π, K, ρ, ϕ as $q\bar{q}$ bound states of the Salpeter equation in a $^3P_0$ condensed vacuum

Pedro J. de A. Bicudo and José E. F. T. Ribeiro
Centro de Física da Matéria Condensada, Av. Prof. Gama Pinto 2, 1699 Lisboa, Portugal
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With the help of the Feynman rules, we derive and solve the Salpeter equation (in the presence of the condensed vacuum) for the mesonic bound states π, K, ρ, and ϕ. For these mesons, we give wave functions and bare masses (that is, prior to decay). For S waves, Gaussian approximations—the cluster sizes—are also given. These cluster sizes depend on the chiral angle. Whereas pseudoscalar wave functions are shown to behave like doublets in an abstract spinlike energy space, vectorial mesons are shown to possess a more complicated structure involving couplings to spatial excitations as well. Already at this stage it is shown that, in order to accommodate decays, we face strong bounds on the possible values for the potential strength and quark current masses.

I. INTRODUCTION

In this paper we derive and solve the Salpeter equation for mesonic bound states[1] in a condensed vacuum,[2,3] in particular for π, K, ρ, and ϕ. We only consider bare mesons, i.e., noninteracting mesons. The knowledge of masses and wave functions of bare mesons is fundamental as a starting point for the study of hadronic physics (spectroscopy, form factors, strong interactions,. . .). For instance this is a necessary step, if one wants to compute the influence of particle decay in hadronic spectroscopy. This influence is known to be considerable.[4,5] The chiral condensation mechanism, described in Ref. 6, can also be envisaged to be the mechanism responsible for such decays.[7] In order to answer the question of whether this claim can be quantitatively substantiated, bare mesons must be studied first.

Unlike previous works which make use of the Feynman field operators Ψ (Ref. 3) we work in the formalism of quark and antiquark Fock-space operators $(b^\dagger;b)$ and $(d^\dagger;d)$ usual in solid-state physics. The two formalisms are, of course, equivalent. In the second formalism, the role of quark and antiquarks is rendered, we think, more transparent and it helped us to interpret the condensed quark-antiquark pairs, the quark masses, the creation and annihilation amplitudes, and the Salpeter functions, when in the presence of a condensed vacuum. It allows a unified treatment of mesonic decay, which is under study. It also turns out that Salpeter equations are directly obtained in their simplest form, while in the other formalism[3] it is necessary to guess what variables yield the simpler equations. We used both formalisms, and found the second one less cumbersome for π, and even less so for ρ or ϕ, where we have four coupled channels, or for the K, where the quark and the antiquark have different current masses.[8]

This paper is divided as follows. Section II is devoted to the discussion and setting of the Bethe-Salpeter equation for mesons. In Sec. III we derive, for our potential, the associated Salpeter equation. The numerical solutions of these equations are given in Sec. IV. Finally the results and conclusions are given in Sec. V.

II. BETHE-SALPETER EQUATION FOR MESONIC FOUR-FERMION GREEN'S FUNCTIONS

As is well known, the existence of a bound state of two fermions corresponds to a pole in the Green's function with four fermionic legs. With the adopted interaction (see the Feynman rules[6]), among the ten independent amplitudes with four fermionic legs we can find three, connecting quark-antiquark pairs, with the quantum numbers of a meson, i.e., color singlets. In Fig. 1 we give an example of an irreducible diagram, whereas in Fig. 2 we depict schematically these three amplitudes together with the three corresponding Green's functions. The first is responsible for the interaction inside the $q\bar{q}$ pair and the other two for the annihilation and creation of two of these pairs, respectively.

The Bethe-Salpeter equation is just the self-consistent Dyson equation for these Green's functions. The corresponding diagrams are written in Fig. 3.

In the neighborhood of a bound state, the above considered Green's functions can be written as a product of three functions: one for the initial relative $q\bar{q}$ momentum, another for the final relative $q\bar{q}$ momentum, and

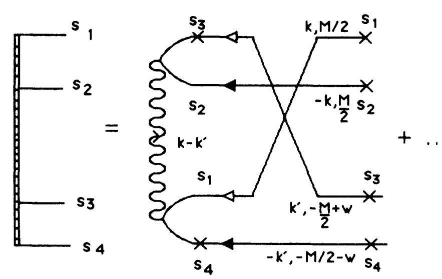

FIG. 1. Irreducible diagram for the annihilation of two $q\bar{q}$ pairs.

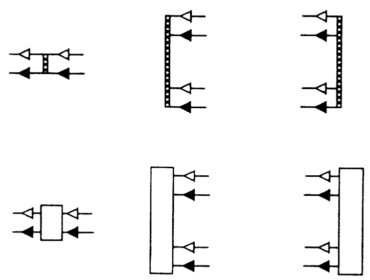

FIG. 2. The interactions we consider between color-singlet $q\bar{q}$ pairs, together with the corresponding Green functions.

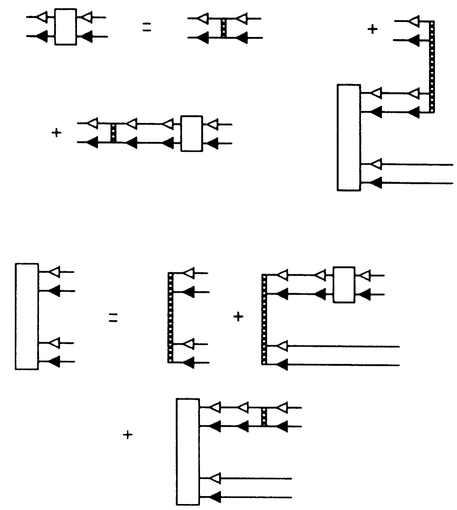

FIG. 3. Bethe-Salpeter equations for $q\bar{q}$ Green's functions with quantum numbers of mesons.

finally a third function for the center-of-mass momentum. This last function possesses a pole corresponding to the mesonic bound-state propagator. The two functions of the relative momenta, initial and final, correspond to the microscopic wave function of the mesons when considered in terms of their $q\bar{q}$ content (see Fig. 4).

III. BETHE-SALPETER EQUATION FOR $q\bar{q}$ BOUND STATES

A. Salpeter equation

In the neighborhood of the pole the inhomogeneous term of the Bethe-Salpeter equation can be discarded because it is finite. The inhomogeneous term is only necessary when normalizing the wave functions. The equation is then homogeneous and can be factorized in order to yield an equation for the wave functions. In our case we have coupled-channel equations for the two functions ϕ^+ and ϕ^-. This is shown diagramatically in Fig. 5. Notice that both quarks (the upper line entering ϕ^+ or leaving ϕ^-) and antiquarks (the lower line) propagate from left to right, in the direction of time. With the help of Feynman rules, the diagrams of Fig. 5 read

