Relativistic description of quark-antiquark bound states. II. Spin-dependent treatment

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We present the results of a study of light- and heavy-quark–antiquark bound states in the context of the reduced Bethe-Salpeter equation, including the full spin dependence. We obtain good fits to the observed spin splittings in the $b\bar{b}$ and $c\bar{c}$ systems using a short-distance single-gluon-exchange interaction, and a long-distance scalar confining interaction. However, we cannot obtain satisfactory fits to the centers of gravity of the $b\bar{b}$ and $c\bar{c}$ spin multiplets at the same time, and the splittings calculated for $q\bar{Q}$ mesons containing the lighter quarks are very poor. The difficulty appears to be intrinsic to the reduced Salpeter equation for reasons which we discuss.

I. INTRODUCTION

In an earlier paper,[1] we presented the results of a detailed relativistic analysis of the spin-averaged spectra of light- and heavy-quark–antiquark ($q\bar{q}$) bound states based on the reduced Salpeter equation, a standard approximation to the full Bethe-Salpeter equation. The results of that analysis were somewhat unexpected. We could obtain (apparently) good fits to the $b\bar{b}$, $c\bar{c}$, and $s\bar{s}$ spectra using an instantaneous interaction which contained a short-range Lorentz-vector one-gluon-exchange term, and a long-range Lorentz-scalar confining interaction as expected theoretically, both with quite reasonable parameters. However, close examination showed that the discrepancies between our best fits and the data had a small but significant dependence on the quark masses. The model failed completely for light-quark systems: the slopes of the calculated $l\bar{l}$ Regge trajectories were two to three times larger than the experimental slopes. This problem was shown, in the context of the reduced Salpeter equation, to result from the scalar nature of the confining interaction. The difficulties noted are just those one would expect in a description of light-quark systems in QCD which omits the dynamical energy of the gluon fields, and, hence, misses the essentially stringlike behavior of the light mesons.[2–4]

We noted in Ref. 1 that the problems with the calculated spectra could be eliminated by using a roughly equal mixture of scalar and vector components in the confining interaction. While such a mixture would contradict current theoretical ideas,[4,5] it cannot be excluded phenomenologically in a spin-independent analysis. A number of spin-dependent analyses of the $q\bar{q}$ data have been given with various starting points, e.g., Schrödinger[6] or relativistic[7] wave equations with spin effects treated as perturbations, and treatments based on the reduced Salpeter or other relativistic equations with spin included exactly.[8–10] While the results generally favor a Lorentz-scalar form of the confining interactions with a short-range Lorentz-vector interaction, some fits favor a vector-scalar mixture in the confining potential,[11] and the phenomenological situation remains unclear. However, spin effects have been used in nonperturbative calculations in lattice QCD to show conclusively that the interactions between heavy quarks are predominantly scalar at large distances, and are consistent with vector single-gluon exchange at short distances.[12] We will adopt this picture below.

In this paper we summarize briefly the results of a complete spin-dependent analysis of the data on ($q\bar{q}$) and unlike ($q\bar{Q}$) quark-antiquark systems. The analysis is based on exact numerical solution of the reduced Salpeter equation. We encountered problems similar to those encountered in the spin-averaged analysis when we attempted to fit the complete spectra. In particular, we could obtain excellent fits to the spin splittings in the $c\bar{c}$ and $b\bar{b}$ systems, but only at the expense of increased discrepancies in the spin-averaged energy levels. The situation was not improved significantly by allowing a vector component in the confining interaction. On the basis of the systematics of the fits and calculations of Regge trajectories for light-quark mesons, we have concluded, as in Ref. 1, that the starting point of the calculations, the reduced Salpeter equation with static interactions, does not give an adequate description of the basic physics of the lighter $q\bar{q}$ systems. This conclusion is further substantiated by the results on the $q\bar{Q}$ systems. Our treatment of spin effects may nevertheless be of some interest for future calculations since we avoid the usual expansions in inverse powers of the quark masses—thus allowing a sensible treatment of light-quark systems—and use new matrix methods to solve the resulting coupled, nonlocal differential equations.[13]

The organization of the rest of the paper is as follows. In Sec. II, we describe the transformation of the reduced Salpeter equation in momentum space into the set of coupled, nonlocal equations in position space which we have used in our calculations. We discuss our parametrization of the $q\bar{q}$ interaction in Sec. III A, and our numerical procedures in Sec. III B. Our results and conclusions are summarized in Secs. III C and III D. Some results useful in the reduction of the spin-dependent wave equation to radial form are summarized in the Appendix.

II. THE REDUCED SALPETER EQUATION IN POSITION SPACE

Our analysis of quark-antiquark bound states will be based on the reduced Salpeter equation used in a number of studies of relativistic bound states. For a mixture of scalar and vector interactions between the quark and the antiquark (possibly with different flavors), the equation assumes the form

$$(M-\omega_1-\omega_2)\Phi(\mathbf{p})=\Lambda^+(\mathbf{p})\gamma^0\int\frac{d^3p'}{(2\pi)^3}[V_V(|\mathbf{p}-\mathbf{p}'|)\gamma_\mu\Phi(\mathbf{p}')\gamma^\mu+V_S(|\mathbf{p}-\mathbf{p}'|)\Phi(\mathbf{p}')]\gamma^0\Lambda^-(-\mathbf{p}) , \tag{1}$$

where V_V and V_S are scalar potentials, $\Lambda^\pm$ are positive- and negative-energy projection operators, $\omega_i=(\mathbf{p}^2+m_i^2)^{1/2}$, M is the total mass of the bound state, and Φ is a 4×4 matrix wave function. We will represent Φ in block matrix form as

$$\Phi=\begin{bmatrix}\phi^{+-} & \phi^{++}\\ \phi^{--} & \phi^{-+}\end{bmatrix} , \tag{2}$$

where each element is a 2×2 matrix with the first index associated with the quark and the second with the antiquark.

The relations

$$\phi^{+-}=\phi^{++}\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{\phi_2+m_2},\quad \phi^{-+}=\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{\omega_1+m_1}\phi^{++},\quad \phi^{--}=\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{\omega_1+m_1}\phi^{++}\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{\omega_2+m_2} , \tag{3}$$

which follow from the properties of the projection operators in Eq. (1) allow us to express the "small components" of Φ in terms of the "large component" ϕ^{++}, and reduce Eq. (1) to an equation for ϕ^{++} alone. After a change to a normal spin basis using the definition[1]

$$\phi_{m_1m_2}(\mathbf{p})=[\phi^{++}(\mathbf{p})i\sigma_2]_{m_1m_2} , \tag{4}$$

that equation can be converted to a matrix wave equation for the wave functions $\phi_{m_1m_2}$ with spin projections m_1, m_2:[1]

$$(M-\omega_1-\omega_2)\phi(\mathbf{p})=\int\frac{d^3p'}{(2\pi)^3}\sum_{i=S,V}F_i(\mathbf{p},\mathbf{p}',\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2)V_i(|\mathbf{p}-\mathbf{p}'|)\phi(\mathbf{p}') . \tag{5}$$

