Current-quark model in a $^3P_0$ condensed vacuum

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In this work we assume that quarks are described by Dirac spinors, with current masses which eventually can be set to zero, interacting through a confining and chirally invariant potential. Other than the strength of the interquark potential and the current masses of the quarks we have no free parameters. $^3P_0$ quark-antiquark vacuum condensation is allowed and the mass gap equation is solved for the chosen potential. The solution, and even the mere existence of it, depends quite sharply on the chosen potentials. Vacuum condensation is shown to be responsible for partial conservation of axial-vector current and for the constituent scale. The mass gap equation also ensures us that quark annihilation is obtained in a consistent way. In our formalism quarks and antiquarks appear explicitly, which greatly simplifies the derivation of both the Salpeter equations for meson bound states and the resonating-group-method equations for meson decays and scattering.

I. INTRODUCTION

The commonly used adiabatic and nonrelativistic quark models are simple to use but they have limited theoretical support and either leave many problems unexplained or use many *ad hoc* parameters. We know that the constituent quark mass is not fully understood[1] and that the origin of the hyperfine quark-quark potential is not clear.[2] Moreover, the small mass of π is not easily obtained. Recently, using a resonating-group-method calculation,[3] we have shown[4] that this class of model failed to reproduce phase shifts in K-N exotic scattering, and thus fails to extend beyond simple spectroscopic calculations.

On the other hand, the success of PCAC[5] (partial conservation of axial-vector current) provides evidence that the splitting between the masses of $^1S_0$ and $^3S_1$ mesons need not be explained by the use of a hyperfine potential but can be accounted for as a manifestation of the Goldstone mechanism. We also know[6] that most hadrons are broad resonances and coupled-channel equations are, therefore, mandatory. For mesons it has been shown, assuming an annihilating quark-antiquark amplitude,[7] that no hyperfine potential is needed as splitting appears naturally when the coupling between resonances and open channels is turned on. However, the strength and form of this amplitude was introduced in an *ad hoc* fashion, without linking it to the assumed confining potential. This prompted us to see whether the quark-antiquark creation and annihilation amplitudes yielded by the confining potential had any bearing on the observed hadronic spectra. As a test, a preliminary study of the decay of ϕ into kaons was made.[8]

The success of this test motivated us to engage in a complex project (see Fig. 1) to study consistently the dynamical symmetry breaking of chiral invariance, the bound states of quarks and antiquarks (bare mesons), and finally real mesonic resonances. This paper is devoted to studying the effects on quarks of the quark-antiquark condensate. Unlike previous works that use Feynman field operators Ψ (Refs. 9 and 10) we work in the more explicit formalism of quark and antiquark field operators $b^\dagger d^\dagger$, that is more usual in condensed-matter physics than in particle physics. In this equivalent formalism, not only are calculations simpler, but we are able to interpret clearly the condensed pairs in the vacuum. This framework also greatly simplifies the study of Dyson equations for quark-antiquark systems, and this is fundamental for our project.

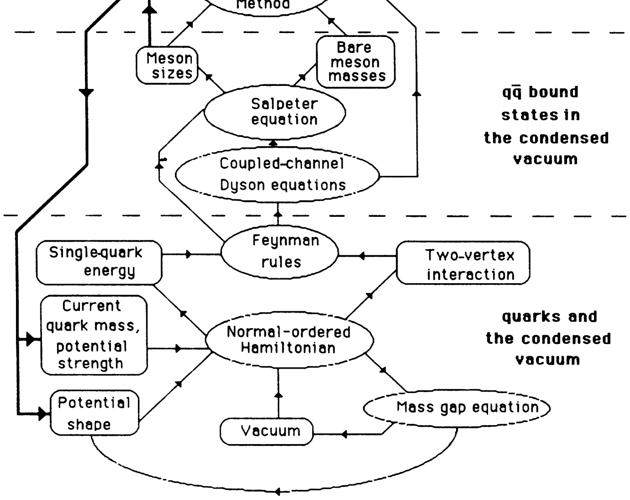

FIG. 1. Diagram of the organization of this project.

In this work, we will be making two assumptions. The first is that quarks have zero or, at least, a small current mass (as is indicated by QCD sum rules) and, second, that they are bound by a specific confining potential. Other than the strength of the potential and the current masses of the quarks we have no free parameters. Then chiral symmetry is shown to be dynamically broken and the vacuum to be a condensate of $^3P_0$ quark-antiquark Cooper-like pairs. Among the innumerable mesonic bound states, supported by the confining potential, and indeed found in nature, the π plays the special role of being the Goldstone boson.[9,10] As a bonus, indeed as a consequence of the chiral-symmetry-breaking mechanism, a quark-antiquark annihilation (and creation) term is also obtained.

We choose the simple $\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}$ color-confining[10,11] potential. A fundamental assumption of PCAC[5] is that any quark-quark interaction that comes from the Lagrangian of QCD should, in principle, be chirally invariant. This motivates[10,11] the use of the "Coulomb-like" potential (with Lorentz structure γ^0,γ^0) and the "vectorlike" potential (with Lorentz structure $\boldsymbol{\gamma},\boldsymbol{\gamma}$), that are chiral invariant and candidates to reproduce the results of the quark sector of QCD sum rules.[12] A third potential is motivated[11,13] by a calculation in a quenched lattice gauge theory[14] showing that the instantaneous part of the confining potential is scalarlike (chiral noninvariant). Being instantaneous, this class of potentials is not covariant. This fact constitutes another approximation.

As is well known, we can have particle-antiparticle creation even in the presence of stationary (instantaneous, "nonrelativistic") potentials. This is the typical case of strong external fields. In QED itself, it has long been recognized (first by Schwinger) that strong electric fields which extend over a sufficiently large area of space can continuously produce electron-positron pairs.

Related to this phenomenon we have (at least theoretically) electron-positron pair production in the presence of a strong and localized well produced, for instance, by a heavy, positively charged, ion (in fact by the sum of two overlapping heavy ions). In all the cases mentioned above, retardation and magnetic effects are negligible or small and yet we have pair production of electrons and positrons. This is due to the fact that such a strong well, and the correspondingly large binding energy, "plunges" into the Dirac negative-energy continuum, therefore mixing the free positive and negative Dirac solutions. Common to all these cases is the fact that we have a natural privileged nonrelativistic frame: the center of mass of the external potential. Admittedly, such a frame is not readily available in the case of quark-antiquark systems. If we were in QED such a system would display, no doubt, strong retardation effects. But this *need not* be the case, at least for bulk properties, with the confining phase of QCD. Being non-Abelian, the gluons interact among themselves and get modified to such an extent that they, too, together with the quarks, get confined.

Many other models inspired in QCD, addressing the long-wavelength (and stationary), confining regime of QCD, exist in the literature. In a certain sense, everything happens as if the quarks have dug a hole in a color-electric confining vacuum. The bigger the hole, the bigger the energy it will cost to dig, which means that quarks cannot be pulled apart indefinitely. In this picture a privileged frame appears: the center of mass of the quark-antiquark system, the vacuum-plus-a-hole "providing" the external confining field. But, and this is the essential point, be it a bag model, vacuum dielectrics, or strong potentials, whatever confines the quarks also violates (at least spontaneously) chiral symmetry; it is bound to upset the Dirac sea, and (ultimately) is responsible for quark-antiquark annihilation (creation): To put it shortly, it seems that what confines also produces mesonic decay. Our work is precisely concerned with the decay phenomenological consequences of a specific confining mechanism, but the methodology and the formalism could be applied to other mechanisms as well. Whatever picture nature has chosen to confine quarks, we feel that in view of the complexities of the gluonic sector of QCD and, due to the absence of any "real" derivation of realistic effective quark microscopic interaction from QCD, it seems reasonable to start by studying a simple stationary confining potential. In any case our work provides a natural framework to implement retarded (and magnetic) corrections if need arises. To include box diagrams systematically, one could follow a treatment similar to the one adopted by Gross[15] to treat the deuteron.

A separate issue, in addition to the theoretical uncertainties mentioned above (which are common to all phenomenological models we know), consists in the correct treatment of center-of-mass boosts. We have not, in this work, attempted to solve the covariantized mass gap equation. We know that center-of-mass boosts will have some influence on the mesonic decay. Because of this, and at this stage, we have decided to study an extreme limit in order to encompass the uncertainties brought in by having a no covariantized mass gap equation. This limit is discussed in Ref. 16.

In Sec. II, we study the vacuum condensation of $^3P_0\ q\bar{q}$ pairs. Section III is devoted to show how spinors and energy projectors are rotated, using a Valatin-Bogoliubov canonical transformation. In Sec. IV, the Hamiltonian is discussed, both in the original and in the Valatin-Bogoliubov (VB) rotated form, whereas in Sec. V the Feynman rules are given. We study the solution of the mass gap equation both for a linear and for a quadratic confining potential in Sec. VI, and in Sec. VII we discuss the quark constituent mass. We give our conclusions in Sec. VIII.

