(n)n!\left[\frac{c}{\eta}\right]^n \; ,\end{aligned} \tag{60}$$

where $\zeta(n)$ is a function of n weaker than an exponential and c is a number of order unity.

With the accuracy adopted here, this expression has the same functional dependence as that in Eq. (53). However when the energy is larger than the height of the barrier between different topological sectors, $E \gg \eta^2/M_H$, the total cross section is dominated by states with $n \approx E/M_H$ for which the relativistic estimate of the phase space is invalid. For n nonrelativistic particles we find that

$$\tau_n \sim \frac{1}{n!}\left[\frac{\sqrt{2M_H}}{6\pi^2}\left[\frac{\Delta}{n}\right]^{3/2}\right]^n , \tag{61}$$

where $\Delta = E - nM_H$. It is now straightforward to estimate that the maximal contribution to the total cross section arises from states with

$$\frac{\Delta}{nM_H} \approx \frac{2}{3\ln(EM_H/\eta^2)} \tag{62}$$

and that $\sigma_{\rm tot}(E)$ may be evaluated to exponential accuracy as

$$\sigma_{\rm tot}(E) \sim \left[\frac{AEM_H}{\eta^2[\ln(EM_H/\eta^2)]^{3/2}}\right]^{E/2M_H} , \quad E \gg \eta^2/M_H . \tag{63}$$

We thus find no cutoff of the energy growth of the total cross section in the one instanton sector.

IV. DISCUSSION AND SUMMARY

As was the case in the two-dimensional Abelian Higgs model, the single-instanton computation of the total cross section is not valid when the cross section becomes large in electroweak theory. There must be unitarity corrections arising at some energy scale due to multi-instanton–anti-instanton configurations. This happens when the center-of-mass energy reaches $E \sim \eta^2/M_H$, the height of the barrier separating the different topological sectors of electroweak theory. To estimate the total cross section when the exponential growth is saturated, we first note that the pointlike instanton interaction of left-handed fermions has only one inelastic helicity amplitude $f^0_{(-1/2)(1/2)}$ generated for processes such as fermion+fermion$\to \Delta(B+L)+X$. At first therefore only one helicity amplitude becomes of order unity at $E \sim E_{\rm sp} \sim \eta^2/M_H$ which corresponds to the cross section

$$\sigma_{\rm tot}(E \sim E_{\rm sp}) \approx \frac{4\pi}{E_{\rm sp}^2} . \tag{64}$$

The saturation of the unitarity bound in one helicity amplitude implies a strong interaction in this partial wave. It is however impossible to keep the interaction strong only in one partial wave over all the energy range when $E \ge E_{\rm sp}$. Iterations of the instanton amplitudes in the t channel will give rise to a proliferation of the strong interaction to higher partial waves. We expect therefore at $E \ge E_{\rm sp}$ that a picture typical of ordinary strong interactions arises from t-channel unitarization, and that asymptotically the total cross section grows as

$$\sigma_{\rm tot}(E) \sim \frac{4\pi}{E_{\rm sp}^2}\left[\ln\left[\frac{E}{E_{\rm sp}}\right]\right]^{\kappa} , \tag{65}$$

where κ is some power $\kappa \le 2$. The multiplicity of Higgs and gauge bosons is likely to grow up logarithmically at $E \gg E_{\rm sp}$.

Also the iteration of the $(B+L)$-violating amplitude in various channels will give rise as well to strong $(B+L)$-conserving elastic and inelastic amplitudes. This conclusion, though it looks somewhat unusual, can probably be traced to the long ago expected breakdown of perturbation theory in higher orders (see, e.g., Ref. 15 and references therein).

As we saw the onset of strong interaction in the lowest helicity inelastic amplitude is very sharp, in fact it can be approximated by the step function $\theta(E-E_0)$. This issue which is directly related to experiment is what the precise value of E_0 is. To clarify this point one needs more accurate estimates than ours. However, keeping in mind the picture of Fig. 2 with the barrier separating sectors with different values of $N_{\rm CS}$, one might guess that E_0 coincides with the height of the barrier: $E_0 = E_{\rm sp}$. We can recall in this connection that $E_{\rm sp}$ is computed[5] to be

$$E_{\rm sp} = 2AM_W/\alpha_w \approx 7\text{–}13\ \text{TeV} , \tag{66}$$

where the coefficient A varies from 1.52 to 2.70 for $0 < \lambda/g^2 < \infty$.

In the previous analysis we have considered multiple production of only Higgs bosons and have in fact ignored the analogous processes for the multiple emission of vector gauge bosons. The reason was that it was technically simpler to consider the distortion of the instanton by the scalar source than by a strong source of the gauge fields. This was sufficient for our "existence proof," that is to show that some baryon-number-violating amplitudes become of order unity above the sphaleron energy threshold. We naturally expect that at that energy the gauge bosons are also copiously produced with a typical multiplicity of order $1/\alpha_w$ and typical momenta of order M_W. From the perspective of an experimental search for such processes, it is of primary importance to know what would typically be the ratio of multiplicities of the Higgs to gauge bosons. To answer this question and find the dependence of n_H/n_W on the ratio of coupling constants λ/g^2 one has to perform a more refined analysis than we have done. We would however expect that for $\lambda/g^2 \sim 1$ multiplicities of both should be of order $1/\alpha_w$.

Another more theoretical question arises in connection with the unusual behavior of the instanton-induced vertices which was first observed by Ringwald. Specifically, the vertices have form factors in the virtuality of individual particles p^2, but not in the pair invariants such as $p_i \cdot p_j$. In principle this behavior is invalidated by radiative corrections, that is by the exchange of virtual particles between external legs of the vertex. We have however seen that the more than exponential growth of the cross section in two-particle collisions arises due to configurations in which only the two incoming particles have large energy, while each of the final-state particles is soft. For purposes of discussion it is convenient to consider the process in which the incoming particles are two fermions. The modification of the amplitude by the collective effect of the n soft particles is accounted for in the distorted instanton action, and the wave functions of the energetic incoming fermions are those of the zero modes of the distorted instanton. So the only unaccounted radiative effect is the exchange between the lines of energetic particles which at most gives rise to a power dependence on the logarithm of the total energy, and which cannot compete with the strong growth of the amplitude which we have found.

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*Permanent address: Institute for Nuclear Physics, Novosibirsk, U.S.S.R.

†Permanent address: Institute for Theoretical and Experimental Physics, Moscow, U.S.S.R.

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