$$\begin{aligned}\phi^{+}_{s_1s_2}(\mathbf{k},0)=&\frac{2}{2}\int\frac{d^3\mathbf{k}'dw}{(2\pi)^4}S\left(\mathbf{k}',\frac{M}{2}+w\right)S\left(-\mathbf{k}',\frac{M}{2}-w\right)[-iV(\mathbf{k}-\mathbf{k}')]u^\dagger_{s_1}(\mathbf{k})u_{s_3}(\mathbf{k}')[-v^\dagger_{s_4}(\mathbf{k}')v_{s_2}(\mathbf{k})]\phi^{+}_{s_3s_4}(\mathbf{k}',0)\\&+\frac{-2}{2}\int\frac{d^4\mathbf{k}'}{(2\pi)^4}S\left(\mathbf{k}',-\frac{M}{2}+w\right)S\left(-\mathbf{k}',-\frac{M}{2}-w\right)[-iV(\mathbf{k}-\mathbf{k}')]u^\dagger_{s_1}(\mathbf{k})v_{s_4}(\mathbf{k}')u^\dagger_{s_3}(\mathbf{k}')v_{s_2}(\mathbf{k})\phi^{-}_{s_3s_4}(\mathbf{k}',0)\,,\end{aligned}\tag{3.1a}$$

$$\begin{aligned}\phi^{-}_{s_1s_2}(\mathbf{k},0)=&\frac{2}{2}\int\frac{d^3\mathbf{k}'dw}{(2\pi)^4}S\left(\mathbf{k}',-\frac{M}{2}+w\right)S\left(-\mathbf{k}',-\frac{M}{2}-w\right)[-iV(\mathbf{k}-\mathbf{k}')]u^\dagger_{s_3}(\mathbf{k}')u_{s_1}(\mathbf{k})[-v^\dagger_{s_2}(\mathbf{k})v_{s_4}(\mathbf{k}')]\phi^{-}_{s_3s_4}(\mathbf{k}',0)\\&+\frac{-2}{2}\int\frac{d^3\mathbf{k}'dw}{(2\pi)^4}S\left(\mathbf{k}',\frac{M}{2}+w\right)S\left(-\mathbf{k}',\frac{M}{2}-w\right)[-iV(\mathbf{k}-\mathbf{k}')]v^\dagger_{s_4}(\mathbf{k}')u_{s_1}(\mathbf{k})v^\dagger_{s_2}(\mathbf{k})u_{s_3}(\mathbf{k}')\phi^{\dagger}_{s_3s_4}(\mathbf{k}',0)\,,\end{aligned}\tag{3.1b}$$

where the numerators 2 and -2 come from the sign and the number of diagrams.

Being instantaneous, the potential does not depend on w. It is preferable to use untruncated functions if we do not want to integrate the propagators S_q and $S_{\bar{q}}$ in momentum coordinates. As Eq. (3.1b) can be obtained from (3.1a) by exchanging the plus with a minus in ϕ, and by replacing M with $-M$, from now on we shall only write one such equation. Defining the Salpeter function ϕ^+ as

$$\phi^{+}_{s_1s_2}(\mathbf{k})=\int\frac{dw}{(2\pi)}S\left(\mathbf{k},\frac{M}{2}+w\right)S\left(-\mathbf{k},\frac{M}{2}-w\right)\phi^{+}_{s_1s_2}(\mathbf{k},0)\tag{3.2}$$

and using a similar definition for ϕ^-, we are led from Eqs. (3.1) to

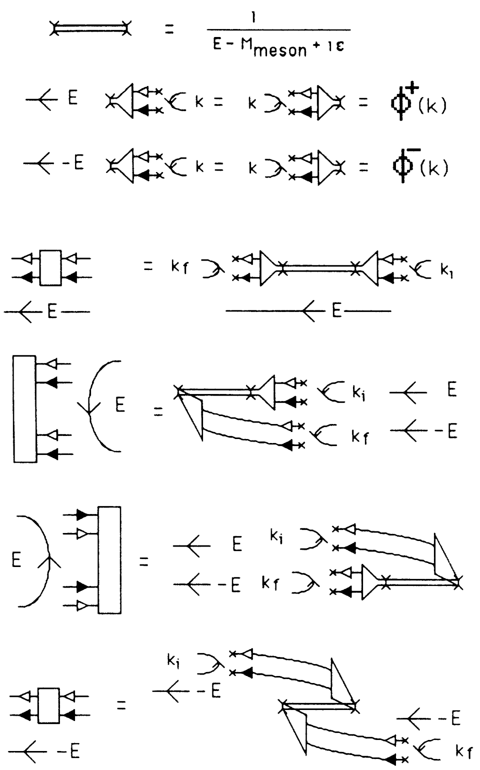

FIG. 4. The Green's functions of a bound state are decomposed as a product of a wave function of the relative initial momentum, a pole in the center-of-mass momentum, and a wave function of the relative final momentum.

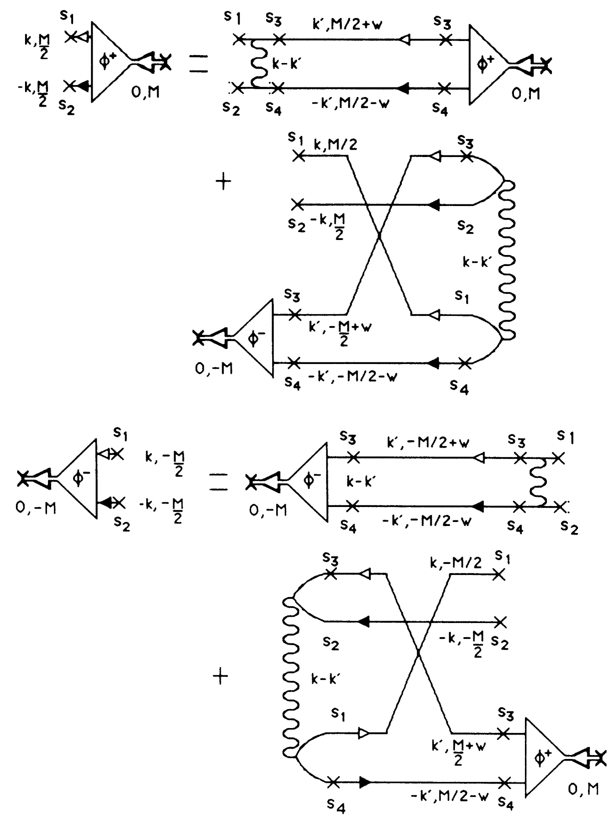

FIG. 5. Salpeter equations for $q\bar{q}$ wave functions.

$$
\begin{aligned}
\phi^{+}_{s_1 s_2}(\mathbf{k}) = &\left[\int \frac{dw}{(2\pi)} S_q\left[\mathbf{k}, \frac{M}{2}+w\right] S_{\bar{q}}\left[-\mathbf{k}, \frac{M}{2}-w\right]\right] \\
&\times \left\{ \int \frac{d^3\mathbf{k}'}{(2\pi)^3}[-iV(\mathbf{k}-\mathbf{k}')]u^{\dagger}_{s_1}(\mathbf{k})u_{s_3}(\mathbf{k}')[-v^{\dagger}_{s_4}(\mathbf{k}')v_{s_2}(\mathbf{k})]\phi^{+}_{s_3 s_4}(\mathbf{k}') \right. \\
&\left. - \int \frac{d^3\mathbf{k}'}{(2\pi)^3}[-iV(\mathbf{k}-\mathbf{k}')]u^{\dagger}_{s_1}(\mathbf{k})v_{s_4}(\mathbf{k}')u^{\dagger}_{s_3}(\mathbf{k}')v_{s_2}(\mathbf{k})\phi^{-}_{s_3 s_4}(\mathbf{k}') \right\}
\end{aligned}
\tag{3.3}
$$

and to a similar equation for ϕ^-. The propagators S_q and $S_{\bar{q}}$ can be trivially integrated to yield

$$
\int \frac{dw}{2\pi} S_q\left[\mathbf{k}, \frac{M}{2}+w\right] S_{\bar{q}}\left[-\mathbf{k}, \frac{M}{2}-w\right] = \frac{i}{M - E_q(\mathbf{k}) - E_{\bar{q}}(\mathbf{k})} \ . \tag{3.4}
$$