II. VACUUM CONDENSATION WITH $^3P_0\ q\bar{q}$ PAIRS

A. Generator of the condensed vacuum

In this section, and following rather closely the BCS theory, we study the condensation of $q\bar{q}$ pairs in the vacuum. These pairs must have the vacuum J^{PC} quantum numbers 0^{++} and thus can only have ^SL_J equal to $^3P_0$. In this way we differ from the usual superconductivity case where Cooper pairs are $^1S_0$ bound states of two electrons. The generator of the new vacuum is defined to be

$$Q_0^\dagger = \sum_{cf} \int d^3k\, \phi_k C_{cfk}^\dagger ,$$

$$C_{cfk}^\dagger = (b_{\mathbf{k}_\uparrow}^\dagger, b_{\mathbf{k}_\downarrow}^\dagger)_{cf} \mathbb{M} \begin{bmatrix} d_{\mathbf{k}_\uparrow}^\dagger \\ d_{\mathbf{k}_\downarrow}^\dagger \end{bmatrix}_{cf} , \tag{2.1}$$

with a similar expression for the Hermitian conjugate Q_0. In expression (2.1), c stands for the color and f for the flavor index. The operator Q_0 transforms as a singlet both in color and in flavor. The matrix $\mathbb{M}$ contains the $^3P_0$ coupling,

$$\mathbb{M}_{\sigma_1\sigma_2} = (-\sqrt{6}) \sum_{\sigma m} \begin{bmatrix} 1 & 1 & 0 \\ m & \sigma & 0 \end{bmatrix} \hat{\mathbf{k}}_{1m} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \sigma_1 & \sigma_2 & \sigma \end{bmatrix} , \tag{2.2}$$

and has the following interesting properties:

$$\mathbb{M} = \begin{bmatrix} -\hat{\mathbf{k}}_x + i\hat{\mathbf{k}}_y & \hat{\mathbf{k}}_z \\ \hat{\mathbf{k}}_z & \hat{\mathbf{k}}_x + i\hat{\mathbf{k}}_y \end{bmatrix} ,$$
$$\mathbb{M}\mathbb{M}^* = \mathbb{1} , \tag{2.3}$$
$$\mathbb{M} = \boldsymbol{\sigma} \cdot \hat{\mathbf{k}} (i\sigma_2) .$$

The new vacuum $|\tilde{0}\rangle$ is generated from the trivial (empty) vacuum $|0\rangle$ as

$$|\tilde{0}\rangle = S|0\rangle, \quad S = e^{(Q_0^\dagger - Q_0)} . \tag{2.4}$$

This new vacuum $|\tilde{0}\rangle$ is of the form $\exp(C^\dagger - C)|0\rangle$, where it can be quite easily shown that

$$(C^\dagger - C)|0\rangle = C^\dagger|0\rangle ,$$
$$(C^\dagger - C)^2|0\rangle = (C^{\dagger 2} - 2)|0\rangle , \tag{2.5}$$
$$(C^\dagger - C)^3|0\rangle = -4C^\dagger|0\rangle = -4(C^\dagger - C)|0\rangle .$$

After expanding $S|0\rangle$ and reordering the powers of $(C^\dagger - C)$, we get

$$|\tilde{0}\rangle = \prod_{cfk} \left[\frac{1+\cos 2\phi_k}{2} + \frac{\sin 2\phi_k}{2} C_{cfk}^\dagger + \frac{1-\cos 2\phi_k}{4} C_{cfk}^{\dagger 2} \right] dk |0\rangle , \tag{2.6}$$

and thus, the two vacua are orthogonal if the condensation angle ϕ_k differs from 0:

$$\langle 0|\tilde{0}\rangle = \prod_{cfk} \frac{1+\cos 2\phi_k}{2} dk = 0 . \tag{2.7}$$

In a broader sense it can be proven that the Hilbert space one builds from $|\tilde{0}\rangle$ is orthogonal to the usual Hilbert space containing $|0\rangle$. Hence the assertion that, whenever one has vacuum condensates, one must go beyond perturbation theory.

B. The Fock-space operators b and d, in the new vacuum

The annihilator of the new vacuum $|\tilde{0}\rangle$ is not b_k but is instead

$$\tilde{b}_k = S b_k S^{-1}, \quad \tilde{b}_k |\tilde{0}\rangle = 0 . \tag{2.8}$$

Using a simplified notation,

$$S = e^{(\phi A)}, \quad A = C^\dagger - C , \tag{2.9}$$

together with the relation

$$\begin{bmatrix} b_\sigma \\ d_\mu^\dagger \end{bmatrix} A = \begin{bmatrix} A\delta_{\sigma e} & M_{\sigma v} \\ -M_{\mu e}^* & A\delta_{\mu v} \end{bmatrix} \begin{bmatrix} b_e \\ d_v^\dagger \end{bmatrix} , \tag{2.10}$$

we are able to relate the two sets of Fock-space operators

$$\widetilde{\begin{bmatrix} b \\ d^\dagger \end{bmatrix}} = \exp(\phi A) \begin{bmatrix} b \\ d^\dagger \end{bmatrix} \exp(-\phi A)$$
$$= \exp(\phi A) \exp\left[-\phi \begin{bmatrix} A\mathbb{1} & \mathbb{M} \\ -\mathbb{M}^* & A\mathbb{1} \end{bmatrix} \right] \begin{bmatrix} b \\ d^\dagger \end{bmatrix}$$
$$= \exp\left[\phi \begin{bmatrix} 0 & -\mathbb{M} \\ \mathbb{M}^* & 0 \end{bmatrix} \right] \begin{bmatrix} b \\ d^\dagger \end{bmatrix}$$
$$= \begin{bmatrix} \cos\phi \mathbb{1} & -\sin\phi \mathbb{M} \\ \sin\phi \mathbb{M}^* & \cos\phi \mathbb{1} \end{bmatrix} \begin{bmatrix} b \\ d^\dagger \end{bmatrix} , \tag{2.11}$$

or, when written in full,

$$\widetilde{\begin{bmatrix} b_{cfk\sigma} \\ b_{cfk\sigma}^\dagger \\ d_{cf-k\mu} \\ d_{cf-k\mu}^\dagger \end{bmatrix}} = \sum_{\sigma v} \begin{bmatrix} \cos\phi_k \delta_{\sigma e} & 0 & 0 & -\sin\phi_\kappa M_{\sigma v} \\ 0 & \cos\phi_k \delta_{\sigma e} & -\sin\phi_\kappa M_{\sigma v}^* & 0 \\ 0 & \sin\phi_\kappa M_{\mu e} & \cos\phi_k \delta_{\mu v} & 0 \\ \sin\phi_\kappa M_{\mu e}^* & 0 & 0 & \cos\phi_k \delta_{\mu v} \end{bmatrix} \begin{bmatrix} b_{cfke} \\ b_{cfke}^\dagger \\ d_{cf-kv} \\ d_{cf-kv}^\dagger \end{bmatrix} . \tag{2.12}$$

III. ROTATION OF SPINORS AND NEW ENERGY PROJECTORS

Following Ref. 10, the Feynman field operator is defined to be

$$\psi(x) = \sum_s \int \frac{d^3k}{\sqrt{2\pi}^3} [u_s(k) b_{ks} + v_s(k) d_{-ks}^\dagger] e^{ik \cdot x} . \tag{3.1}$$

We want this form to be invariant under the VB transformation. In other words, we want that (we will use simplified notations)

$$ub + vd^\dagger = \tilde{u}\,\tilde{b} + \tilde{v}\,\tilde{d}^\dagger \ . \tag{3.2}$$

In order for this relation to be true, we must compensate for the rotation of the operators b and d, by counterrotating u and v. To accomplish this, we define the new spinors as

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} = \begin{bmatrix} \cos\phi \mathbb{1} & -\sin\phi \mathbb{M}^* \\ \sin\phi \mathbb{M} & \cos\phi \mathbb{1} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \ . \tag{3.3}$$

We will be working with fermions with very small masses and thus it is convenient to use a definition[10] for the spinors $u(k)$ and $v(k)$ which is different from the usual definition of Bjorken and Drell,[17] where, for instance, the energy projectors were given by

$$\begin{aligned} &\Lambda_k^{\pm} = \tfrac{1}{2}(\pm\cosh\theta + \beta \pm \sinh\theta \cdot \boldsymbol{\alpha}\cdot\hat{\mathbf{k}})\beta \ , \\ &\cosh\theta \cong \sinh\theta = \frac{k}{m} \cong \infty \ . \end{aligned} \tag{3.4}$$

Clearly, this normalization is not very convenient. For the massless case we prefer, then, to use the noncovariant projectors

$$\tilde{\Lambda}_+ = \tilde{u}\,\tilde{u}^\dagger, \quad \tilde{\Lambda}_- = \tilde{v}\,\tilde{v}^\dagger \ . \tag{3.5}$$

We also prefer, as was done in Ref. 10, to use for $v(k)$ the momentum with a sign opposite to the usual one. We have, in the massless case,

$$\begin{aligned} u_s(\mathbf{k}) &= \frac{1}{\sqrt{2}}(1 + \boldsymbol{\alpha}\cdot\hat{\mathbf{k}})u_{0_s} \ , \\ v_s(\mathbf{k}) &= \frac{1}{\sqrt{2}}(1 - \boldsymbol{\alpha}\cdot\hat{\mathbf{k}})v_{0_s} \ , \end{aligned} \tag{3.6}$$

where charge conjugation imposes the condition

$$v_s(-\mathbf{k}) = +i\gamma_2 u_s^*(\mathbf{k}) \ . \tag{3.7}$$

In particular, for $k=0$, we have

$$\begin{aligned} &u_{0\uparrow} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad u_{0\downarrow} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \\ &v_{0\uparrow} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_{0\downarrow} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \ . \end{aligned} \tag{3.8}$$