Here,

$$\begin{aligned}F_S(\mathbf{p},\mathbf{p}',\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2)=\frac{1}{4\omega_1\omega_2}\Bigg[&(\omega_1+m_1)(\omega_2+m_2)-\left[\frac{\omega_1+m_1}{\omega_2'+m_2}+\frac{\omega_2+m_2}{\omega_1'+m_1}\right]\left[\mathbf{p}\cdot\mathbf{p}'+\frac{i}{2}(\mathbf{p}\times\mathbf{p}')\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\right]\\&-\left[\frac{\omega_1+m_1}{\omega_2'+m_2}-\frac{\omega_2+m_2}{\omega_1'+m_1}\right]\left[\frac{i}{2}\mathbf{p}\times\mathbf{p}'\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\right]\\&+\frac{1}{(\omega_1'+m_1)(\omega_2'+m_2)}[(\mathbf{p}\cdot\mathbf{p}')^2+i\mathbf{p}\cdot\mathbf{p}'(\mathbf{p}\times\mathbf{p}')\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)-(\mathbf{p}\times\mathbf{p}')\cdot\boldsymbol{\sigma}_1(\mathbf{p}\times\mathbf{p}')\cdot\boldsymbol{\sigma}_2]\Bigg] ,\end{aligned} \tag{6a}$$

$$\begin{aligned}F_V(\mathbf{p},\mathbf{p}',\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2)=\frac{1}{4\omega_1\omega_2}\Bigg[&(\omega_1+m_1)(\omega_2+m_2)+\mathbf{p}^2-\mathbf{p}^2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+\mathbf{p}\cdot\boldsymbol{\sigma}_1\mathbf{p}\cdot\boldsymbol{\sigma}_2\\&+\left[\frac{\omega_1+m_1}{\omega_2'+m_2}+\frac{\omega_2+m_2}{\omega_1'+m_1}\right]\left[\mathbf{p}\cdot\mathbf{p}'+\frac{i}{2}(\mathbf{p}\times\mathbf{p}')\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\right]\\&+\left[\frac{\omega_1+m_1}{\omega_2'+m_2}-\frac{\omega_2+m_2}{\omega_1'+m_1}\right]\left[\frac{i}{2}(\mathbf{p}\times\mathbf{p}')\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\right]\\&+\left[\frac{\omega_1+m_1}{\omega_1'+m_1}+\frac{\omega_2+m_2}{\omega_2'+m_2}\right][\mathbf{p}\cdot\mathbf{p}'(1+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)+i(\mathbf{p}\times\mathbf{p}')\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)-\mathbf{p}\cdot\boldsymbol{\sigma}_1\mathbf{p}'\cdot\boldsymbol{\sigma}_2]\\&+\frac{(\omega_1+m_1)(\omega_2+m_2)}{(\omega_1'+m_1)(\omega_2'+m_2)}(\mathbf{p}'^2-\mathbf{p}'^2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+\mathbf{p}'\cdot\boldsymbol{\sigma}_1\mathbf{p}'\cdot\boldsymbol{\sigma}_2)\\&+\frac{1}{(\omega_1'+m_1)(\omega_2'+m_2)}[(\mathbf{p}\cdot\mathbf{p}')^2+i\mathbf{p}\cdot\mathbf{p}'(\mathbf{p}\times\mathbf{p}')\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)-(\mathbf{p}\times\mathbf{p}')\cdot\boldsymbol{\sigma}_1(\mathbf{p}\times\mathbf{p}')\cdot\boldsymbol{\sigma}_2]\Bigg] .\end{aligned} \tag{6b}$$

The Pauli matrices σ_1 and σ_2 act on the first and second spin indices of $\phi_{m_1 m_2}$.

It is convenient in fitting $q\bar{q}$ bound states to transform Eq. (5) to position space where the potentials V_i are local and easy to handle. A Fourier transform gives the simple-appearing, but nonlocal, equation

$$(M-E_1-E_2)\psi(\mathbf{r}) = \sum_{i=S,V} \mathcal{F}_i(-i\nabla,-i\nabla',\sigma_1,\sigma_2)V_i(r)(r)\psi(\mathbf{r}) , \quad (7)$$

where the $\mathcal{F}$'s are obtained by replacing $\mathbf{p}$ and $\mathbf{p}'$ in Eqs. (6) by the gradient operators $-i\nabla$ and $-i\nabla'$, and the ω's by the nonlocal operators $E_i=(-\nabla^2+m_i^2)^{1/2}$. The E's are defined as the formal Fourier transforms of $\omega_1=(\mathbf{p}^2+m_i^2)^{1/2}$, $V_i(r)$ is the Fourier transform of $V_i(|\mathbf{p}-\mathbf{p}'|)$, and $\psi_{m_1 m_2}(\mathbf{r})$ is the transform of $\phi_{m_1 m_1}(\mathbf{p})$. It is understood that the gradient operators ∇ and ∇' in Eq. (7) are to be written respectively on the left- and right-hand sides of $V(r)$; both act on the coordinate $\mathbf{r}$. For example, the term of the form

$$\frac{\omega_1+m_1}{\omega_2'+m_2}\mathbf{p}\cdot\mathbf{p}'V(|\mathbf{p}-\mathbf{p}'|)\Phi(\mathbf{p}') \quad (8a)$$

in Eq. (5) becomes

$$-(E_1+m_1)\nabla\cdot[V(r)(E_2+m_2)^{-1}\nabla\psi(\mathbf{r})] \quad (8b)$$

in Eq. (7), where $(E_2+m_2)^{-1}$ is the operator inverse of (E_2+m_2).

The total angular momentum j and the parity of the $q\bar{q}$ system are conserved by the interaction terms in Eq. (7). We can therefore separate that equation into a set of coupled equations for states of given j and parity, but differing values of the orbital and spin angular momenta l and s. In the case of equal-mass particles on which we will concentrate for simplicity, the terms in Eqs. (6a) and (6b) which involve $\sigma_1-\sigma_2$ vanish, and there is no mixing of singlet- and triplet-spin states. The total spin is therefore a good quantum number, and we can make the standard separation into singlet- and triplet-spin systems, $\psi\rightarrow\psi_{jms}$, with

$$\psi_{jm0}(\mathbf{r})=R_{j,0}(r)\mathcal{Y}_{jmj0}(\hat{\mathbf{r}}) \quad \text{with } P=-(-1)^j ,$$
$$\psi_{jml}(\mathbf{r})=R_{j+1,1}(r)\mathcal{Y}_{jm,j+1,1}(\hat{\mathbf{r}})+R_{j-1,1}(r)\mathcal{Y}_{jm,j-1,1}(\hat{\mathbf{r}}) \quad \text{for } P=(-1)^j , \quad (9)$$
$$\psi_{jm1}(\mathbf{r})=R_{j,1}(r)\mathcal{Y}_{jmj1}(\hat{\mathbf{r}}) \quad \text{for } P=-(-1)^j ,$$

where the $\mathcal{Y}$'s are normalized spin-angle functions:

$$\mathcal{Y}_{jmls}(\hat{\mathbf{r}})=\sum_{m_1 m_2}(-1)^{l-s+m}(2j+1)^{1/2}\begin{pmatrix} l & s & j \\ m_l & m_s & -m \end{pmatrix} \times Y_{lm_l}(\hat{\mathbf{r}})\chi^s_{m_s}(s_1,s_2) . \quad (10)$$

After a rather tedious calculation, we can reduce Eq. (7) to a set of nonlocal equations for the radial wave functions $R_{ls}(r)$ [coupled equations for $s=1$, $P=(-1)^j$]. The results are too complicated to record, and were actually constructed in our calculations as sums of terms corresponding to the different types of spin operators which appear in the $\mathcal{F}$'s. The relations necessary for this construction are summarized in the Appendix.