Next, considering the form of the potential $V(k)$,

$$
V(k) = -K_0^3(2\pi)^3\Delta_{\mathbf{k}}\delta^3(\mathbf{k})
$$

and working in units of K_0 we finally arrive at

$$
[M - E_q(\mathbf{k}) - E_{\bar{q}}(\mathbf{k})]\phi^{+}_{s_1 s_2}(\mathbf{k}) = \Delta_{\mathbf{k}'}\{u^{\dagger}_{s_1}(\mathbf{k})u_{s_3}(\mathbf{k}')[-v^{+}_{s_4}(\mathbf{k}')v_{s_2}(\mathbf{k})]\phi^{\dagger}_{s_3 s_4}(\mathbf{k}') - u^{\dagger}_{s_1}(\mathbf{k})v_{s_4}(\mathbf{k}')u^{\dagger}_{s_3}(\mathbf{k}')v_{s_2}(\mathbf{k})\phi^{-}_{s_3 s_4}(\mathbf{k}')\}_{\mathbf{k}'=\mathbf{k}} \ . \tag{3.5}
$$

B. Salpeter equation for $^1S_0$ mesons

As an example we shall study the case of the K meson, where q and $\bar{q}$ have different masses (e.g., m_u and m_s) corresponding to different vacuum condensation angles, and which has spin 0. In this case the wave function can be cast as

$$\phi_{ss'}(\mathbf{k})=\mu_{ss'}\frac{\nu(k)}{k}, \quad \mu=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{3.6}$$

where the $1/k$ was introduced to simplify the use of the Laplacian:

$$\Delta_{\mathbf{k}}\frac{\nu}{k}=\frac{1}{k}\frac{d^2\nu}{dk^2} . \tag{3.7}$$

We have also[6]

$$\begin{aligned} u_s^\dagger(\mathbf{k})u_{s'}(\mathbf{k}')=v_{s'}^\dagger(\mathbf{k}')v_s(\mathbf{k})&=\tfrac{1}{2}\{[\sqrt{1+S}\sqrt{1+S'}+\sqrt{1-S}\sqrt{1-S'}(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')]\delta_{ss'}+\sqrt{1-S}\sqrt{1-S'}(i\sigma\cdot\hat{\mathbf{k}}\times\hat{\mathbf{k}}')_{ss'}\} , \\ u_s^\dagger(\mathbf{k})v_{s'}(\mathbf{k}')=[v_s^\dagger(\mathbf{k}')u_{s'}(\mathbf{k})]^*&=-\tfrac{1}{2}\{(\sqrt{1-S}\sqrt{1+S'}\hat{\mathbf{k}}-\sqrt{1+S}\sqrt{1-S'}\hat{\mathbf{k}}')\cdot(\sigma i\sigma_2)_{ss'}\} . \end{aligned} \tag{3.8}$$

In expressions (3.8), for equal $\mathbf{k}$, the first expression becomes the identity and the second one vanishes. Some derivatives also vanish:

$$\left[\hat{\mathbf{k}}\cdot\frac{\partial\hat{\mathbf{k}}}{\partial k_i}\right]=0; \quad (\hat{\mathbf{k}}\cdot\nabla_{\mathbf{k}})\hat{\mathbf{k}}=0 \tag{3.9}$$

with

$$\begin{aligned} &\frac{\partial}{\partial k_i}\hat{\mathbf{k}}=\frac{1}{k}\left[\mathbf{e}_i-\frac{\mathbf{k}k_i}{k^2}\right], \quad \hat{\mathbf{k}}\times\frac{\partial\hat{\mathbf{k}}}{\partial k_i}=\hat{\mathbf{k}}\times\frac{\mathbf{e}_i}{k}, \quad \Delta\hat{\mathbf{k}}=-\frac{2}{k^2}\hat{\mathbf{k}} , \\ &\Delta[f(k)\hat{\mathbf{k}}]=\Delta(f)\hat{\mathbf{k}}+f\Delta(\hat{\mathbf{k}})=\left[\frac{d^2f}{dk^2}+\frac{2}{k}\frac{df}{dk}-f\frac{2}{k^2}\right]\hat{\mathbf{k}} . \end{aligned} \tag{3.10}$$

Thus the Salpeter equation can be written

$$\begin{aligned} [E_q(\mathbf{k})+E_{\bar{q}}(\mathbf{k})-M]\phi^+(\mathbf{k})=\{&\Delta_{\mathbf{k}'}+\Delta_{\mathbf{k}'}[u^\dagger(\mathbf{k})u(\mathbf{k}')_q+u^\dagger(\mathbf{k})u(\mathbf{k}')_{\bar{q}}]+2\nabla_{\mathbf{k}'}[u^\dagger(\mathbf{k})u(\mathbf{k}')]_q\nabla_{\mathbf{k}'}[u^\dagger(\mathbf{k})u(\mathbf{k}')]_{\bar{q}}\}\phi^+(\mathbf{k}') \\ &-2\{\nabla_{\mathbf{k}'}[u^\dagger(\mathbf{k})v(\mathbf{k}')]_q\nabla_{\mathbf{k}'}[u^\dagger(\mathbf{k})v(\mathbf{k}')]_{\bar{q}}\}\phi^-(\mathbf{k})|_{\mathbf{k}'=\mathbf{k}} , \end{aligned} \tag{3.11}$$

where the spin indices were abbreviated. With the help of formulas (3.8) and (3.9) we can evaluate the derivatives in expression (3.11). We have

$$\begin{aligned} \Delta_{\mathbf{k}'}[u^\dagger(\mathbf{k})u(\mathbf{k}')_q+u^\dagger(\mathbf{k})u(\mathbf{k}')_{\bar{q}}]|_{\mathbf{k}'=\mathbf{k}}=&\tfrac{1}{2}\delta_{s_1s_3}\delta_{s_2s_4}[(\sqrt{1+S}\,\Delta\sqrt{1+S}+\sqrt{1-S}\,\Delta\sqrt{1-S}+(1-S)\hat{\mathbf{k}}\Delta\hat{\mathbf{k}})_q \\ &+(\sqrt{1+S}\,\Delta\sqrt{1+S}+\sqrt{1-S}\,\Delta\sqrt{1-S}+(1-S)\hat{\mathbf{k}}\Delta\hat{\mathbf{k}})_{\bar{q}}] , \\ 2[u^\dagger(\mathbf{k})\nabla u(\mathbf{k})]_q\cdot[u^\dagger(\mathbf{k})\nabla u(\mathbf{k})]_{\bar{q}}=&\tfrac{1}{2}\delta_{s_1s_3}\delta_{s_2s_4} \\ &\times(\sqrt{1+S}\,\nabla\sqrt{1+S}+\sqrt{1-S}\,\nabla\sqrt{1-S})_q\cdot(\sqrt{1+S}\,\nabla\sqrt{1+S}+\sqrt{1-S}\,\nabla\sqrt{1-S})_{\bar{q}} \\ &+\tfrac{1}{2}\nabla(i\sigma\cdot\hat{\mathbf{k}}\times\hat{\mathbf{k}})_{s_1s_3}\cdot\nabla(i\sigma\cdot\hat{\mathbf{k}}\times\hat{\mathbf{k}})_{s_2s_4}(1-S)_q(1-S)_{\bar{q}} , \end{aligned} \tag{3.12}$$