Using these definitions, we apply the VB rotation on u and v, to obtain

$$\begin{aligned} \tilde{u}_s(\mathbf{k}) &= \frac{1}{\sqrt{2[1+\sin(2\phi)]}} \\ &\quad \times [1 + \sin(2\phi)\beta + \cos(2\phi)\boldsymbol{\alpha}\cdot\hat{\mathbf{k}}]u_{0_s} \ , \\ \tilde{v}_s(\mathbf{k}) &= \frac{1}{\sqrt{2[1+\sin(2\phi)]}} \\ &\quad \times [1 - \sin(2\phi)\beta - \cos(2\phi)\boldsymbol{\alpha}\cdot\hat{\mathbf{k}}]v_{0_s} \ . \end{aligned} \tag{3.9}$$

If the quarks have an initial current mass m, we should start with

$$\begin{aligned} u_s(\mathbf{k}) &= \frac{1}{\sqrt{2E(E+m)}}(E + m\beta + |k|\boldsymbol{\alpha}\cdot\hat{\mathbf{k}})u_{0_s} \ , \\ v_s(\mathbf{k}) &= \frac{1}{\sqrt{2E(E+m)}}(E - m\beta - |k|\boldsymbol{\alpha}\cdot\hat{\mathbf{k}})v_{0_s} \ . \end{aligned} \tag{3.10}$$

They are equivalent to the preceding formulas (3.9), when one replaces the vacuum angle 2ϕ by the mass angle $\arctan(m/k)$. Therefore, whenever the vacuum condenses, we are going to obtain a total rotation angle φ, which is the sum of the mass angle plus the dynamical angle,

$$\varphi = \arctan\left[\frac{m}{k}\right] + 2\phi \ . \tag{3.11}$$

The spinors and energy projectors are now

$$\begin{aligned} &\tilde{u}_s(\mathbf{k}) = \frac{1}{\sqrt{2(1+\sin\varphi)}}(1 + \sin\varphi\beta + \cos\varphi\boldsymbol{\alpha}\cdot\hat{\mathbf{k}})u_{0_s} \ , \\ &\tilde{v}_s(\mathbf{k}) = \frac{1}{\sqrt{2(1+\sin\varphi)}}(1 - \sin\varphi\beta - \cos\varphi\boldsymbol{\alpha}\cdot\hat{\mathbf{k}})v_{0_s} \ , \\ &\tilde{\Lambda}_{\mathbf{k}}^{\pm} = \tfrac{1}{2}(1 \pm \sin\varphi\beta \pm \cos\varphi\boldsymbol{\alpha}\cdot\hat{\mathbf{k}}) \ . \end{aligned} \tag{3.12}$$

In the remainder of this paper, we will use the simplified notation $S = \cos\varphi$ and $C = \sin\varphi$. Because we will be always working in the condensed vacuum regime, the tilde will also be suppressed. A set of useful properties, of which we will make repeated use, is listed below:

$$\begin{aligned} (u;v) &= \left[\frac{2}{1+S}\right]^{1/2} \tilde{\Lambda}_{\pm}(u_0;v_0) \\ v_0 &= (-i\alpha_2)u_0 \ , \\ &= \frac{1}{\sqrt{2}}(\sqrt{1+S} \pm \sqrt{1-S}\,\boldsymbol{\alpha}\cdot\hat{\mathbf{k}})(u_0;v_0) \ . \end{aligned} \tag{3.13}$$

IV. NORMAL-ORDERED HAMILTONIAN OF (QUASI) QUARKS

In this section we introduce the microscopic Hamiltonian

$$\begin{aligned} H = \Big[&\int d^3x\, \psi^\dagger(\mathbf{x})(m\beta - i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla})\psi(\mathbf{x}) \\ &+ \int d^3x\, d^3y\, \psi^\dagger(\mathbf{x})\lambda\psi(\mathbf{x})V_C(x-y)\psi^\dagger(\mathbf{y})\lambda\psi(\mathbf{y}) \\ &+ \int d^3x\, d^3y\, \psi^\dagger(\mathbf{x})\alpha\lambda\psi(\mathbf{x})V_V(x-y)\psi^\dagger(\mathbf{y})\alpha\lambda\psi(\mathbf{y}) \\ &+ \int d^3x\, d^3y\, \psi^\dagger(\mathbf{x})\beta\lambda\psi(\mathbf{x})V_S(x-y)\psi^\dagger(\mathbf{y})\beta\lambda\psi(\mathbf{y}) \Big] \ . \end{aligned} \tag{4.1}$$

In expression (4.1), V_C, V_V, and V_S stand, respectively, for the Coulomb, vector, and scalarlike potentials. As we said, these potentials are considered to be instantaneous and, therefore, only the space dependence needs to be explicitly written in (4.1). The time dependence, which is trivial, will be considered in Sec. V when dealing with the Feynman rules. In formula (4.1), λ represents the color Gell-Mann matrices, and α and β the Dirac matrices.

We need to write the Hamiltonian in normal-ordered form with respect to the new vacuum. It is true that one can consider, in addition to a four-fermion interaction, other types of microscopic interactions. An example of such interactions [to lift the $U_A(1)$ symmetry] is studied in a very interesting paper by Bernard *et al.*,[18] in the framework of a Hartree-Fock, Nambu–Jona-Lasinio model. Of course, it will be very interesting to go through the last stage of "resonances in the condensed vacuum"—see Fig. 1 (which also introduces, via coupled processes such as $u \to sK^+ \to u$, flavor mixing) to study the influence of such terms in the scattering phase shifts. Unfortunately because of the lack of a "fool-proof" derivation (from QCD) of these effective terms, appropriate parameters have (as they were in the aforementioned paper) to be introduced. Then considering such six-fermion microscopic interactions will amount to putting bounds on such constants. In view of the already involved calculations, we preferred to defer the consideration of such terms to a forthcoming paper.

As is well known, the Wick contraction technique offers[10] the handiest way to perform such a task. The Hamiltonian turns out to be decomposable in three terms: one without operators (H_0); another containing two operators (H_2); and still another (H_4) with four operators, i.e.,

$$H = H_0 + :H_2: + :H_4: . \tag{4.2}$$

In short, if we use the simplified notation

$$H = T(\psi^\dagger K\psi + \psi^\dagger\Gamma\psi V\psi^\dagger\Gamma\psi) \tag{4.3}$$

and knowing that the traces of the Gell-Mann matrices are zero, we obtain the terms

$$\begin{aligned} H_0 &= \overline{\psi^\dagger K\psi} + \psi^\dagger\Gamma\overline{\psi V\cdot\psi^\dagger}\Gamma\psi ,\\ H_2 &= \psi^\dagger K\psi + \psi^\dagger\Gamma\overline{\psi V\cdot\psi^\dagger}\Gamma\psi + \psi^\dagger\Gamma\psi V\cdot\overline{\psi^\dagger\Gamma\psi} ,\\ H_4 &= \psi^\dagger\Gamma\psi V\cdot\psi^\dagger\Gamma\psi . \end{aligned} \tag{4.4}$$

The fact that the normal ordering depends upon the vacuum can be clearly seen in the expression for the Wick contraction:

$$\overline{\psi_\alpha(\mathbf{x})\psi^\dagger_\beta(\mathbf{y})} = \int \frac{d^3k}{(2\pi)^3}\Lambda^+_{\mathbf{k}_{\alpha\beta}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} , \tag{4.5}$$

where Λ^+, the positive-energy projector, depends explicitly on the chiral angle.

Because of the fact that the traces of the Gell-Mann matrices are zero, there are no tadpolelike terms in (4.4). In this way the terms H_0 and H_2 appear as a sum of single-flavor terms. This will imply that for each flavor there is an independent mass gap equation. We will now evaluate the above-mentioned terms H_0, H_2, and H_4, where for the sake of simplicity we omit the flavor indices.

For the c-number H_0, we obtain

$$\begin{aligned} H_0 = 3\int \frac{d^3k}{(2\pi)^3} \Bigg\{ & \mathrm{Tr}[(m\beta+\boldsymbol{\alpha}\cdot\mathbf{k})\Lambda^-_{\mathbf{k}}] + 2\int \frac{d^3k'}{(2\pi)^3} \tilde{V}_C(\mathbf{k}-\mathbf{k}')\mathrm{Tr}(\Lambda^+_{\mathbf{k}}\Lambda^-_{\mathbf{k}'}) \\ & + \tilde{V}_V(\mathbf{k}-\mathbf{k}')\mathrm{Tr}(\boldsymbol{\alpha}\Lambda^+_{\mathbf{k}}\cdot\boldsymbol{\alpha}\Lambda^-_{\mathbf{k}'}) + \tilde{V}_S(\mathbf{k}-\mathbf{k}')\mathrm{Tr}(\beta\Lambda^+_{\mathbf{k}}\beta\Lambda^-_{\mathbf{k}'}) \Bigg\} . \end{aligned} \tag{4.6}$$