III. SPIN-DEPENDENT FITS TO $q\bar{q}$ BOUND STATES

A. The quark-antiquark interaction

We will use the parametrization of the scalar and vector components of the quark-antiquark interaction used in Ref 1:

$$V_S(r)=(1-\delta)(-\beta/r+Br)(1-e^{-\mu r})+V_0 +(C_0+C_1 r+C_2 r^2)(1-e^{-\mu r})e^{-\mu r} , \quad (11a)$$
$$V_V(r)=-\frac{4}{3}\frac{\alpha_s(r)}{r}e^{-\mu' r}+\delta(-\beta/r+Br)(1-e^{-\mu r}) . \quad (11b)$$

The scalar potential $V_S(r)$ incorporates the expected long-range linear confining potential with a slope B which is expected naively to be related to the slope α' of the leading light-meson $(I\bar{I})$ Regge trajectory by $B=1/2\pi\alpha'$. We have also included a $1/r$ term (the Lüscher term[14]) which arises from the transverse zero-point oscillations of a string or flux tube joining the quarks. Otto and Stack[15] obtain the value $\beta=(0.95\pm0.08)\pi/12$ for the coefficient of this term in a lattice SU(3) calculation, in good agreement with the value $\beta=\pi/12$ expected for a Nambu string. We will take $\beta=\pi/12$. We have multiplied the confining term in V_S by a factor $(1-\delta)$ and included an identical confining term in V_V, with a coefficient δ to allow us to adjust the scalar-vector mix in the confining interaction. We expect theoretically to have $\delta=0$; the best fits in Ref. 1 corresponded to $\delta\approx 1/2$. The remaining terms in V_S provide a flexible parametrization of this potential at intermediate to small values of r.

In addition to the (possible) vector piece of the confining interaction noted above, the vector potential $V_V(r)$ incorporates the expected short-distance behavior from single-gluon exchange,[16] but with a damping factor $e^{-\mu' r}$ to eliminate this term at large r where the r^{-1} dependence is associated with the Lüscher term. In this parametrization, $1/\mu'$ acts as confinement radius, around which the nature of the $q\bar{q}$ interaction changes. The running coupling constant $\alpha_s(r)$ in the gluon-exhange term is defined, following the position-space analysis of Hagiwara *et al.*[17] as modified in Ref. 1, as the solution of a regularized next-to-leading-order renormalization-group equation

$$b_0\ln\left[\left[\frac{Q(r)}{\Lambda_{\overline{\mathrm{MS}}}}\right]^2+\left[\frac{b_0^2}{2b_1}\right]^{2b_1/b_0^2}\right] =\frac{2\pi}{\alpha_s}+\frac{2b_1}{b_0}\ln\left[\frac{b_0\alpha_s}{2\pi}\Big/\left[1+\frac{b_1\alpha_s}{b_0\pi}\right]\right] . \quad (12)$$

Here

$$b_0=\tfrac{11}{2}-\tfrac{1}{3}n_f, \quad b_1=\tfrac{51}{4}-\tfrac{19}{12}n_f , \quad (13)$$

for a color-singlet $q\bar{q}$ system with n_f active quarks, and $\Lambda_{\overline{\rm MS}}^{(n_f)}$ is the corresponding QCD scale factor in the modified minimal subtraction scheme. We will use $n_f=4$. The function $Q(r)$ is defined by Hagiwara *et al.*;

$$Q(r)=\frac{1}{r}e^{-A(r)/b_0}\ , \tag{14a}$$

where the function $A(r)$ is given to good approximation over the range of r of interest by

$$A(r)\approx\tfrac{25}{6}\gamma_E+\tfrac{197}{333}+\tfrac{1}{3}[\gamma_E+\ln(m_c r e^{5/6})+E_1(m_c r e^{5/6})]\ . \tag{14b}$$

Here $\gamma_E=0.5772\ldots$ is Euler's constant and $E_1(z)$ is an exponential integral function.[18]

B. Data and fitting procedures

The data included in our analysis consisted of the masses of the well-established $b\bar{b}$, $c\bar{c}$, and $s\bar{s}$ states given in the first column in Table I.[19] Because of theoretical uncertainties in our model, which are certainly large compared to the experimental uncertainties in many of the masses, we did not use the latter to weight the data, but weighted all states equally.

Our theoretical calculations of the $q\bar{q}$ spectra are performed by reducing Eq. (7) to a set of radial equations for fixed j, s, and parity as sketched in the preceding section. After the angular dependence of the wave functions has been extracted, the nonlocal operators $E=(-\nabla^2+m^2)^{1/2}$ appear in a modified form which depends on the orbital angular momentum l in the term in question:

$$E\to E_l=(-\nabla_l^2+m^2)^{1/2}\ ,\qquad \nabla_l^2=\frac{1}{r}\frac{d^2}{dr^2}r-\frac{l(l+1)}{r^2}\ . \tag{15}$$

The crux of the calculation is the generation of effective matrix representations of this operator which allow us to reduce the solution of the radial eigenvalue problem to a small matrix problem. The method used is described in detail in a separate paper.[13] We will only sketch the principal features here.

For equal-mass $q\bar{q}$ systems, we can scale the quark mass out of such operators as E_l+m. We then choose a finite basis $\{L_j(x),\ j=1,\ldots,n\}$ of associated Laguerre functions (these are appropriate for the solution of Coulomb-like problems), and construct a matrix $\mathbf{D}_l$ which gives the exact action of $(-\nabla_l^2+1)$ on the basis functions at the zeros $\{x_{n+1,i},\ i=1,\ldots,n\}$ of $L_{n+1}(x)$. The matrix representation of the square-root operator $(-\nabla_l^2+1)^{1/2}$ is then constructed as

TABLE I. Fits to the $s\bar{s}$, $c\bar{c}$, and $b\bar{b}$ spectra using the full spin-dependent relativistic interaction in Eq. (7). The experimental masses and mass differences were taken from Ref. 19. Comparisons are made where possible with the spin-weighted averages of multiplets, and the splittings within multiplets. δ measures the amount of scalar-vector mixing; $\delta=0$ for pure scalar confinement.