and finally,

$$\begin{aligned} -2[u^\dagger(\mathbf{k})\nabla v(\mathbf{k})]_q\cdot[u^\dagger(\mathbf{k})\nabla v(\mathbf{k})]_{\bar{q}}=&-\tfrac{1}{2}(\hat{\mathbf{k}}\cdot\sigma i\sigma_2)_{s_1s_4}(\hat{\mathbf{k}}\cdot\sigma i\sigma_2)_{s_3s_2} \\ &\times(\sqrt{1+S}\,\nabla\sqrt{1+S}-\sqrt{1+S}\,\nabla\sqrt{1-S})_q\cdot(\sqrt{1-S}\,\nabla\sqrt{1+S}-\sqrt{1+S}\,\nabla_{\sqrt{1-S}})_{\bar{q}} \\ &-\tfrac{1}{2}(\nabla\hat{\mathbf{k}}\cdot\sigma i\sigma_2)_{s_1s_4}\cdot(\nabla\hat{\mathbf{k}}\cdot\sigma i\sigma_2)_{s_3s_2}\sqrt{1-S_q^2}\sqrt{1-S_{\bar{q}}^2} . \end{aligned} \tag{3.13}$$

In spin space, Eq. (3.13), can be summarized as

$$\mu_{s_1s_2}=(\mathbf{A}_{s_1s_3}\mathbf{B}_{s_2s_4}+\mathbf{C}_{s_1s_4}\mathbf{D}_{s_3s_2})\mu_{s_3s_4} \tag{3.14}$$

which, when written in matrix notation, looks like

$$\mu=\mathbf{A}\mu\mathbf{B}^T+\mathbf{C}\mu^T\mathbf{D}^T . \tag{3.15}$$

It is easy to see from the structure of (3.13) that we have only three cases to consider. We have the case ($A=B=1$; $C=D=0$)

$$\mu=\mu \tag{3.16a}$$

or

$$[\sigma\cdot\mathbf{V}\times\mu\times(\sigma\cdot\mathbf{V})^T]=-V^2\mu \tag{3.16b}$$

or else

$$[\mu\sigma\cdot\mathbf{V}\times\mu^T\times(\mu\sigma\cdot\mathbf{V})^T]=V^2\mu \ . \tag{3.16c}$$

After factorizing out the spin matrix μ, and using the derivatives

$$\begin{aligned}\nabla\sqrt{1\pm S}&=\pm\tfrac{1}{2}\hat{\mathbf{k}}\varphi'\sqrt{1\mp S}\ ,\\ \Delta\sqrt{1\pm S}&=-\frac{\varphi'^2}{4}\sqrt{1\pm S}\pm\left[\frac{\varphi''}{2}+\frac{\varphi'}{k}\right]\sqrt{1\mp S}\ ,\end{aligned}\tag{3.17}$$

we finally obtain the Salpeter equation for the radial part of the kaon wave function:

$$\left[\left[-\frac{d^2}{dk^2}+E_q(k)+E_{\bar q}(k)+\frac{\varphi_q'^2+\varphi_{\bar q}'^2}{4}+\frac{1-S_qS_{\bar q}}{k^2}\right]\begin{bmatrix}1&0\\0&1\end{bmatrix}+\left[\frac{\varphi_q'\varphi_{\bar q}'}{2}+\frac{C_qC_{\bar q}}{k^2}\right]\begin{bmatrix}0&1\\1&0\end{bmatrix}-M\begin{bmatrix}1&0\\0&-1\end{bmatrix}\right]\begin{bmatrix}v^+(k)\\v^-(k)\end{bmatrix}=0\ , \tag{3.18}$$

where S stands for $\sin(\varphi)$ and C for $\cos(\varphi)$.

The Salpeter equation for the π is just a particular case of the previous one where φ_q is equal to $\varphi_{\bar q}$. The equation reads

$$\left[\left[-\frac{d^2}{dk^2}+2E\right]\begin{bmatrix}1&0\\0&1\end{bmatrix}+\left[\frac{\varphi'^2}{2}+\frac{C^2}{k^2}\right]\begin{bmatrix}1&1\\1&1\end{bmatrix}-M\begin{bmatrix}1&0\\0&-1\end{bmatrix}\right]\begin{bmatrix}v^+(k)\\v^-(k)\end{bmatrix}=0\ . \tag{3.19}$$

C. Salpeter equation for $^3S_1$ mesons

We also present the equations for the ρ and ϕ case, where q and $\bar q$ have the same masses but the spin is 1. In this case we have the double of equations because S and D waves are coupled (see also Ref. 8). If we call the radial S wave functions v_0 and the radial D wave functions v_2, the final form of Salpeter equations for ρ and ϕ is

$$\begin{aligned}&\left[\left[-\frac{d^2}{dk^2}+2E\right]\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}+\frac{6}{k^2}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}+\frac{\varphi'^2}{6}\begin{bmatrix}3&-1&0&-2\sqrt2\\-1&3&-2\sqrt2&0\\0&-2\sqrt2&3&1\\-2\sqrt2&0&1&3\end{bmatrix}\right.\\&\quad-\frac{C^2}{3k^2}\begin{bmatrix}2&-2&\sqrt2&-\sqrt2\\-2&2&-\sqrt2&\sqrt2\\\sqrt2&-\sqrt2&1&-1\\-\sqrt2&\sqrt2&-1&1\end{bmatrix}+\frac{2\sqrt2(1-S)}{3k^2}\begin{bmatrix}-2\sqrt2&0&1&0\\0&-2\sqrt2&0&1\\1&0&-2\sqrt2&0\\0&1&0&-2\sqrt2\end{bmatrix}\\&\quad\left.-M\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}\right]\times\begin{bmatrix}v_2^+(k)\\v_2^-(k)\\v_0^+(k)\\v_0^-(k)\end{bmatrix}=0\ .\end{aligned}\tag{3.20}$$

IV. NUMERICAL SOLUTION FOR SALPETER EQUATIONS

A. The numerical method

The Salpeter equation is solved when we calculate the bound-state mass M and the functions v. The condition for M to be accepted is that v exists and has the correct shape for a bound state. So we keep the bound-state mass M fixed and solve the linear system for n (2 or 4) second-order equations to look for a proper v.

We have $2n$ degrees of freedom. The $2n$ boundary conditions are that both $v(0)$ and $v(\infty)$ must be zero. The condition at $k=0$ depends on the angular momentum L of v_L and must ensure that the actual radial wave function v/k is well behaved. The condition at $k=\infty$ is necessary for a bound state.