Because of the fact that the Dirac and Pauli matrices have null traces, we get

$$H_0 = \int \frac{d^3k}{(2\pi)^3}(2C_k - 6E_k), \quad C_k = \int \frac{d^3k'}{(2\pi)^3}[\tilde{V}_C(\mathbf{k}-\mathbf{k}') + 3\tilde{V}_V(\mathbf{k}-\mathbf{k}') + \tilde{V}_S(\mathbf{k}-\mathbf{k}')] , \tag{4.7}$$

where E_k will turn out, as we will see later when studying H_2, to be the energy of a quark. C_k does not depend on φ and is, therefore, independent of the vacuum structure. H_0 is the constant energy of the vacuum. The fact that it seems to be infinite should not worry us because a constant shift of this energy is not physically observable. For the term H_2, with two operators, we have

$$\begin{aligned} H_2 = & \int d^3x\, \psi^\dagger(\mathbf{x})(m\beta - i\boldsymbol{\alpha}\cdot\nabla)\psi(\mathbf{x}) \\ & + \int d^3x\, d^3y\, \psi^\dagger(\mathbf{x}) \int \frac{d^3k}{(2\pi)^3}\{ V_C(\mathbf{x}-\mathbf{y})e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}(\Lambda^+_{\mathbf{k}} - \Lambda^-_{\mathbf{k}}) + V_V(\mathbf{x}-\mathbf{y})e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}[\beta(\Lambda^+_{\mathbf{k}} - \Lambda^-_{\mathbf{k}})\beta] \\ & + V_S(\mathbf{x}-\mathbf{y})e^{ik(x-y)}[\boldsymbol{\alpha}\cdot(\Lambda^+_k - \Lambda^-_k)\boldsymbol{\alpha}]\}\psi(\mathbf{y}) . \end{aligned} \tag{4.8}$$

After expanding the ψ, and performing the space integrations, we obtain

$$H_2 = \int \frac{d^3k}{(2\pi)^3}[u^\dagger_s(\mathbf{k})b^\dagger_s(\mathbf{k}) + v^\dagger_s(\mathbf{k})d_s(-\mathbf{k})]H_k[u_s(\mathbf{k})b_s(\mathbf{k}) + v_s(\mathbf{k})d^\dagger_s(-\mathbf{k})] , \tag{4.9}$$

where H_k has the form

$$H_k = A_k\beta + B_k\boldsymbol{\alpha}\cdot\hat{\mathbf{k}} \tag{4.10}$$

and

$$A_k = m + \tfrac{2}{3}\int \frac{d^3k'}{(2\pi)^3}[V_C(\mathbf{k}-\mathbf{k}') - 3V_V(\mathbf{k}-\mathbf{k}') + V_S(\mathbf{k}-\mathbf{k}')]\sin\varphi_{k'} ,$$
$$B_k = k + \tfrac{2}{3}\int \frac{d^3k'}{(2\pi)^3}[V_C(\mathbf{k}-\mathbf{k}') - V_V(\mathbf{k}-\mathbf{k}') - V_S(\mathbf{k}-\mathbf{k}')]\cos\varphi_{k'}\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}' . \tag{4.11}$$

Expanding the spinors, we obtain

$$H_2 = \int \frac{d^3k}{(2\pi)^3} E_k \sum_s [b_s^\dagger(\mathbf{k})b_s(\mathbf{k}) + d_s^\dagger(\mathbf{k})d_s(\mathbf{k})] + \sum_{ss'}(\cos\varphi_k A_k - \sin\varphi_k B_k)[\mathbb{M}_{ss'}b_s^\dagger(\mathbf{k})d_{s'}^\dagger(-\mathbf{k}) + \mathbb{M}_{ss'}^* d_{s'}(-\mathbf{k})b_s(\mathbf{k})] , \tag{4.12}$$

and the dispersion law for a quark (or an antiquark) now reads

$$E_k = \sin\varphi_k A_k + \cos\varphi_k B_k . \tag{4.13}$$

Notice that the second term of Eq. (4.12) has the form of the vacuum generator Q_0 (see Sec. II). This second term, known in the literature as an anomalous Bogoliubov term, destabilizes the vacuum. Hence, we must get rid of it. The simplest way of achieving this is to set[19]

$$[\cos\varphi_k A_k - \sin\varphi_k B_k] = 0 . \tag{4.14}$$

This is, precisely, the mass gap equation. In Sec. V, this equation will be studied in detail. Having (4.14) in mind, H_2 is simply given by

$$H_2 = \int d^3k\, E_k(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + d_{\mathbf{k}}^\dagger d_{\mathbf{k}}), \quad A_k = \sin\varphi_k E_k, \quad B_k = \cos\varphi_k E_k, \quad E_k = (A_k^2 + B_k^2)^{1/2} . \tag{4.15}$$

For the term H_4, we have

$$H_4 = :\tfrac{1}{2}\int d^3x\, d^3y\, \psi^\dagger(\mathbf{x})\lambda\psi(\mathbf{x})V_C(\mathbf{x}-\mathbf{y})\cdot\psi^\dagger(\mathbf{y})\lambda\psi(\mathbf{y}): . \tag{4.16}$$

Now, if we expand $\psi(x)$ in terms of b_k and d_k, we obtain several terms with four fields:

$$H_4 = :\tfrac{1}{2}\sum_{s_1s_2s_3s_4}\int \frac{d^3k_1 d^3k_2 d^3k_3 d^3k_4}{(2\pi)^3}\delta^3(\mathbf{k}_1+\mathbf{k}_3-\mathbf{k}_2-\mathbf{k}_4)V_C(\mathbf{k}_1-\mathbf{k}_2)$$
$$\times[u_{s_1}^\dagger(\mathbf{k}_1)b_{\mathbf{k}_1s_1}^\dagger + v_{s_1}^\dagger(\mathbf{k}_1)d_{-\mathbf{k}_1s_1}]\lambda[u_{s_2}(\mathbf{k}_2)b_{\mathbf{k}_2s_2} + v_{s_2}(\mathbf{k}_2)d_{-\mathbf{k}_2s_2}^\dagger]$$
$$\times[u_{s_3}^\dagger(\mathbf{k}_3)b_{\mathbf{k}_3s_3}^\dagger + v_{s_3}^\dagger(\mathbf{k}_3)d_{-\mathbf{k}_3s_3}]\lambda[u_{s_4}(\mathbf{k}_4)b_{\mathbf{k}_4s_4} + v_{s_4}(\mathbf{k}_4)d_{-\mathbf{k}_4s_4}^\dagger]: . \tag{4.17}$$

After evaluating the inner products of spinors $u(k)$ and $v(k)$, we will be able to find ten different four-quark amplitudes ultimately responsible for mesonic decays. An example is shown in Fig. 2.

Such amplitudes can be summarized, for all intents and purposes, by a much more restricted set of vertices. These vertices are depicted in Fig. 3. Figure 3(a) represents the interaction with a quark, whereas Fig. 3(b) represents the interaction with an antiquark. Figure 3(c) depicts the creation of a $q\bar{q}$ pair and Fig. 3(d) the annihilation of a $q\bar{q}$ pair. We recall that both quarks and antiquarks propagate forward in time and have the same dispersion law [see (4.15)]. As a diagrammatic convention we choose that the time flows from right to left (in the bra-ket direction).

It is sufficient, in regard to Eq. (4.17), to consider the following vertex:

$$:\psi_{s_1}^\dagger(\mathbf{k}_1)\psi_{s_2}(\mathbf{k}_2): = :[u_{s_1}^\dagger(\mathbf{k}_1)b_{\mathbf{k}_1s_1}^\dagger + v_{s_1}^\dagger(\mathbf{k}_1)d_{-\mathbf{k}_1s_1}][u_{s_2}(\mathbf{k}_2)b_{\mathbf{k}_2s_2} + v_{s_2}(\mathbf{k}_2)d_{-\mathbf{k}_2s_2}^\dagger]: . \tag{4.18}$$

Then, the terms depicted in Figs. 3(a), 3(b), 3(c), and 3(d) correspond, respectively, to

$$[u_{s_1}^\dagger(\mathbf{k}_1)u_{s_2}(\mathbf{k}_2)]b_{\mathbf{k}_1s_1}^\dagger b_{\mathbf{k}_2s_2}, \quad -[v_{s_1}^\dagger(\mathbf{k}_1)v_{s_2}(\mathbf{k}_2)]d_{-\mathbf{k}_2s_2}^\dagger d_{-\mathbf{k}_1s_1} ,$$
$$[v_{s_1}^\dagger(\mathbf{k}_1)u_{s_2}(\mathbf{k}_2)]d_{-\mathbf{k}_1s_1}b_{\mathbf{k}_2s_2}, \quad [u_{s_1}^\dagger(\mathbf{k}_1)v_{s_2}(\mathbf{k}_2)]b_{\mathbf{k}_1s_1}^\dagger d_{-\mathbf{k}_2s_2}^\dagger . \tag{4.19}$$

It suffices to perform the above inner products to obtain the momentum dependence of these microscopic vertices.