State	Data (MeV)	Spin-averaged potential, $\delta=0$ (MeV)	Error (MeV)	Spin-dependent fit, $\delta=0$ (MeV)	Error (MeV)	Spin-dependent fit, $\delta\neq0$ (MeV)	Error (MeV)
				$s\bar{s}$ states			
$\phi(1\,^3S_1)$	1019	1052	+33	1098	+79	1019	0
$\phi(2\,^3S_1)$	1685^{+75}_{-15}	1520	−165	1616	−69	1510[a]	−175
				$c\bar{c}$ states			
$1S$ avg.	3068	3082	+14	3078	+10	3067	−1
$\psi-\eta_c$	117	141	+24	97	−20	100	−17
$1\,^3P$ avg.	3525	3514	−11	3512	−13	3513	−12
$\chi_{c1}-\chi_{c0}$	96	143	+47	88	−8	91	−5
$\chi_{c2}-\chi_{c0}$	141	249	+108	141	0	146	+5
$\psi(2\,^3S_1)$	3686	3671	−15	3665	−21	3668	−18
$\eta_c(2\,^1S_0)$		3628		3626		3621	
				$b\bar{b}$ states			
$\Upsilon(1^3S_1)$	9460	9441	−19	9434	−26	9426	−34
1^3P avg.	9899	9895	−4	9895	−4	9903	+4
$\chi_{b1}-\chi_0$	32	43	+11	33	+1	30	−2
$\chi_{b2}-\chi_{b0}$	53	77	+24	57	+4	55	+2
$\Upsilon(2^3S_1)$	10023	10021	−2	10027	+4	10028	+5
2^3P avg.	10261	10257	−4	10273	+12	10262	+1
$\chi_{b1}-\chi_{b0}$	20	26	+6	22	+2	18	−2
$\chi_{b2}-\chi_{b0}$	34	45	+11	36	+2	32	−2
$\Upsilon(3\,^3S_1)$	10356	10349	−7	10381	+25	10359	+3

[a]Not included in fit.

$\mathbf{S}_l=\mathbf{D}_l^{1/2}=\mathbf{U}_l\boldsymbol{\lambda}_l^{1/2}\mathbf{U}_l^{-1}$, where $\boldsymbol{\lambda}_l$ is the diagonal matrix of eigenvalues of $\mathbf{D}_l$ and $\mathbf{U}_l$ is the matrix of eigenvectors. The radial wave functions are represented as column vectors containing their values at the points $x_{n+1,i}$, and operators such as E_l, E_l^{-1}, or $(E_l+m)^{-1}$ are represented by the matrices $\mathbf{S}_l$, $\mathbf{S}_l^{-1}$, or $(\mathbf{S}_l+\mathbf{1})^{-1}$. Finally, ordinary differential operators are represented by matrices which give their exact action on the basis set at the points $x_{n+1,i}$.

This construction, which is closely related to the method of orthogonal collocation, allows us to reduce the solution of our very complicated set of nonlocal equations to a matrix eigenvalue problem. The eigenvalues converge extremely rapidly with increasing matrix size, and it is sufficient in the calculations reported here to use 25×25 matrices. These can be manipulated sufficiently rapidly that we can use a standard nonlinear least-squares fitting routine to search for the potential parameters and quark masses which give the best fit to the data, within reasonable physical constraints. The results are discussed in the next section.

C. Spin-dependent fits

In Table I we show the results obtained for the $s\bar{s}$, $c\bar{c}$, and $b\bar{b}$ spectra in three complete spin-dependent calculations. In the first set, we simply calculated the spin-dependent spectra using the scalar and vector potentials obtained in the $\delta=0$ (scalar confinement) fit to the spin-averaged data discussed in Ref. 1. The results show clearly the necessity of including spin in the analysis from the beginning. The spin averages of the calculated masses agree very well with the results of Ref. 1, with shifts of only 2–5 MeV, thus suggesting the approximate validity of the lowest-order perturbative treatment of the spin-dependent interactions used in many previous analyses.[6,7] However, the calculated mass splittings within spin multiplets are quite poor.

The second set of results in Table I is a best fit to the data using the potentials in Eqs. (11) with $\delta=0$, that is, pure scalar confinement at large distances. The improvement in the fit is substantial. In particular, the P-wave splittings in the $c\bar{c}$ and $b\bar{b}$ systems are within a few MeV of the correct values. However, as would be expected, the problems encountered in Ref. 1 with the spacings of the multiplets (e.g., the spacings of the centers of gravity, or of the $n\,^3S_1$ states) persist. We have not been able to obtain completely satisfactorily fits to all the states within our constraints that (i) the potentials be flavor-independent, (ii) the short-range interaction be described by the one-gluon-exchange potential including vacuum polarization corrections, i.e., that the running of $\alpha_s(r)$ in Eq. (11b) be determined by Eq. (12), and (iii) the dominant confining interaction be Lorentz scalar, $\delta=0$ in Eqs. (11).

The parameters of this fit and the corresponding parameters in Ref. 1 are

$$m_s=0.496\pm0.034\ \text{GeV}\quad(0.482\ \text{GeV}),$$

$$m_c=1.681\pm0.042\ \text{GeV}\quad(1.636\ \text{GeV}),$$

$$m_b=5.052\pm0.044\ \text{GeV}\quad(4.962\ \text{GeV}),$$

$$\Lambda=0.342\pm0.011\ \text{GeV}\quad(0.430\ \text{GeV}),$$

$$B=0.209\pm0.061\ \text{GeV}^2\quad(0.177\ \text{GeV}^2),$$

$$V_0=-0.612\pm0.454\ \text{GeV}\quad(-0.366\ \text{GeV}),$$

$$C_0=1.03\pm2.10\ \text{GeV}\quad(2.45\ \text{GeV}),$$

$$C_1=0.62\pm1.00\ \text{GeV}^2\quad(-0.074\ \text{GeV}^2),$$

$$C_2=0.24\pm0.41\ \text{GeV}^2\quad(0.34\ \text{GeV}^3),$$

$$\mu=1.19\pm0.74\ \text{GeV}\quad(0.933\text{GeV}),$$

$$\mu'=0.825\pm0.041\ \text{GeV}\quad(0.740\ \text{GeV}).$$

The main changes are in the slope B of the linear confining potential, and in the poorly determined (and strongly correlated) parameters associated with the intermediate range potential. The other parameters—Λ, the μ's which determine where the vector-scalar transition occurs, and the quark masses—are reasonable, though we note that m_s is unexpectedly large, a sign the model would prefer nonrelativistic to relativistic quarks as discussed in Ref. 1.