This system could be solved directly if we discretized k on a lattice. However, in order to reduce CPU time and increase precision, we use an iterative[3] program with step h that builds $v(k)$ from the initial $v(0)$ and $v(h)$. Before checking $v(\infty)$, and because there are n degrees of freedom, we have to choose a basis for v. Let us choose the basis

$$v^i_\alpha(h)=\text{const}\times\delta_{i\alpha}\ ,\qquad i=1,\ldots,n,\quad \alpha=1,\ldots,n\ , \tag{4.1}$$

where α stands for the component index.

We have a bound state when there exists a linear combination of the $v^i(k)$ that goes asymptotically to zero when k tends to infinity. Thus we are led to suppose that the determinant of the matrix of the components $v^i_\alpha(k)$ tends to zero when k tends to infinity. However, in this region the equation is given approximately by

$$\ddot{v}\cong(2k)\mathbb{1}v \tag{4.2}$$

and thus the components v^i_α either diverge or converge to zero faster than exponentially. If the bound state is not degenerate then the determinant will have more divergent than convergent components and will eventually diverge. Fortunately, if we try an M slightly different from the exact one, then the solution is no longer compact and, sooner than later, will explode.[9] Either with a positive or negative change in M the solution, while exploding crosses the abscissa and the determinant goes through zero very steeply. In this way, if we find the mass M yielding a null determinant of $v^i_\alpha(k)$, for an arbitrarily high k, we approach arbitrarily close to the exact M.

The numerical iterative program has to solve systems of linear equations of the form

$$\ddot{v}=\mathbb{A}(k)v\ . \tag{4.3}$$

We use the version of Runge-Kutta based on Numerov. If we define

$$w(k)=\left[\mathbb{1}-\frac{h^2}{12}\mathbb{A}(k)\right]v(k) \tag{4.4}$$

then with finite differences we can show that

$$w(k+h)-2w(k)+w(k-h)=h^2\mathbb{A}(k)v-\frac{h^6}{240}v^{(6)} \tag{4.5}$$

and we have an iterative method to determine w,v:

$$w(k+h)=\left[-10+\frac{12}{1-h^2\mathbb{A}(k)/12}\right]w(k)-w(k-h)\ . \tag{4.6}$$

The numerical procedure goes through the following steps.

(1) As an input we take the quark and antiquark masses and the quantum numbers of the bound state (we work in units of $K_0=1$).

(2) With the Runge-Kutta method we solve the mass gap equation[6] in order to obtain the vacuum angles corresponding to the quark and antiquark masses, and build the Salpeter matrix $\mathbb{A}(k)$ corresponding to the bound-state quantum numbers (in fact we do it step by step).

(3) For each bound state mass M we run the Runge-Kutta-Numerov routine in quadruple precision with a step of 0.002 until k reaches $k_{\max}=6\gg0$ [the size of the bound state is of the order of 1 and $v(6)$ should be quite small as it decreases faster than exponentially] and we obtain the determinant of $v^i_\alpha(6)$.

(4) This determinant is a function of M which, in general, diverges very fast and crosses the abscissa very steeply. Hence the Newton method is best suited to obtain iteratively the zeros of the function $\det(M)$, and we can use the relation

$$M_{i+1}=\frac{\det(M_i)M_i-\det(M_{i-1})M_{i-1}}{\det(M_i)-\det(M_{i-1})} \tag{4.7}$$

which gives an excellent convergence.

(5) The solution v is now a linear combination of the previously obtained $v^i(k)$:

$$v=\sum_{i=1}^{n}c_i v^i(k)\ . \tag{4.8}$$

It is zero at $k=6$. For example, if we fix the normalization of v with $c_1=1$ then we have to solve the linear

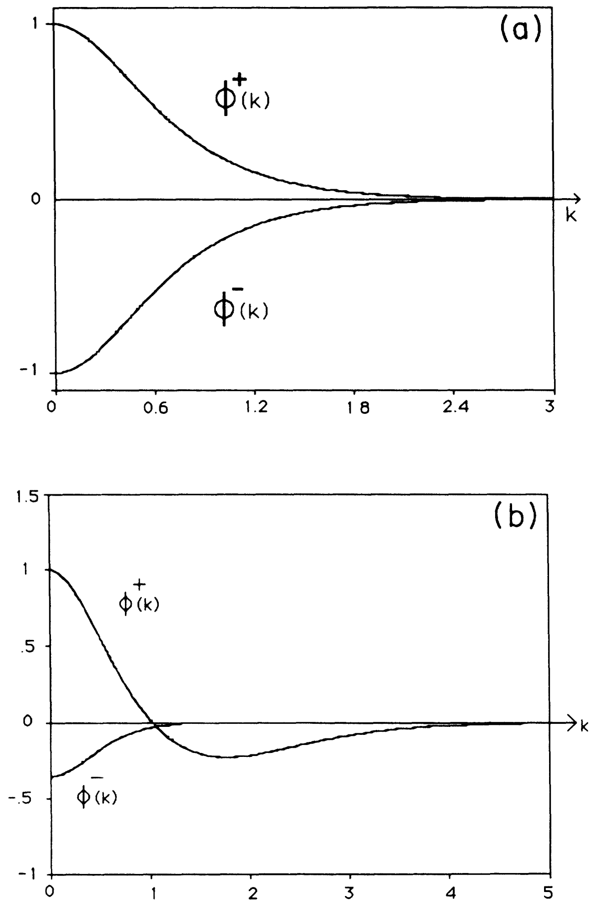

FIG. 6. Wave functions for the ground state (a) and first radial excitation (b) of the π. We show the positive-energy and negative-energy components in arbitrary normalization, as a function of the momentum in $K_0=1$ units.

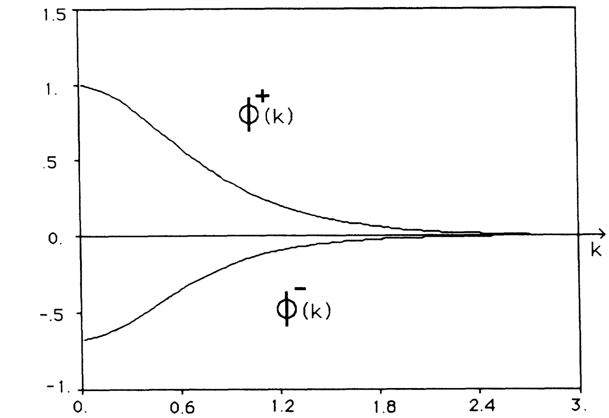

FIG. 7. Wave functions of the K when $m_u = m_d = 0$ and $m_s = 0.1$ in $K_0 = 1$ units. The wave functions are an arbitrary normalization.

equation for the rest of the c_i:

$$\sum_{i=2}^{n} c_i v^i(6) = -v^1(6) \ . \tag{4.9}$$

That the function v ought to behave properly, furnishes the last check on this numerical method.

B. Solutions of the Salpeter equation and results

Now we will proceed to show the solutions of the Salpeter equation for the π ground state and first radial excitation (Fig 6), for the K (Fig. 7), for the ρ (Fig. 8), and for the ϕ (Fig. 9). For the K we suppose that one of the quarks has zero mass (up or down) while the other (strange) has not.