They are

$$u_{s_1}^\dagger(\mathbf{k}_1)u_{s_2}(\mathbf{k}_2)=\tfrac{1}{2}\{[\sqrt{1+S_1}\sqrt{1+S_2}+\sqrt{1-S_1}\sqrt{1-S_2}(\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2)]\delta_{s_1s_2}+\sqrt{1-S_1}\sqrt{1-S_2}(i\sigma\cdot\hat{\mathbf{k}}_1\times\hat{\mathbf{k}}_2)_{s_1s_2}\}\ ,$$

$$-v_{s_1}^\dagger(\mathbf{k}_1)v_{s_2}(\mathbf{k}_2)=-\tfrac{1}{2}\{[\sqrt{1+S_1}\sqrt{1+S_2}+\sqrt{1-S_1}\sqrt{1-S_2}(\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2)]\delta_{s_1s_2}+\sqrt{1-S_1}\sqrt{1-S_2}(i\sigma\cdot\hat{\mathbf{k}}_1\times\hat{\mathbf{k}}_2)^*_{s_1s_2}\}\ ,$$

$$u_{s_1}^\dagger(\mathbf{k}_1)v_{s_2}(\mathbf{k}_2)=-\tfrac{1}{2}[(\sqrt{1-S_1}\sqrt{1+S_2}\hat{\mathbf{k}}_1-\sqrt{1+S_1}\sqrt{1-S_2}\hat{\mathbf{k}}_2)\cdot(\sigma i\sigma_2)_{s_1s_2}]\ ,\tag{4.20}$$

$$v_{s_1}^\dagger(\mathbf{k}_1)u_{s_2}(\mathbf{k}_2)=\tfrac{1}{2}[(\sqrt{1-S_1}\sqrt{1+S_2}\hat{\mathbf{k}}_1-\sqrt{1+S_1}\sqrt{1-S_2}\hat{\mathbf{k}}_2)\cdot(\sigma i\sigma_2)^*_{s_1s_2}]\ .$$

V. FEYNMAN RULES

In this section we define the Feynman rules to be used later[20] when deriving the Salpeter and RGM equations. A possible set of Feynman rules, see Ref. 10, is given below.

(i) Fermion propagator:

$$(2\pi)^4\delta(\omega-\omega)\delta^3(\kappa-\kappa')[D(k,w)]_{\alpha\alpha}$$

with

$$D_{\alpha\alpha}(k,w)=\frac{i}{w-E_k+i\epsilon\Lambda(\kappa)}\ .$$

(ii) Four-fermion amplitudes:

$$-i(2\pi)^4\delta(\omega_1+\omega_2-\omega_1-\omega_2)\delta^2(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_1-\mathbf{k}_2)\times\tfrac{4}{3}V(\mathbf{k}_1-\mathbf{k}_2)\sum_a[\lambda^a]_{ii}[\lambda^a]_{jj}\ .\tag{5.1}$$

However, it is more convenient when deriving the Dyson equations for mesons, to introduce an alternative, but equivalent, set of Feynman rules.

A. Propagators

The time dependence of the Fock-space operators b and d is rendered, after getting rid of the anomalous Bogoliubov terms, quite trivial:

$$b_{kt}^\dagger=e^{iH_2}b_k^\dagger e^{-iH_2}=b_k^\dagger e^{iE_kt}\ .\tag{5.2}$$

We define the propagators, for quarks and antiquarks, as follows:

$$S(k,k';t',t)=\langle 0|Tb_{k't'}b_{kt}^\dagger|0\rangle$$
$$=\langle 0|Td_{k't'}d_{kt}^\dagger|0\rangle\ .\tag{5.3}$$

Using the Fourier transforms of b_{kt} and d_{kt},

$$b_{kw}^\dagger=\int\frac{dt}{\sqrt{2\pi}}b_k^\dagger e^{i(E_k-w)t}\ ,\tag{5.4}$$

we obtain

$$S(k,w,k',w';E_k)=(2\pi)^4\delta^3(k-k')\times\delta(w-w')\frac{i}{w-E_k+i\epsilon}\ .\tag{5.5}$$

B. Vertices

To define the new Feynman vertices, we must extend $:H_4:$ to different times. As the potential is instantaneous, we have

$$i\int H_{\rm int}=:\frac{i}{2}\int d^3x\,d^3x'dt\,dt'\psi^\dagger(x,t)\lambda\psi(x,t)\cdot V_C(x-x')\delta(t-t')\psi^\dagger(x',t')\lambda\psi(x',t'):\ ,\tag{5.6}$$

where the time-dependent field operators now read

$$\psi(x,t)=\sum_s\int\frac{d^3k\,dw}{(2\pi)^{4/2}}[u_s(k)b_{kws}+v_s(k)d^\dagger_{-k-ws}]\times e^{-i(wt-k\cdot x)}\ .\tag{5.7}$$

After we integrate over the space and time variables, we

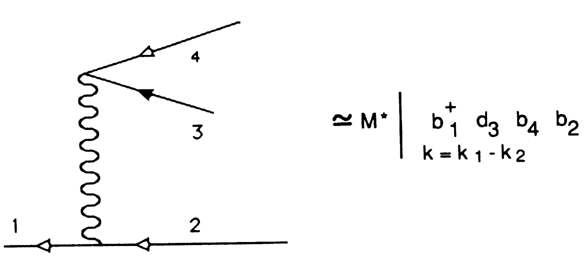

FIG. 2. A $q\bar{q}$ annihilation in the potential H_4.

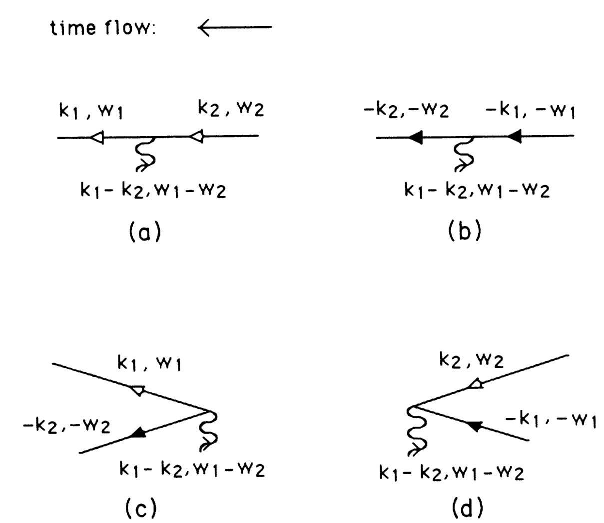

FIG. 3. Vertices in the potential H_4.

can see that the vertices obtained in the formalism of the $b_{k,w}$ are, with the exception of an extra Dirac δ in the energies, the same as those already obtained in expression (4.19):

$$\begin{aligned}&[u_{s_1}^\dagger(\mathbf{k}_1)u_{s_2}(\mathbf{k}_2)]b^\dagger_{\mathbf{k}_1w_1s_1}b_{\mathbf{k}_2w_2s_2}\ ,\\ &-[v_{s_1}^\dagger(\mathbf{k}_1)v_{s_2}(\mathbf{k}_2)]d^\dagger_{-\mathbf{k}_2-w_2s_2}d_{-\mathbf{k}_i-w_1s_1}\ ,\\ &[v_{s_1}^\dagger(\mathbf{k}_1)u_{s_2}(\mathbf{k}_2)]d_{-\mathbf{k}_1-w_1s_1}b_{\mathbf{k}_2w_2s_2}\ ,\\ &[u_{s_1}^\dagger(\mathbf{k}_1)v_{s_2}(\mathbf{k}_2)]b^\dagger_{\mathbf{k}_1w_1s_1}d^\dagger_{-\mathbf{k}_2-w_2s_2}\ .\end{aligned} \tag{5.8}$$

Finally, for each insertion of a potential line, between two of these vertices, one should consider the function

$$V(k-k')\sum(\lambda^a)_{ii}(\lambda^a)_{jj}\ , \tag{5.9}$$

where k and k' are, as usual, the momenta flowing in the potential line. As we have already said, although the potential is not quantized, and in fact we are dealing with four quark amplitudes as a whole, it is not difficult to see that we can subdivide these amplitudes in two quark-quark-potential subvertices, each of them conserving both momenta and energy. The different two-fermion vertices are shown in Fig. 3, and their coefficients have already been calculated in the study of H_4.

VI. MASS GAP EQUATION AND SOLUTIONS

The mass gap equation can be derived in various different ways. Some authors derived it as the condition[10] for the vacuum energy H_0 to be minimum, or in the form of a Dyson equation for a fermion propagator, or else as[11] a Ward identity. We obtain the mass gap equation simply by imposing[19] that the nondiagonal terms in H_2 corresponding to the direct creation (or annihilation) of a $q\bar{q}$ pair must be zero. The equation is

$$\cos\varphi_k A_k-\sin\varphi_k b_k=0\ . \tag{6.1}$$

If we consider a potential with components of the form Coulomb+vectorial+scalar (see Sec. IV), we are thus led to the equation

$$\begin{aligned}k\sin\varphi_k-m\cos\varphi_k=\tfrac{2}{3}\int\frac{d^3k'}{(2\pi)^3}\{&[V_C(\mathbf{k}-\mathbf{k}')-3V_V(\mathbf{k}-\mathbf{k}')+V_S(\mathbf{k}-\mathbf{k}')]\sin\varphi_{k'}\cos\varphi_k\\ &-[V_C(\mathbf{k}-\mathbf{k}')-V_V(\mathbf{k}-\mathbf{k}')-V_S(\mathbf{k}-\mathbf{k}')]\cos\varphi_{k'}\sin\varphi_k\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}'\}\ .\end{aligned} \tag{6.2}$$

This is, clearly, a nonlinear integral equation.