The third set of results in Table I is the best spin-dependent fit to the data from Ref. 9. In this analysis δ in Eqs. (11) was treated as a free parameter to determine if the addition of a Lorentz-vector component to the confining potential would improve the fit, as it had in the spin-independent case (Refs. 1 and 9). The parametrization of $V_V(r)$ also included an extra energy shift obtained by replacing $-4\alpha_s/3r$ in Eq. (11b) by $(-4\alpha_s/3r+V_0')$. Despite the extra freedom in the potential, there is not a significant improvement in the fit to the $c\bar{c}$ and $b\bar{b}$ states, or to the $1S$-$2S$ splitting in the $s\bar{s}$ system. [The $2\,^3S_1$ $s\bar{s}$ state was not included in the fit, so that $1\,^3S_1$ energy could be fitted exactly by varying m_s; in the $\delta=0$ fit, the (smaller) error in the splitting was divided between the two states.] The best value of δ was not large, $\delta=0.29\pm0.08$, i.e., $a\sim30$–70 vector-scalar mixture.

There is some improvement in the mass splittings *between* multiplets with $\delta>0$, but the errors in these splittings remain large. These errors could be reduced to a few MeV in the spin-independent analysis in Ref. 1 with a 50-50 vector-scalar mixture in the confining interaction ($\delta=\frac{1}{2}$). However, the spin splittings *within* multiplets evidently favor at most a small vector component in the confining interaction. Even that component is somewhat suspect as there is a natural overlap of the vector interaction from single-gluon exchange and the (scalar) confining interaction at intermediate quark-antiquark separations where the wave functions are largest. The separation of the total vector interaction in this region into gluon-exchange and confining components is ambiguous since the former becomes singular at a finite radius where $\alpha_s(r)$ diverges and must be regularized[1,17] and, hence, does not have a well-determined form. The $1S$-$2S$ splitting in the $s\bar{s}$ system, which is most sensitive to the long range part of the interaction, is actually worse with $\delta=0.29$ than with $\delta=0$. We therefore do not regard the evidence for a vector component of the confining interac-

TABLE II. Calculated masses for unequal-mass $q\bar{Q}$ states obtained using the $\delta > 0$ potential of Ref. 9. The corresponding results for the $q\bar{q}$ states are given in Table I.

State	Quarks	Data (MeV)	Calculated (MeV)	Error (MeV)
$\bar{B}^0$	$b\bar{d}$	5279	5381	+102
D_s^+	$c\bar{s}$	1969	2097	+128
D_s^{*+}	$c\bar{s}$	2113	2148	+35
D^+	$c\bar{d}$	1869	1983	+114
D^{*+}	$c\bar{d}$	2010	2010[a]	
$\bar{K}^0$	$s\bar{d}$	498	743	+245
$\bar{K}^{*0}$	$s\bar{d}$	892	870	−22
D_s^{*+}-D_s^+		144	51	−93
D^{*+}-D^+		141	27	−114
$\bar{K}^{*0}$-$\bar{K}^0$		394	127	−267

[a]Used to fix m_d.

tion as particularly significant.

We have also calculated the masses of the mixed $q\bar{Q}$ ($b\bar{d}$, $c\bar{s}$, $c\bar{d}$, and $s\bar{d}$) quark-antiquark states using the $\delta > 0$ potentials with m_d adjusted to $m_d \equiv 76$ MeV to fit the mass of the D^{*+}. The results[9] are given in Table II. The calculated masses and spin splittings are generally quite poor, with the $1\,^3S_1$-$1\,^1S_0$ splittings too small by large factors, e.g., 2.7 for $D_s^{*+}-D_s^+$, and more for the states containing light quarks. These results cast serious doubt on the usefulness of the Salpeter equation with static potentials in describing systems containing light or even strange quarks.

We have also investigated briefly the Regge trajectory for the highest triplet spin states for light-quark systems. The results are essentially unchanged from those given for the spin-averaged interaction in Ref. 1: the trajectory is much steeper than the observed ρ or ω trajectories for any reasonable value of the parameter B in the confining potential.

D. Discussion

The results above—the failure to obtain satisfactory fits to the $c\bar{c}$ and $b\bar{b}$ spectra and the $s\bar{s}$ states, the very bad results for the $q\bar{Q}$ states containing strange or light quarks, and the unsatisfactory light-quark $q\bar{q}$ Regge trajectories—lead us to the same conclusion as we reached in Ref. 1, that the reduced Salpeter equation with the expected static scalar and vector potentials does not give an adequate description of relativistic bound states. The basic reason was discussed in Ref. 1. The spacings between spin multiplets are determined mainly by the spin-independent parts of the interactions in Eqs. (15) and (16). In particular, the spin-independent part of the scalar operator F_S, Eq. (6a), is given for equal quark masses by

$$F_S^{\rm SI} = \left[\frac{\omega+m}{2\omega}\right]^2 \left[1 - \frac{\mathbf{p}\cdot\mathbf{p}'}{(\omega+m)(\omega'+m)}\right]^2 . \qquad (16)$$

A linear confining potential varies in momentum space as $(\mathbf{p}-\mathbf{p}')^{-4}$, so kinematic configurations with $\mathbf{p}' \approx \mathbf{p}$ are strongly weighted in Eq. (5). For $\mathbf{p}'=\mathbf{p}$, $F_S^{\rm SI}=(m/\omega)^2$, and the effective strength of the confining interaction is sharply reduced for relativistic quarks, i.e., for $|\mathbf{p}| > m$. Although the corresponding vector operator $F_V^{\rm SI}$ has its maximum for $\mathbf{p}'=\mathbf{p}$, the addition of spin splittings to the analysis prevents the addition of enough vector confinement to offset the kinematic suppression in the scalar confining potential, and the overall interaction becomes weaker for relativistic than for nonrelativistic quarks. This effect shows up clearly in the unphysical compression of the excited states evident in the calculated steep Regge trajectories, and in the consistent failure of the reduced Salpeter equation to fit the $1S$-$2S$ spacings in the $c\bar{c}$ and $s\bar{s}$ systems: the spacings are consistently too small.

The very small $^3S_1$-$^1S_0$ splittings obtained for the $q\bar{Q}$ systems containing light quarks are another consequence of the weakness of the binding potential in these systems. The calculated wave functions are very long range (see, e.g., Fig. 1 in Ref. 1 for the $s\bar{s}$ wave function), and the probability of the quarks being close together where the $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$ terms in the vector interaction in Eq. (6b) can act is small. More precisely, the splitting is given by $(16\pi\alpha_s/9m_q^2)|\psi(0)|^2$ in the usual approximation in which the interaction expanded to leading order in $1/m_q^2$, and $|\psi(0)|^2$ is simply too small as calculated here. The situation is not changed significantly for the exact $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$ interaction, although the exact interaction extends over a distance $\sim 1/m_q$ rather than being concentrated at $\mathbf{r}=0$. It is easily checked that the reduction in the magnitude of ψ near the origin (estimated simply from the greater volume occupied by the wave function) can account for the remarkably small hyperfine splittings of the light-quark systems.