We recall that the components $\phi^+ = v^+/k$ are wave functions for a quark-antiquark pair which is produced by a meson, whereas the components $\phi^- = v^-/k$ represent wave functions for a quark-antiquark pair that

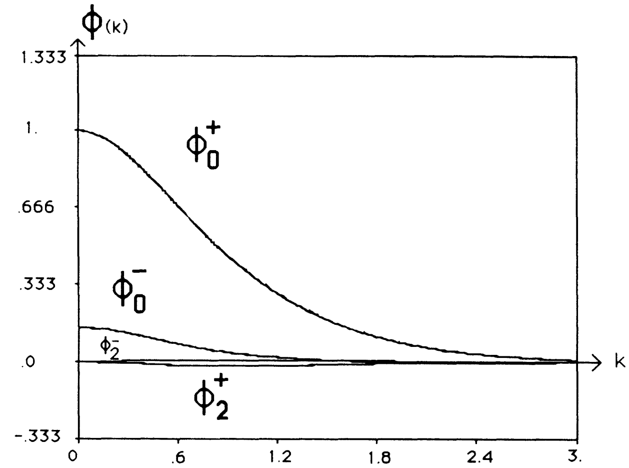

FIG. 9. Wave functions of the ϕ for $m_s = 0.1$. The four components of positive or negative energy and S or D wave are shown in units of $K_0 = 1$.

is annihilated within a meson (in this formalism both quarks, antiquarks, mesons, and antimesons propagate forward in time).

It should be noticed that the obtained wave functions ϕ are bare in the sense that they are not yet coupled to other mesonic channels and, hence, are not allowed to decay. Thus the masses that we will obtain as eigenstates of the Salpeter equations are not supposed to be the physical ones, for such a coupling would induce, in most cases, large negative and imaginary mass shifts.

In Fig. 8 we give the S and D wave functions for the ρ, with zero mass quarks, and in Fig. 9 we provide the wave functions for the ϕ with $m_S = 0.1 K_0$.

In what follows we denote by m_h the heavy-quark mass. In Fig. 10 we show the masses $M_{\pi,\eta}$, M_K, and $M_{\rho,\phi}$ for running m_h. For massless quarks we verified

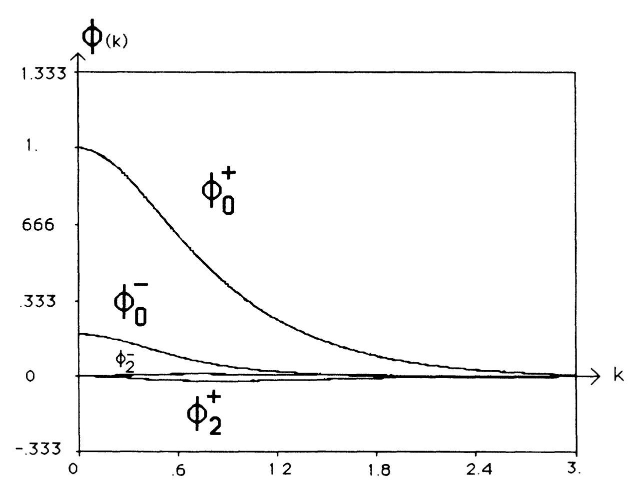

FIG. 8. Wave functions of the ρ for $m_q = 0$ in units of $K_0 = 1$. There are four components of positive or negative energy and of S or D wave angular momentum.

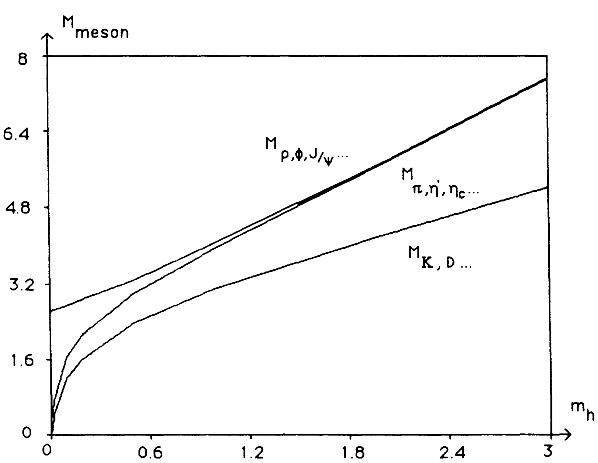

FIG. 10. Masses of the mesons $\pi - \eta' - \eta_c$, $K - D$, and $\rho - \phi - J/\psi$ as a function of the quark mass (in the second case only one of the two quarks is massive). All masses are in units of $K_0 = 1$.

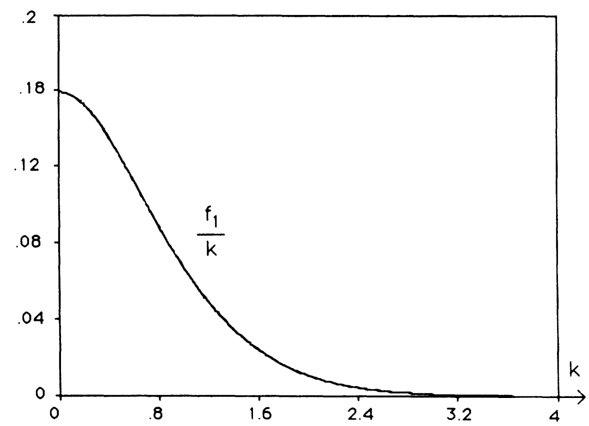

FIG. 11. Function f_1/k, where k is in $K_0=1$ units.

that these masses agree with the ones givens in Ref. 3.

We see that M_π and M_K initially follow a curve proportional to $\sqrt{m_q+m_{\bar{q}}}$ and, therefore, they obey the Gell-Mann[11] rule. This is an excellent test to the numerical precision of our method. Numerically, for massless quarks, we do not get a bound π, but only for a quark mass of the order of the step in the Runge-Kutta-Numerov method. However, it can be shown analytically[3] that for massless quarks we have the solution

$$\nu^+=\nu^-=k\sin(\varphi) . \tag{4.10}$$

This numerical error arises because of the stiffness of the Salpeter equation.[9] In Ref. 3 a relation is derived between the quark masses and the bound-state mass of the form of the Gell-Mann[11] relation which is very accurate for small quark masses:

$$(m_q+m_{\bar{q}})\int dk\, k^2\sin\varphi=M^2\int dk\, k\sin\varphi f_1+O(m^2) . \tag{4.11}$$

This equation is an integral equation and does not depend strongly on small fluctuations of the functions φ and f_1. After evaluating the integrals we obtain

$$M=\sqrt{17.143\,29\,(m_q+m_{\bar{q}})} . \tag{4.12}$$

The function f_1 is shown in Fig. 11 and is a solution of the differential equation[3]

$$\left[-\frac{d^2}{dk^2}+2k\cos\varphi\right]f_1=k\sin\varphi . \tag{4.13}$$

We can use this very same equation to obtain the wave functions of the $^1S_0$ mesons with vanishing mass M:[3]

$$\nu^\pm=Mf_1\pm k\sin(\varphi)+O(Mm) . \tag{4.14}$$

It is also interesting to study the limit of very high current quark masses. Doing so we study in fact the conditions and limits for nonrelativistic approximations. In Fig. 10 we can see that $M_{\pi,\eta}$ reaches $M_{\rho,\phi,J/\psi}$ while $M_{K,D}$ remains close to $M_{\pi,\eta}/\sqrt{2}$.