A. Linear potential

The linear potential has been "derived" from QCD in the framework of lattice gauge theories in the quenched approximation. It also corresponds to the intuitive model of flux tubes. Although these potentials contain also a $1/r$ term, here we will only be considering the linear confining term. If we choose the potential (where K_0^2 already contains the color contribution of $\frac{4}{3}$), to be of the form

$$V(x)=K_0^2x\ , \tag{6.3}$$

then the Fourier transform of this potential reads

$$V(k)=-K_0^2\frac{8\pi}{k^4}\ . \tag{6.4}$$

If we parametrize the different components of the potential as

$$V_C=\gamma V,\quad V_V=-\nu V,\quad V_S=\sigma V\ , \tag{6.5}$$

then the mass gap equation becomes, after integration over the spherical angles,

$$\begin{aligned}k\sin\varphi_k-m\cos\varphi_k=\frac{K_0^2}{2\pi}\int dq\, q^2&\sin\varphi_q(\gamma+3\nu+\sigma)\left[\frac{4}{(q^2-k^2)^2}\right]\cos\varphi_k\\ &-\cos\varphi_q(\gamma+\nu-\sigma)\left[\frac{q}{k}\frac{2k^2+2q^2}{(q^2-k^2)^2}-\frac{1}{k^2}\ln\left|\frac{q+k}{q-k}\right|\right]\sin\varphi_k\ .\end{aligned} \tag{6.6}$$

This integral exists only when

$$\gamma+3\nu+\sigma=\gamma+\nu-\sigma\ . \tag{6.7}$$

Otherwise the nonintegrable term in $1/(q-k)^2$ cannot be canceled. Thus, for a pure linear potential, only the Coulomb-like term with an equal mixture of scalarlike and vectorlike terms may exist (chiral angle different from zero), if the vacuum is to be condensed. Although this conclusion is reached in the BCS level of approach, it presumably means that in the other cases, we have a strongly broken chiral symmetry, and therefore it is only natural that the chiral angle should be zero. The mass gap equation for the linear potential case has been studied in the literature but the situation is not yet clear. Unlike Ref. 10, we find that there is no solution for a Coulomb plus vectorlike component. Although the solution probably exists for the simple Coulomb-like potential, we find that the existing[11] solution does not verify the mass gap equation.

B. Quadratic (harmonic-oscillator) potential

This type of potential[10] has the simplifying feature of supporting a simple differential mass gap equation instead of an integro-differential one. Also, one hopes that these potentials are still close enough to the "real physical potential" to give us a sufficiently good illustration of the chiral physics involved in hadronic spectroscopy and scattering. The extension of this potential to more "realistic" potentials will be mainly a matter of cumbersome calculations and, in principle, will not involve "new physics."

We introduce the potential as (where K_0^3 already contains the color contribution of $\frac{4}{3}$)

$$V(x)=K_0^3x^2 \ . \tag{6.8}$$

Then the Fourier transform is

$$V(k)=-K_0^3(2\pi)^3\Delta_{\mathbf{k}}\delta^3(\mathbf{k}) \ , \tag{6.9}$$

and the Dirac delta allows us to transform (6.2) into a differential equation.

If we parametrize the potential as

$$V_C=\gamma V, \quad V_V=-\nu V, \quad V_S=\sigma V \ , \tag{6.10}$$

and working in units of $K_0=1$, we get

$$\begin{aligned}(k^2\varphi')'[\gamma+2\nu+(\nu+\sigma)\cos2\varphi] &= 2k^3\sin\varphi-2mk^2\cos\varphi \\ &\quad +(\nu+\sigma)k^2\varphi'^2\sin2\varphi-(\gamma+\nu-\sigma)\sin2\varphi \ . \end{aligned} \tag{6.11}$$

It is convenient to define the boundary conditions of (6.11) before attempting to solve it. If we suppose that φ has no pole at the origin, expanding it in a power series of k, $\varphi=\varphi_0+\varphi_0'k+\varphi_0''k^2+\cdots$, and replacing it back in the mass gap equation, we obtain an infinite set of recursive relations, of which we write the first two

$$\begin{aligned}&(\gamma+\nu-\sigma)\sin2\varphi_0=0 \ , \\ &\varphi_0'(\gamma+2\nu)(1+\cos2\varphi_0)=0 \ .\end{aligned} \tag{6.12}$$

In this way (except for some very particular values of γ,ν,σ that would not give a numerical solution) we have either φ_0 and φ_0' vanishing (which will correspond to the trivial noncondensed vacuum), or else

$$2\varphi_0=\pm\pi \ , \tag{6.13}$$

and we will choose, for consistency, the positive φ_0. This is because we want to reproduce the nonrelativistic limit of a very massive quark. If we also require that the energy of one quark should be bounded from below we need, for large k, the limit of $\cos\varphi$ to be one. This means that the limit of φ, for large k, is zero.

The mass gap equation, being nonlinear, has no obvious analytical solution. We will solve it numerically. Before doing this, we can determine the set of parameters γ,ν,σ allowing a solution. The most stringent condition on these parameters is that

$$\gamma+2\nu+(\nu+\sigma)\cos2\varphi\neq0 \ . \tag{6.14}$$

In principle, (6.11) could still have a solution when both sides vanish, and in this way the above condition is not absolute. However, we found no exception when solving it numerically. As we just showed, $\cos2\varphi$ takes all values from -1 to 1, thus condition (6.14) reads

$$\frac{\gamma+2\nu}{\nu+\sigma}<-1 \quad \text{or} \quad 1<\frac{\gamma+2\nu}{\nu+\sigma} \ . \tag{6.15}$$

Some particular cases of inequalities (6.15) are given as

$$\begin{aligned}&\gamma=1, \quad \nu=0, \quad \sigma=0 \ , \\ &\gamma=1, \quad \nu<-1 \ \text{or} \ -\tfrac{1}{3}<\nu, \quad \sigma=0 \ , \\ &\gamma=1, \quad \nu=0, \quad -1<\sigma<1 \ .\end{aligned} \tag{6.16}$$

We see that although the Coulomb-like and vectorlike components are not very restricted, a pure scalarlike term or even a dominant scalarlike term seems (at the BCS level it is not) not to be allowed. It presumably means that in this case, the chiral symmetry is strongly broken, and therefore the chiral angle is null. One should also keep in mind that the solution φ must correspond to the minimum of H_0 (for example, the potential cannot be repulsive for $^3P_0$ $q\bar{q}$ color singlets). This constitutes a true constraint on the parameters γ,ν,σ (for example, it excludes $\gamma=1$ and $\nu<-1$). We do not verify the results of Ref. 10 for the Coulomb-like plus vectorlike case ($\gamma=1$ and $\nu\in R$).

C. Numerical solution of the mass gap differential equation

The solution $\varphi(k)$ of the equation

$$\varphi''=\frac{-2\varphi'}{k}+\frac{2k^3\sin\varphi-2mk^2\cos\varphi-k^2\varphi'^2\sin2\varphi(\nu+\sigma)-(\gamma+\nu-\sigma)\sin2\varphi}{k^2[\gamma+2\nu+(\nu+\sigma)\cos2\varphi]} \tag{6.17}$$

starts at $\varphi=\pi/2$ and has to converge to $\varphi=0$. We use the Runge-Kutta method to obtain, starting with $\varphi(0)=\pi/2$ and $\varphi'(0)$ arbitrary, the whole of $\varphi(k)$ and $\varphi'(k)$. In this iterative method we used a step of 0.004 in units of K_0 (the error is of the order of the step to the fourth power). Equation (6.17) being nonlinear, it is difficult to shoot directly at the value of $\varphi'(0)$ yielding $\varphi(\infty)=0$. Our function would rather oscillate around π or $-\pi$. However, if we know a k such that $\varphi(k)$ is already close to 0, then the equation is almost linear in the neighborhood of this k. Then the Newton method can be used and the value of $\varphi'(0)$ that yields a vanishing $\varphi(k)$ is obtained iteratively with the sequence $\varphi'(0)_i$:

$$\varphi'(0)_{i+1}=\frac{\varphi(k)_i\varphi'(0)_i-\varphi(k)_{i-1}\varphi'(0)_{i-1}}{\varphi(k)_i-\varphi(k)_{i-1}} . \quad (6.18)$$

We start by setting φ to 0 at a small k ($k=1.5$). Then we use the corresponding $\varphi'(0)$ as a starting value to find a better one which sets φ to 0 at a larger (better) k ($k=2.5$). We repeat the process for successively higher values of k until we reach a k that is big enough when compared with the region where $\varphi(k)$ is nonvanishing ($k=6.5$). This method is illustrated in Fig. 4, where we show the different $\varphi(k)$ vanishing at successive values of k.

In the case when the quarks have a mass, $\varphi(k)$ does not converge to zero exponentially any longer, because the mass contribution is $\arctan(m/k)$ (see Sec. II). Instead we should apply this method to the difference between the vacuum angle and the mass angle, $\varphi(k)-\arctan(m/k)$.

D. Solutions for a quadratic potential

In Fig. 5 we depict the simplest case[10] where we only have a Coulomb-like potential ($m=0$, $\gamma=1$, $\nu=0$, $\sigma=0$). This solution is positive and the initial derivative is -2.0375. . . . We also show in Fig. 5, for the same potential, the first excited solution displaying one node. In this case, the set of different solutions correspond to different vacuum energies H_0. From now on we shall be only interested in the ground states of the vacuum.

In Fig. 6 we study the case where the potential is still Coulomb-like but where the quarks have a finite mass ($m\neq 0$, $\gamma=1$, $\nu=0$, $\sigma=0$). In Fig. 6(a) we show the total angle $\varphi(k)$ together with the mass angle $\arctan(m/k)$ and the vacuum angle 2φ. In Fig. 6(b) the total angle for different masses ($m=0$ or 0.01 or 0.1 or 0.5 or 1 or 2 or 3) is shown.