We emphasize that the effective suppression of the confining interaction for relativistic quarks is contrary to the behavior expected for stringlike confinement. The more relativistic the quarks in a string model, the closer the Regge trajectories are to being linear with a slope $\alpha' = 1/2\pi B$ given directly by the energy per unit length B in the string, that is, the slope of the linear confining potential for static (or nonrelativistic) quarks.[2,3] The problem in the Salpeter approach, we believe, is in the omission in the Hamiltonian of the kinetic energy of the rotating color flux tube (string) connecting the quarks. We note in this connection that Pisarski and Stack[4] have derived a consistent string picture for quark confinement which includes spin effects naturally, and gives the spin-orbit interactions corresponding to scalar confinement for large quark separations.

There are of course other relativistic contributions which are omitted in the present treatment of the Bethe-Salpeter equation, such as the retardation effects dropped in the instantaneous approximation used to obtain the Salpeter equation, and the further omission of pair effects in the reduction of the latter.[20] However, it is difficult to see how these effects could eliminate the problems with the Regge trajectories which are explained naturally in the flux tube or string picture.

Other authors have noted that one can obtain better overall fits to the $c\bar{c}$ and $b\bar{b}$ spectra if the spin-independent relativistic corrections in the operators F_S and F_V in Eqs. (6) are eliminated[21] or treated phenomenologically.[10] For example, Olsson and Suchyta[10] found that they could obtain good fits to the $c\bar{c}$ and $b\bar{b}$ spectra only by treating the relativistic correction terms in $F_S^{\rm SI}$ (i.e., the difference of $F_S^{\rm SI}$ from unity) phenomenologically, with the strength of the corrections fitted to the data. In a different approach, Gupta, Repko, and Suchyta[11] used a modified quasistatic potential proposed by Gupta[21] and a flavor-dependent vector-scalar mixture for the confining potential to obtain a nearly perfect fit to the $c\bar{c}$ and $b\bar{b}$ spectra. However, the derivation of the modified potential in Ref. 20 is suspect for a linear confining interaction since, as noted above, small values of the momentum difference $(\mathbf{p}-\mathbf{p}')$ are emphasized, while the approximations used in Ref. 20 assume instead that $(\mathbf{p}+\mathbf{p}')$ is small and $(\mathbf{p}-\mathbf{p}')$, large. The approximations, in any case, amount to setting $F_S^{\rm SI}$ and $F_V^{\rm SI}$ equal to unity so that the spin-independent potential is just V_S+V_V, without corrections, and the problems discussed above are eliminated. However, these approaches depart from the approach based on the reduced Salpeter equation which is adopted here, as does the use of flavor-dependent interactions by Gupta *et al.*[11] The conclusion remains, that the reduced Salpeter equation is not an adequate starting point for the discussion of a relativistic $q\bar{q}$ (or $q\bar{Q}$) system.

We recall finally (and more positively) that the spin-dependent shifts in particle masses calculated with the full interaction average very nearly to zero over the various $c\bar{c}$ and $b\bar{b}$ multiplets, suggesting that first-order perturbation theory is approximately valid. This conclusion is supported by direct perturbative calculations of the splittings.[9] Also, it is certainly necessary when dealing with the lighter quarks to avoid expansions in powers of $1/m_q$, and, hence, to use the exact (or appropriately smoothed[20]) interactions for numerical calculations. The methods presented here allow a straightforward treatment of these nonlocal perturbations.

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APPENDIX

In this Appendix, we collect a number of relations which are needed in the reduction of the position-space form of Eq. (7) to a set of (coupled) radial wave equations for states with specific angular momentum and parity.

1. Spin-independent operators

The spin-independent operators which appear in $F_S(\mathbf{p},\mathbf{p}',\sigma_1,\sigma_2)$ and $F_V(\mathbf{p},\mathbf{p}',\sigma_1,\sigma_2)$ in Eqs. (6a) and (6b) are

$$\begin{aligned}
&\mathbf{p}^2,\mathbf{p}'^2\rightarrow-\nabla^2,-\nabla'^2\ ,\\
&\omega,\omega'\rightarrow(-\nabla^2+m^2)^{1/2},\ \ (\nabla'^2+m^2)^{1/2}\ ,\\
&\mathbf{p}\cdot\mathbf{p}'\rightarrow-\nabla\cdot\nabla'\ ,\\
&(\mathbf{p}\cdot\mathbf{p}')^2\rightarrow(\nabla\cdot\nabla')^2=\nabla_i\nabla_j\nabla'_i\nabla'_j\ ,
\end{aligned}\tag{A1}$$

where the second form given in each case is that which appears in the Fourier-transformed operators $\mathcal{F}_S(-i\nabla,-i\nabla',\sigma_1,\sigma_2)$ and $\mathcal{F}_V(-i\nabla,-i\nabla',\sigma_1,\sigma_2)$ in Eq. (7). We recall that the primed operators in $\mathcal{F}_S$ and $\mathcal{F}_V$ stand to the right of $V_i(r)$, hence act only on $\psi(\mathbf{r})$, while the unprimed operators act on both $V_i(r)$ and $\psi(\mathbf{r})$.

The reduction of Eq. (7) to a set of radial wave equations, some of which are coupled, requires that we extract the angular dependence of the wave functions, that is, move the spin-angle functions $\mathcal{Y}_{jmls}(\hat{r})$ in Eqs. (9) to the left through the operators which appear. The relations necessary in the case of the operators above are

$$\nabla^2F(r)\mathcal{Y}_{jmls}(\hat{\mathbf{r}})=\mathcal{Y}_{jmls}(\hat{\mathbf{r}})\nabla_l^2F(r)\ ,\tag{A2a}$$

$$(-\nabla^2+m^2)^{1/2}F(r)\mathcal{Y}_{jmls}(\hat{\mathbf{r}})=\mathcal{Y}_{jmls}(\hat{\mathbf{r}})(-\nabla_l^2+m^2)^{1/2}F(r)\ ,\tag{A2b}$$

$$[\nabla_iV(r)\nabla_i]R_{ls}(r)\mathcal{Y}_{jmls}(\hat{\mathbf{r}})=\mathcal{Y}_{jmls}(\hat{\mathbf{r}})\left[\frac{dV(r)}{dr}\frac{d}{dr}+V(r)\nabla_l^2\right]R_{ls}(r)\ ,\tag{A2c}$$

$$[\nabla_i\nabla_jV(r)\nabla_i\nabla_j]R_{ls}(r)\mathcal{Y}_{jmls}(\hat{\mathbf{r}})=\mathcal{Y}_{jmls}(\hat{\mathbf{r}})\left[\frac{d^2V(r)}{dr^2}\frac{d^2}{dr^2}+\frac{dV(r)}{dr}\left[\nabla_l^2\frac{d}{dr}+\frac{d}{dr}\nabla_l^2\right]+V(r)\nabla_l^2\nabla_l^2\right]R_{ls}(r)\ ,\tag{A2d}$$

where $F(r)$ in the first equations may be $R_{lm}(r)$ or $V(r)R_{lm}(r)$, and

$$\nabla_l^2=\frac{1}{r}\frac{d^2}{dr^2}r-\frac{l(l+1)}{r^2}\ .\tag{A3}$$

We note that it is also useful to move some derivative operators to the left of the nonlocal operators $E_l=(-\nabla_l^2+m^2)^{1/2}$. The latter act as smoothing operators precisely because of their nonlocal character, essentially averaging the functions on which they act over regions with characteristic dimensions m^{-1}. It is advantageous to perform this smoothing operation before differentiating numerically when the original functions are singular. The relation

$$(-\nabla_l^2+m^2)\left[\frac{d}{dr}+\frac{l+2}{r}\right]=\left[\frac{d}{dr}+\frac{l+2}{r}\right](-\nabla_{l+1}^2+m^2)^{1/2} \tag{A4}$$

allows us to make these manipulations.