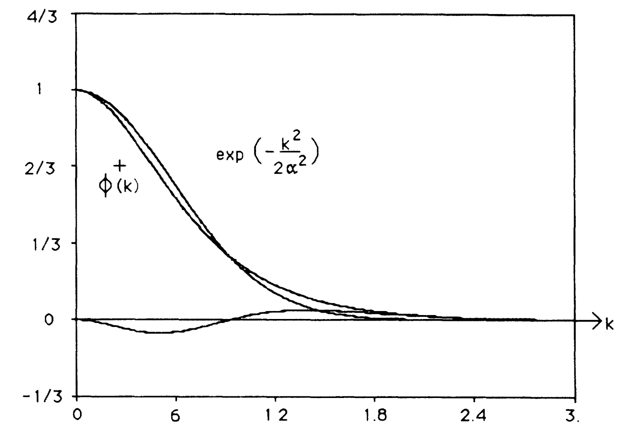

FIG. 13. Component $\phi^+(k)$ of the π wave function in arbitrary normalization, Gaussian approximation, and the difference between them. The momentum k is in units of $K_0=1$.

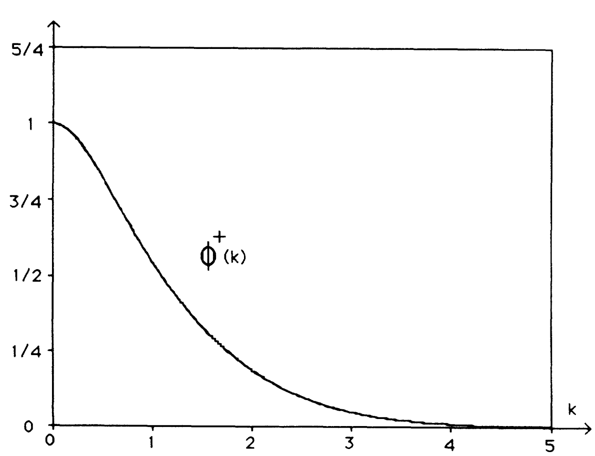

FIG. 12. Limit of the wave function for an $^1S_0$ bound state when a quark has a vanishing mass and the other has an infinite mass.

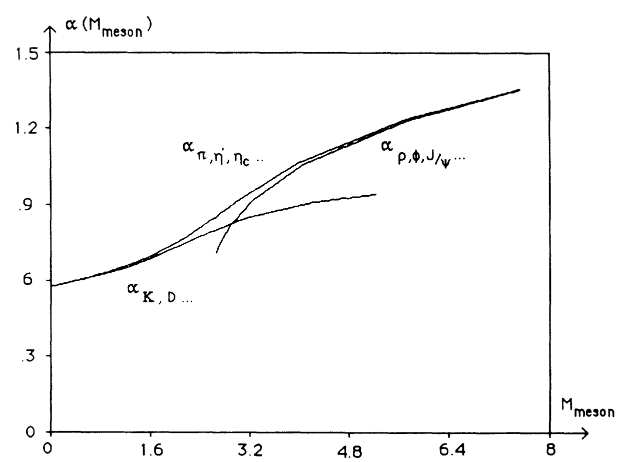

FIG. 14. α parameter as a function of the meson masses.

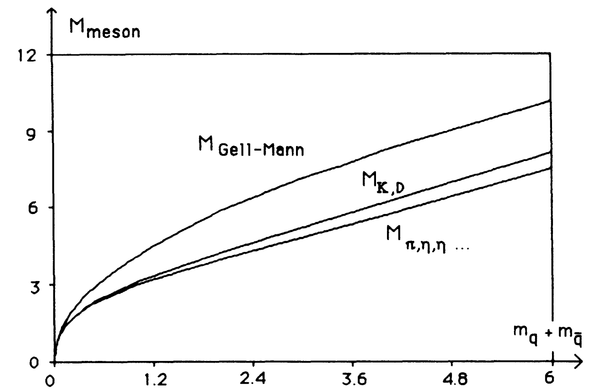

FIG. 15. Mass of the $^1S_0$ mesons compared with the one obtained from the $q\bar{q}$ masses with the Gell-Mann relation.

In this limit, the first two classes of mesons $(\pi,\eta,\ldots)$ and $(\rho,\phi,J/\psi,\ldots)$ have a very heavy $q\bar{q}$ pair which behaves nonrelativistically and there is no vacuum condensate. The Salpeter equation becomes just the harmonic-oscillator Schrödinger equation, and both the meson mass and parameter α, measuring the cluster size, are simply given by

$$M_{\text{meson}}=2m_q+3/\sqrt{m_q}\,,\quad \alpha=(4m_q)^{1/4}\,. \tag{4.15}$$

The third class of mesons $(K,D,\ldots)$ has a $q\bar{q}$ pair with a massless quark and a heavy one, with mass m_h. The Salpeter equation, as in the previous case, also decouples and is given by

$$\left[-\frac{d^2}{dk^2}+E_q(k)+m_h+\frac{\varphi_q'^2}{4}+\frac{1-S_q}{k^2}-M\right]v^+(k)=0\,. \tag{4.16}$$

The corresponding wave function was obtained and it is shown in Fig. 12. It has a shape similar to the one for small quark masses, but α is clearly bigger. We obtain, in

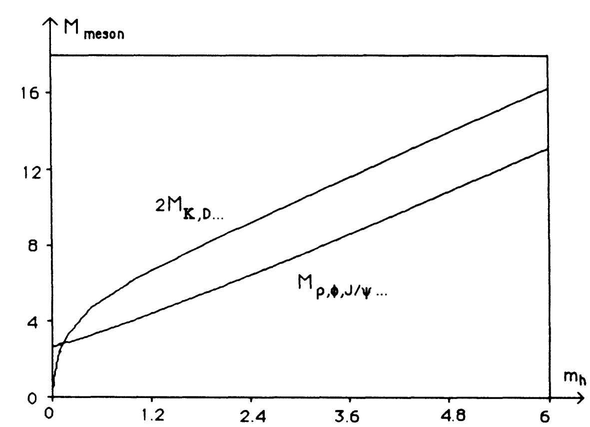

FIG. 16. Masses of the $^3S_1$ mesons with a q and a $\bar{q}$ of equal mass compared with twice the mass of the $^1S_0$ mesons where only the q or the $\bar{q}$ has a nonvanishing mass, as a function of the mass of the massive quarks.