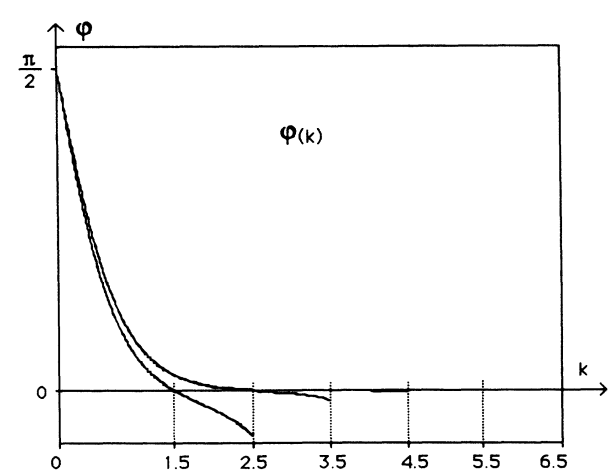

FIG. 4. Convergence of the Runge-Kutta-Newton numerical method for solving the mass gap equation.

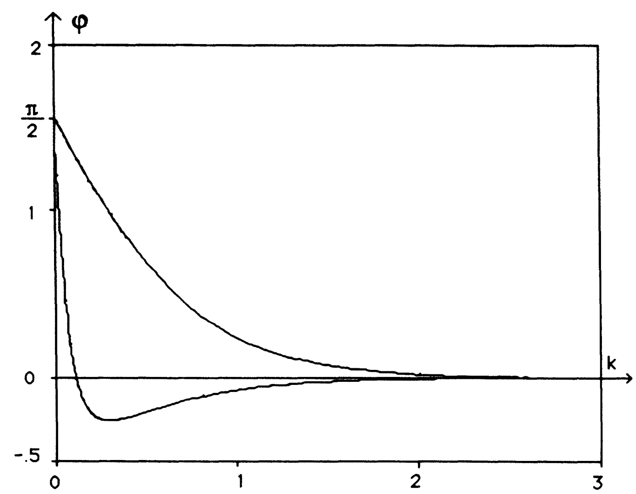

FIG. 5. First two solutions (Ref. 10) of the mass gap equation for a Coulomb-like potential.

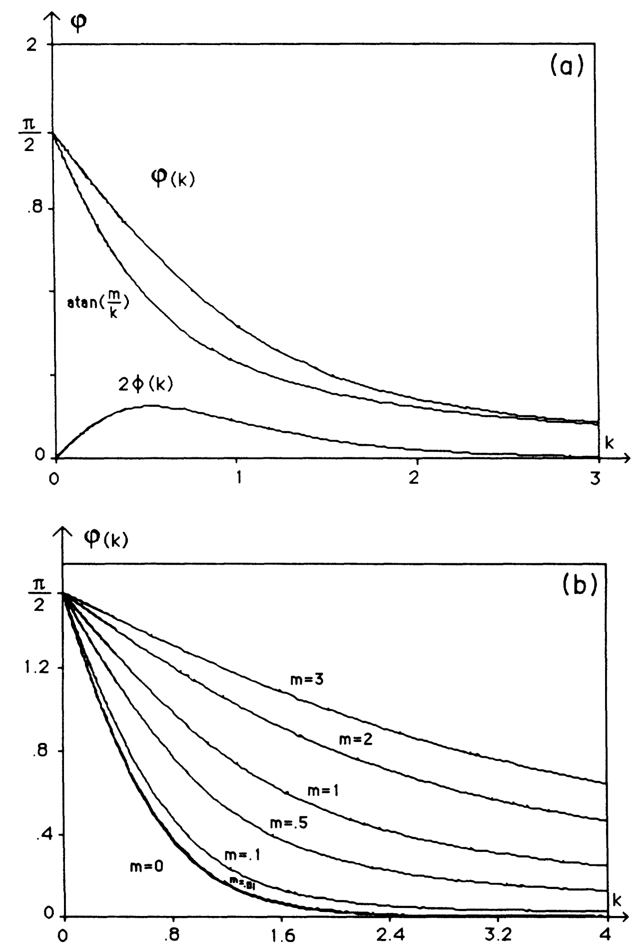

FIG. 6. (a) Vacuum angle 2ϕ, mass angle $\arctan(m/k)$, and solution φ of the mass gap equation in units of $K_0=1$. The potential parameters are $m=0.5$, $\gamma=1$, $\nu=0$, $\sigma=0$. (b) Solution of the mass gap equation when the harmonic-oscillator potential is Coulomb-like from the spinorial point of view and when quarks have a current mass of m. The unit is $k_0=1$.

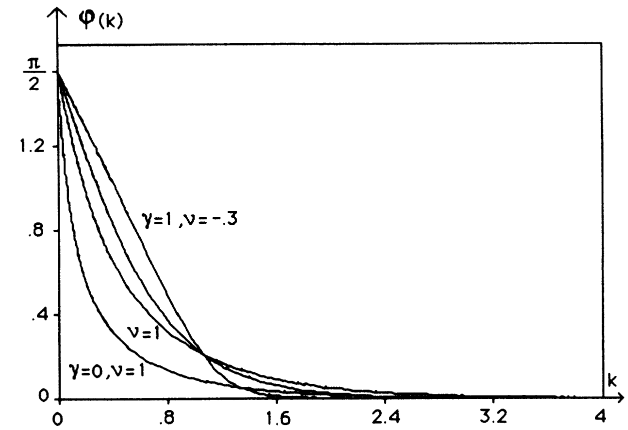

FIG. 7. Solution of the mass gap equation when not only the Coulomb-like term is present but also a vectorlike one. The corresponding parameters are ($m=0$, $\gamma=1$, $\nu=1$ or 0 or -0.3, $\sigma=0$). We also show the solution for the pure vectorlike potential.

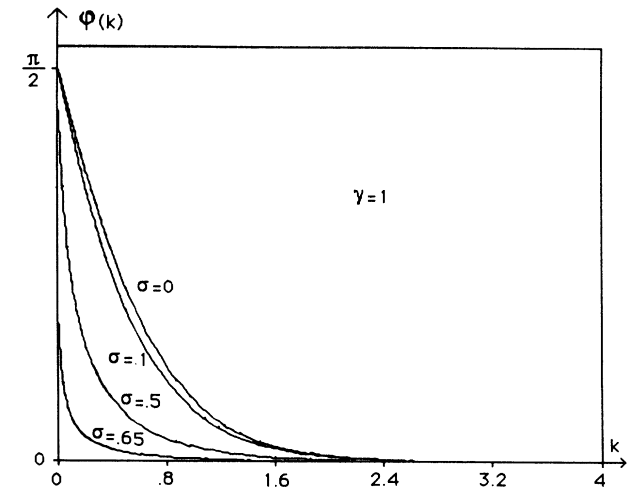

FIG. 8. Solution of the mass gap equation when a scalarlike potential is added to the Coulomb-like one ($m=0$, $\gamma=1$, $\nu=0$, $\sigma=0$ or 0.1 or 0.5 or 0.65).

In Fig. 7 we show the influence of adding to the Coulomb-like potential a vectorlike component ($m=0$, $\gamma=1$, $\nu=1$ or 0 or -0.3, $\sigma=0$). For positive ν the solution is now smaller than before, while for a negative one the effect is the opposite. We also show the solution for a pure vectorlike potential.

For a pure scalar potential we have no solution. However, a small admixture of this potential with the Coulomb-like potential admits a solution. In Fig. 8 we show such a solution ($m=0$, $\gamma=1$, $\nu=0$, $\sigma=0$ or 0.1 or 0.5 or 0.65). This case is interesting for it breaks the chiral symmetry like the massive case, and thus a small admixture of a scalar potential could provide another source for chiral-symmetry breaking.

VII. SINGLE-PARTICLE ENERGY AND DYNAMICAL MASS

Following the discussion in Ref. 10, we will also take the interquark confining potential to be the limit of a sequence of deeper and deeper potentials (but) going to zero whenever $r\to\infty$. This is the proper definition of a confining potential allowing for in and out free states, and amounts to subtract, from any of the potentials mentioned in Sec. VI, the appropriate infinite positive constant to ensure this. As was stated in Refs. 10 and 11 this constant, apart from ensuring that the quark self-energies are infinite, has no effect on the hadronic complex masses (for decay widths we checked this explicitly) and also leaves the mass gap equation invariant. Then, for calculational purposes other than the quark self-energies, which are not physically observable, because they are not infrared finite, we can drop these constants. But in doing so, it should be kept in mind that the quark quasiparticle self-energies will appear shifted downwards, from their "physical infinite value," by an infinite constant amount. Examples of such shifted self-energies are given below. They are simply related with a physical quantity called excitation energy $[E(k)-E(0)]$.