2. Spin-dependent operators

The spin-dependent operators which appear in Eqs. (6) and (7) are, following the conventions above,

$$\begin{aligned}
&\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\rightarrow\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2,\quad i(\mathbf{p}\times\mathbf{p}')\cdot(\boldsymbol{\sigma}_1\pm\boldsymbol{\sigma}_2)\rightarrow -i(\nabla\times\nabla')\cdot(\boldsymbol{\sigma}_1\pm\boldsymbol{\sigma}_2)\ ,\\
&\mathbf{p}\cdot\boldsymbol{\sigma}_1\mathbf{p}\cdot\boldsymbol{\sigma}_2\rightarrow -\nabla\cdot\boldsymbol{\sigma}_1\nabla\cdot\boldsymbol{\sigma}_2,\quad \mathbf{p}\cdot\boldsymbol{\sigma}_1\mathbf{p}'\cdot\boldsymbol{\sigma}_2\rightarrow -\nabla\cdot\boldsymbol{\sigma}_1\nabla'\cdot\boldsymbol{\sigma}_2\ ,\\
&i(\mathbf{p}\cdot\mathbf{p}')(\mathbf{p}\times\mathbf{p}')\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\rightarrow i(\nabla\cdot\nabla')(\nabla\times\nabla')\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\ ,\\
&(i\mathbf{p}\times\mathbf{p}'\cdot\boldsymbol{\sigma}_1)(i\mathbf{p}\times\mathbf{p}'\cdot\boldsymbol{\sigma}_2)\rightarrow(i\nabla\times\nabla'\cdot\boldsymbol{\sigma}_1)(i\nabla\times\nabla'\cdot\boldsymbol{\sigma}_2)\ .
\end{aligned} \tag{A5}$$

The action of several of these operators is more complex than that of the scalar operators and it is useful to express their action in terms of a set of basic operations. In particular, inserting the potential $V(r)$ explicitly and converting from Pauli matrices to spins $\mathbf{s}_i=\frac{1}{2}\boldsymbol{\sigma}_1$, we find that

$$\begin{aligned}
-i(\nabla\times\nabla')\cdot(\boldsymbol{\sigma}_1\pm\boldsymbol{\sigma}_2)&\rightarrow -i\epsilon^{ijk}\nabla_i[V(r)\nabla_j(\boldsymbol{\sigma}_1\pm\boldsymbol{\sigma}_2)_k]\psi(\mathbf{r})\\
&=-i\frac{1}{r}\frac{dV}{dr}(r)(\mathbf{r}\times\nabla)\cdot(\boldsymbol{\sigma}_1\pm\boldsymbol{\sigma}_2)=\frac{2}{r}\frac{dV}{dr}(r)\mathbf{L}\cdot(\mathbf{s}_1\pm\mathbf{s}_2)\ ,
\end{aligned} \tag{A6a}$$

$$\nabla\cdot\boldsymbol{\sigma}_1\nabla'\cdot\boldsymbol{\sigma}_2\rightarrow 4V(r)\nabla\cdot s_1\nabla\cdot s_2+4\frac{dV(r)}{dr}\hat{\mathbf{r}}\cdot s_1\nabla\cdot s_2\ , \tag{A6b}$$

$$-i(\nabla\cdot\nabla')(\nabla\times\nabla')\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\rightarrow 2\mathbf{L}\cdot\mathbf{S}\left[\frac{d^2V(r}{dr^2}\left[\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}\right]+\frac{dV(r)}{dr}\left[\frac{1}{r}\nabla^2-\frac{1}{r^2}\frac{d}{dr}+\frac{1}{r^3}\right]\right],\quad \mathbf{S}=\mathbf{s}_1+\mathbf{s}_2\ , \tag{A6c}$$

$$\begin{aligned}
(-i\nabla\times\nabla'\cdot\boldsymbol{\sigma}_1)(-i\nabla\times\nabla'\cdot\boldsymbol{\sigma}_2)\rightarrow&-4\frac{dV(r)}{dr}\left[\frac{1}{r^2}\left[\mathbf{s}_1\cdot\hat{\mathbf{r}}\mathbf{s}_2\cdot\nabla-\mathbf{s}_1\cdot\mathbf{s}_2\frac{d}{dr}\right]-\frac{1}{r}\mathbf{s}_1\cdot\nabla\mathbf{s}_2\cdot\nabla+\mathbf{s}_1\cdot\mathbf{s}_2\frac{1}{r}\nabla^2+\frac{1}{r^3}\mathbf{L}\cdot\mathbf{s}_1\mathbf{L}\cdot\mathbf{s}_2\right]\\
&+4\frac{d^2V}{dr^2}(r)\left[\frac{1}{r^2}\mathbf{L}\cdot\mathbf{s}_1\mathbf{L}\cdot\mathbf{s}_2+\frac{1}{r}\mathbf{s}_2\cdot\hat{\mathbf{r}}\mathbf{s}_1\cdot\nabla-\mathbf{s}_1\cdot\mathbf{s}_2\frac{1}{r}\frac{d}{dr}\right].
\end{aligned} \tag{A6d}$$