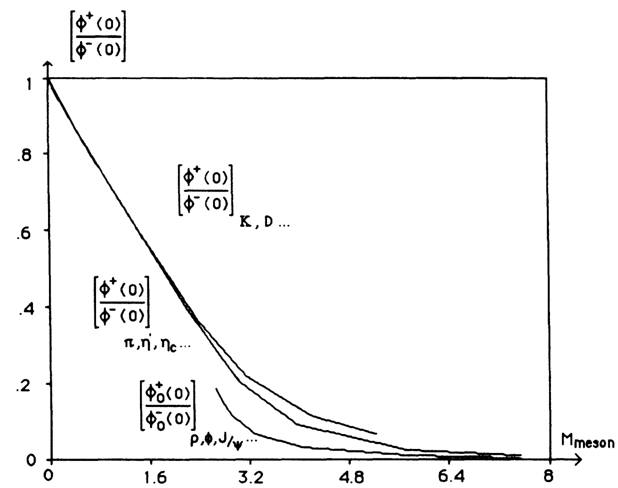

FIG. 17. Ratio of the positive- and negative-energy S wave functions for a vanishing momentum, as a function of the meson masses.

momentum space, for the meson mass M and for the corresponding cluster size (in units of $K_0=1$)

$$\begin{aligned} &M_{\text{meson}}=m_h+2.028\,34,\ldots,\\ &\alpha=1.026\,4,\ldots\,. \end{aligned} \tag{4.17}$$

In this limit of very high m_h, the mass of the $(K,D,\ldots)$ tends to be larger than half of the mass of the $(\pi,\eta,\ldots$ and $\rho,\phi,J/\psi,\ldots)$ by a constant value of $2K_0$. In this limit the cluster size α of the $(K,D,\ldots)$ class tends to a constant value of $1.027K_0$ while the cluster size α of the $\pi,\eta,\ldots$ and $\rho,\phi,J/\psi,\ldots$ remains unbound.

It is clear that the dominant wave functions of Figs. 6(a), 7, 8, and 9 have a shape very close to a Gaussian. The difference comes from the fact that our functions are asymptotic to $\exp(-k^{3/2})$. We chose the Gaussian parameter α such that it yields a Gaussian with the same norm and same value at the origin. In Fig. 13 we compare the exact wave function $\sin(\varphi)$ for a massless π, with its Gaussian approximation. For the cases we have considered (the π,K,ρ and ϕ), the error is about the same. The Gaussian approximation becomes excellent when the quark masses are bigger than the strength K_0, that is when the model becomes nonrelativistic.

In Fig. 14 we show, for different mesons, the α corresponding to different mesonic masses. The limit for the α of the $K,D,\ldots$ [see (4.17)] is reproduced.

V. CONCLUSION

In Fig. 15 we compare the mass of the $^1S_0$ mesons with the mass given by the Gell-Mann relation. We can see that this approximation is only good for small quark masses. For the light u, d, and s quarks we obtain the same ratios of their masses as in QCD sum rules.[8] However, it is interesting that for a wide range of $m_q+m_{\bar{q}}$, the masses of these mesons remain relatively close far beyond the scope of this approximation.

One of the conclusions of this paper is that resonance decay into mesons puts strong bounds on the values of the current quark masses as well as on the potential strength K_0. In Fig. 16 we compare M_ϕ with $2M_{K,D},\ldots,$ both as functions of the heavy-quark mass. The decay of ϕ into two K's ($2M_K < M_\phi$) forces us to consider strange quark current masses less than $0.1K_0$. At the same time we must have for K_0 a value of the order of 400 or 500 MeV if we want the bare mass of the ϕ to be a few hundred MeV above its physical mass of 1020 MeV, in order to make space for the inevitable negative energy shift, due to the coupling with the K's. In this region of K_0, m_s, and $m_{u,d}$ the bare masses M_ϕ and M_ρ, are very close. Had we coupled ρ and ϕ to their related channels, M_ρ would suffer a larger mass shift than M_ϕ both in negative and in imaginary directions, because ρ decays into a larger phase space. As a preliminary result of an ongoing study we obtain the appropriate complex mass for the ρ, with the same parameters used for ϕ. This result gives us additional confidence in this model.

Of course, the harmonic oscillator being a rough approximation to the more realistic potential Coulomb plus linear, is not supposed to describe with sufficient accuracy charmonium and even less bottomium physics. The comparison of $2.M_D$ and $M_{j/\psi}$, $2.M_B$, and M_Υ (Ref. 4) and Fig. 16 [see also (4.17)], favors that $K_0 \cong 300$ MeV, which is lower than the best value of K_0 for light quarks. The fact that the optimal value of K_0 changes with the size of the mesons suggests that our quadratic potential does not have a realistic shape. The wealth of available data in resonance phase shifts, and the degree of agreement, even with such simple potential, obtained so far, should prompt research along these lines with a more realistic potential.

Our strange-quark mass lies in the region of 40 MeV. The absolute values of the current quark masses are not observables and are model dependent.[6] However these masses are below the established ones.[8] For more realistic potentials, of the type Coulomb plus linear, the value for the strange mass that reproduces the correct meson masses is likely to change somewhat. In fact the shape of the potential has a strong influence on the condensation of the vacuum, which in turn affects the masses of the pseudoscalar mesons. The instantaneous approximation, in which the potential is not covariant, may also affect the quark masses.

Another important conclusion derives from the double component nature of mesonic bound states. It implies an essential departure from the Schrödinger equation. For relatively light mesons it is clear that both the number of amplitudes, now having to cope with the negative energy components, and the normalization of mesonic wave functions will be quite different. With the relevant normalization for interacting mesons, if both positive- and negative-energy components have the same norm, then this norm diverges. The ratio between the positive- and negative-energy components at the origin is shown as a function of the meson masses in Fig. 17. The Schrödinger-like limit where the wave function has a single component is reached whenever M_{meson} is a few times bigger than K_0.

When quarks are massless, we know from the Goldstone theorem,[10] that there are n_f^2 massless pseudoscalar mesons (π,η,K,η' in real physics). The microscopic reason for this comes from the fact that these are the only mesons where the quark and antiquark have no radial or angular excitation in their Salpeter equation. The only other candidates would be the $^3S_1$ mesons (ρ,ω,K^*,ϕ in real physics), but in the Salpeter equation S and D waves are coupled naturally and, although the D-wave component is very small, it constrains M to be large. In this way the so-called "constituent mass of the quark" turns out to be the scale of the confining potential appearing via radial or angular excitations.[6]

The Salpeter equation that we solved has no flavor-mixing term. Thus, even if current quark masses no longer vanish, the natural pseudoscalar ground states simply have the flavors [for flavor SU(3)]

$$u\bar{d},\ d\bar{u},\ u\bar{s},\ s\bar{u},\ d\bar{s},\ s\bar{d};\ u\bar{u},\ d\bar{d},\ s\bar{s}\ .$$

If we consider, as is usual in current quark masses,[8] that the quark masses are ordered as $m_u < m_d < m_s$, then the first six mesons can be interpreted as the $\pi^+,\pi^-,K^+,K^-\overline{K^0},K^0$. The last three mesons are a combination of the π^0,η,η'. In our model we find no reason for the nonexistence of an η with small mass. In this way we also are faced with the U(1) problem, and this is natural in PCAC[10] (partial conservation of axial-vector current). In fact, with the parameters that reproduce the masses of the first six mesons, the three mesons $u\bar{u}$, $d\bar{d}$, and $s\bar{s}$ seem to have a smaller mass than the real π^0, η, and η'. It is clear that a flavor-mixing interaction should be used in our model, at least to obtain the correct π^0. In this paper we did not consider the interactions between the ingoing and outgoing legs of the Salpeter Green's functions (these interactions are flavor-mixing). This fact constitutes an approximation.

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