The spin dependence of the operators in Eqs. (5) can therefore be expressed in terms of the seven independent operators $\mathbf{s}_1\cdot\mathbf{s}_2$, $\mathbf{L}\cdot(\mathbf{s}_1\pm\mathbf{s}_2)$, $\mathbf{s}_1\cdot\hat{\mathbf{r}}\mathbf{s}_2\cdot\hat{\mathbf{r}}$, $\mathbf{s}_1\cdot\hat{\mathbf{r}}\mathbf{s}_2\cdot\nabla$, $\mathbf{s}_1\cdot\nabla\mathbf{s}_2\cdot\nabla$, $\mathbf{L}\cdot\mathbf{s}_1\mathbf{L}\cdot\mathbf{s}_2$. The action of these operators on scalar multiples of the spin-angle functions can be determined by rewriting the operators in terms of products of irreducible tensor operators and using the Wigner-Eckart theorem. The results are

$$\mathbf{s}_1\cdot\mathbf{s}_2\mathcal{Y}_{jmls}(\hat{\mathbf{r}})=\tfrac{1}{2}[s(s+1)-\tfrac{3}{2}]\mathcal{Y}_{jmls}(\hat{\mathbf{r}})\ , \tag{A7a}$$

$$\mathbf{L}\cdot\mathbf{S}\mathcal{Y}_{jmls}(\hat{\mathbf{r}})=\tfrac{1}{2}[j(j+1)-l(l+1)-s(s+1)]\mathcal{Y}_{jmls}(\hat{\mathbf{r}})\ , \tag{A7b}$$

$$\mathbf{L}\cdot(\mathbf{s}_1-\mathbf{s}_2)\mathcal{Y}_{jmls}(\hat{\mathbf{r}})=[\tfrac{1}{10}(2l+3)(2l-1)]^{1/2}\delta_{jl}(\delta_{s,0}\mathcal{Y}_{jml1}+\delta_{s,1}\mathcal{Y}_{jml0})\ , \tag{A7c}$$

$$\mathbf{s}_1\cdot\hat{\mathbf{r}}\mathbf{s}_2\cdot\hat{\mathbf{r}}\mathcal{Y}_{jmj0}(\hat{\mathbf{r}})=-\tfrac{1}{4}\mathcal{Y}_{jmj0}(\hat{\mathbf{r}})\ , \tag{A7d}$$

$$\begin{aligned}
\mathbf{s}_1\cdot\hat{\mathbf{r}}\mathbf{s}_2\cdot\hat{\mathbf{r}}\mathcal{Y}_{jml1}(\hat{\mathbf{r}})=&\tfrac{1}{4}\delta_{jl}\mathcal{Y}_{jml1}(\hat{\mathbf{r}})-\frac{1}{4(2l-1)}\delta_{j,l-1}\mathcal{Y}_{jml1}(\hat{\mathbf{r}})+\frac{1}{4(2l+3)}\delta_{j,l+1}\mathcal{Y}_{jml1}(\hat{\mathbf{r}})\\
&-\frac{[(l+1)(l+2)]^{1/2}}{2(2l+3)}\delta_{j,l+1}\mathcal{Y}_{jm,l+2,1}(\hat{\mathbf{r}})-\frac{[l(l-1)]^{1/2}}{2(2l-1)}\delta_{j,l-1}\mathcal{Y}_{jm,l-2,1}(\hat{\mathbf{r}})\ ,
\end{aligned} \tag{A7e}$$

$$\mathbf{s}_1\cdot\hat{\mathbf{r}}\mathbf{s}_2\cdot\nabla\mathcal{Y}_{jmj0}(\hat{\mathbf{r}})=-\tfrac{1}{4}\mathcal{Y}_{jmj0}(\hat{\mathbf{r}})\frac{d}{dr}\ , \tag{A7f}$$

$$\mathbf{s}_1\cdot\hat{\mathbf{r}}\mathbf{s}_2\cdot\nabla\mathcal{Y}_{jml1}(\hat{\mathbf{r}})=\tfrac{1}{4}\delta_{jl}\mathcal{Y}_{jml1}(\hat{\mathbf{r}})\left[\frac{d}{dr}+\frac{1}{r}\right]-\frac{1}{4(2l-1)}\delta_{j,l-1}\mathcal{Y}_{jml1}(\hat{\mathbf{r}})\left[\frac{d}{dr}+\frac{l+1}{r}\right]$$
$$+\frac{1}{4(2l+3)}\delta_{j,l+1}\mathcal{Y}_{jml1}\left[\frac{d}{dr}-\frac{l}{r}\right]-\frac{[(l+1)(l+2)]^{1/2}}{2(2l+3)}\delta_{j,l+1}\mathcal{Y}_{jm,l+2,1}(\hat{\mathbf{r}})\left[\frac{d}{dr}-\frac{l}{r}\right]$$
$$-\frac{[l(l-1)]^{1/2}}{2(2l-1)}\delta_{j,l-1}\mathcal{Y}_{jm,l-2,1}(\hat{\mathbf{r}})\left[\frac{d}{dr}+\frac{l+1}{r}\right], \tag{A7g}$$

$$\mathbf{s}_1\cdot\nabla\mathbf{s}_2\cdot\nabla\mathcal{Y}_{jmj0}(\hat{\mathbf{r}})=-\tfrac{1}{4}\mathcal{Y}_{jmj0}\nabla_j^2\ , \tag{A7h}$$

$$\mathbf{s}_1\cdot\nabla\mathbf{s}_2\cdot\nabla\mathcal{Y}_{jml1}(\hat{\mathbf{r}})=\tfrac{1}{4}\mathcal{Y}_{jml1}(\hat{\mathbf{r}})\nabla_l^2-\frac{1}{4(2l-1)}\delta_{j,l-1}\mathcal{Y}_{jml1}(\hat{\mathbf{r}})\nabla_l^2+\frac{1}{4(2l+3)}\delta_{j,l+1}\mathcal{Y}_{jml1}(\hat{\mathbf{r}})\nabla_l^2$$
$$-\frac{[(l+1)(l+2)]^{1/2}}{2(2l+3)}\delta_{j,l+1}\mathcal{Y}_{jm,l+2,1}(\hat{\mathbf{r}})\left[\frac{d}{dr}-\frac{l+1}{r}\right]\left[\frac{d}{dr}-\frac{l}{r}\right]$$
$$-\frac{[l(l-1)]^{1/2}}{2(2l-1)}\delta_{j,l-1}\mathcal{Y}_{jm,l-2,1}(\hat{\mathbf{r}})\left[\frac{d}{dr}+\frac{l}{r}\right]\left[\frac{d}{dr}+\frac{l+1}{r}\right], \tag{A7i}$$

$$\mathbf{s}_1\cdot\mathbf{L}\mathbf{s}_2\cdot\mathbf{L}\mathcal{Y}_{jmj0}(\hat{\mathbf{r}})=-\tfrac{1}{4}j(j+1)\mathcal{Y}_{jmj0}(\hat{\mathbf{r}})\ , \tag{A7j}$$

$$\mathbf{s}_1\cdot\mathbf{L}\mathbf{s}_2\cdot\mathbf{L}\mathcal{Y}_{jml1}(\hat{\mathbf{r}})=-\tfrac{1}{4}(l^2+l-1)\delta_{j,l}\mathcal{Y}_{jml1}(\hat{\mathbf{r}})+(l+1)^2\delta_{j,l-1}\mathcal{Y}_{j,m,l,1}(\hat{\mathbf{r}})+l^2\delta_{j,l+1}\mathcal{Y}_{j,m,l,1}(\hat{\mathbf{r}})\ . \tag{A7k}$$

It is straightforward though tedious to use the relations above to convert Eq. (9) into a set of radial wave equations. The only coupling in these equations in the case of a quark and its antiquark is between the triplet states with $l=j\pm1$. In the case of a quark and a distinct antiquark, the equations for the triplet and singlet states with $j=l$ are also coupled.

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