In the case of the general quadratic potential, the single-particle energy is given by[10]

$$E_k=A_k/\sin\varphi\ . \tag{7.1}$$

In A_k we use the δ function in the potential to remove the integral. Making use of the mass gap equation to replace φ'', we get

$$E_k=K_0\left[\frac{m}{\sin\varphi}-\frac{\gamma+3\nu+\sigma}{2}\varphi'^2+\frac{\cos\varphi}{2k^2}\frac{\gamma+3\nu+\sigma}{\sin\varphi}\frac{2k^3\sin\varphi-2mk^2\cos\varphi+k^2\varphi'^2\sin2\varphi(\nu+\sigma)-(\gamma+\nu-\sigma)\sin2\varphi}{\gamma+2\nu+(\nu+\sigma)\cos2\varphi}\right]. \tag{7.2}$$

It is easy to obtain this function numerically after finding (see Sec. VI) the functions $\varphi(k)$ and $\varphi'(k)$. In what follows, we shall restrict ourselves to the study of a Coulomb-like potential for massive quarks ($m=m$, $\gamma=1$, $\nu=0$, $\sigma=0$). The energy is now simply

$$E_k=K_0\left[m\sin\varphi-\tfrac{1}{2}\varphi'^2+k\cos\varphi-\frac{\cos^2\varphi}{k^2}\right]. \tag{7.3}$$

This function is shown in Fig. 9 for different masses ($m=0$ or 0.01 or 0.1 or 0.5 or 1 or 2 or 3). In this pic-

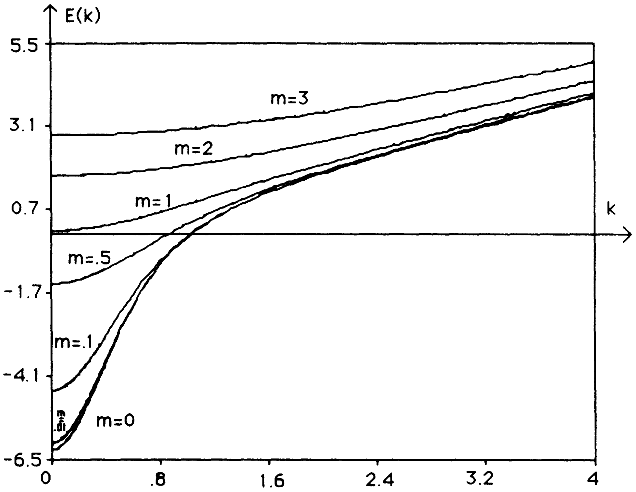

FIG. 9. Single-quark energy for different quark masses m. The units are in $K_0 = 1$.

ture, we see that for current masses bigger than K_0 the dispersion relation is similar to the usual $(k^2+m^2)^{1/2}$. However, for smaller current masses the function E_k acquires a quite different behavior, and even passes below the k axis.

That these energies can be negative should not worry us. In fact[10] if we add to the potential in space coordinates a constant energy shift then the quark energy is shifted by exactly half of that constant with the opposite sign. Indeed, as the potential is confining, in order to have it vanishing for an infinite r, it should be shifted by an infinite negative constant. However, we verified[20] that these two constant shifts exactly cancel against each other in the Bethe-Salpeter equation both for meson and and baryons. This is as it should be, because they are not physically observable. In the neighborhood of $k=0$, we can expand the energy in powers of k to get

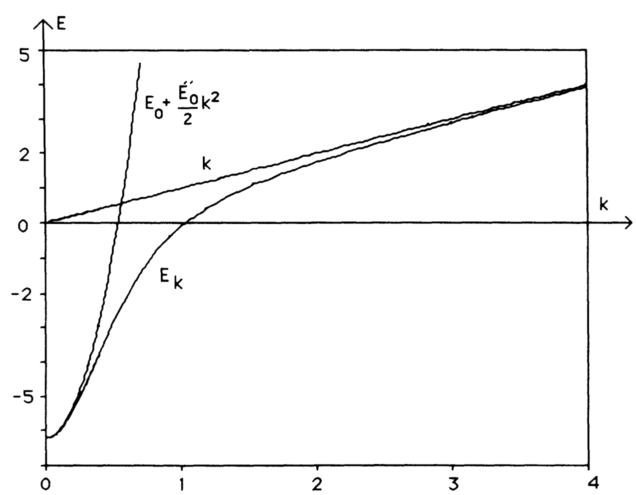

FIG. 10. Single-quark energy for zero current mass. We also show the low-energy quadratic approximation and the high-energy linear approximation.

$$E_k \cong [-\tfrac{3}{2}\varphi'(0)^2]+[\varphi'(0)^4-2\varphi'(0)]k^2+O(k^4) \ . \qquad (7.4)$$

When k is very large, $\varphi(k)$ is approximately 0 and we get, for E_k,

$$E_k \cong k \ . \qquad (7.5)$$

These two different limits of E_k are shown in Fig. 10. Although the expression for E_k for a small k allows a nonrelativistic limit, we see in Fig. 10 that the quadratic approximation for E_k diverges faster from the actual curve than in the usual $(k^2+m^2)^{1/2}$, and thus the chances that we obtain nonrelativistic bound states in this model are low.

A. Dynamical and constituent quark mass

Masses other than the current quark mass m are often used in quark models. When dynamical symmetry is broken, a dynamical mass is usually defined. Also, in nonrelativistic quark models, a constituent quark mass that yields most of the energy of the bound states is often used.

The dynamical mass can be interpreted[10,11] with the help of the propagator. When we work with field operators ψ, the fermion propagator can be written as

$$S=\frac{i}{w-H_{\mathbf{k}}+i\epsilon\Lambda_{\mathbf{k}}} = \frac{\Lambda_{\mathbf{k}}^+}{w-E_k}+\text{analytical} \ , \qquad (7.6)$$

and the dynamical mass m can be interpreted naively in three different ways. If we define m such that the residue[10,11] for a small k behaves like the energy projector for a massive (and noncondensed) situation, we obtain

$$m=\frac{k}{\cos\varphi_0}=-\frac{K_0}{c} \ , \qquad (7.7)$$

where c is the derivative of φ at the origin. However, if we look at the single-particle energy present in the denominator of S, and define m as its value at the origin, we get

$$m=E_0=-\tfrac{3}{2}c^2K_0 \ . \qquad (7.8)$$

It is also possible to define m such that the quadratic term in E_k has the form $K^2/2m$, to obtain

$$m=\left[\frac{d^2}{dk^2}(E_k)_{k=0}\right]^{-1}=\frac{K_0}{2(c^4-2c)} \ . \qquad (7.9)$$

It is clear that in the simple covariant and on mass-shell case all these three definitions of m yield the same result. But this is no longer true for our case. For example, if we have $c=-2$, we ambiguously get three different mass values:

$$m=0.5K_0, \quad m=-6K_0, \quad m=0.024K_0 \ , \qquad (7.10)$$

where the second mass even turns out to be negative.

In the formalism where the field operators are simply the b and the d's, the propagator now reads (the Λ are absorbed in the vertices 5.8)

$$S=\frac{i}{w-E_k+i\epsilon} . \quad (7.11)$$

We are left with the pole E_k, and, in the usual fashion, we parametrize its Taylor expansion with

$$E_k=-\mu+\frac{k^2}{2m^*}+O(k^4) . \quad (7.12)$$

In this way, the second naive definition of m is reinterpreted as being the chemical potential. m can now be defined as the effective dynamical mass:

$$m^*=\frac{K_0}{2(c^4-2c)}\cong 0.02K_0 . \quad (7.13)$$

If we take for K_0 a reasonable value between 300 and 600 MeV, this dynamical mass is clearly smaller than the *constituent* quark mass which is close to K_0. However, the constituent mass is defined from the masses of bound states of quarks in nonrelativistic models (m_{const} is approximately a third of the proton mass or half of the ρ mass). This is further evidence that our model is not likely to have nonrelativistic bound states.

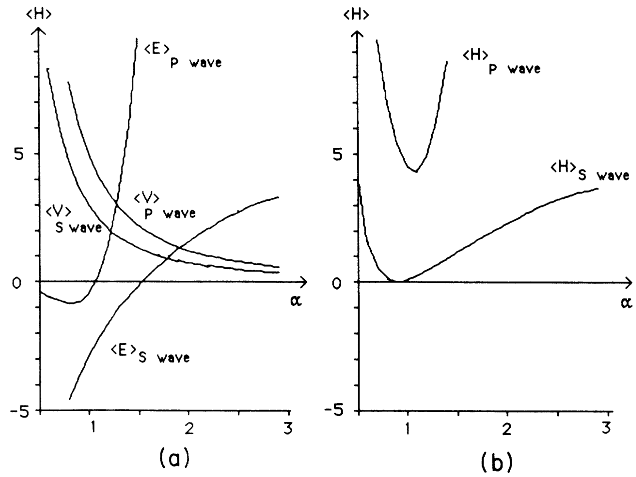

FIG. 11. (a) Mean value, in S-wave and in P-wave harmonic-oscillator wave functions, of the quark and antiquark energies and of the harmonic-oscillator potential. (b) Mean value, in a Gaussian wave function, of the energy of the $q\bar{q}$ in the S-wave and P-wave cases.

VIII. CONCLUSION

For a dominant scalarlike potential we found no vacuum condensation both for a linear and a quadratic confining form for the potential. This is, for instance, the case of the model in Ref. 13, where the spin-orbit component in the Fermi-Breit quark-quark potential vanishes, while having a large spin-spin term.

On the other hand, had we chosen a hyperfine spin-spin potential large enough to accommodate, for instance, the π-ρ mass difference, such a potential would, as we have shown in Ref. 4, produce hadron-hadron phase shifts in complete disagreement with the experiments.

Also, adding a constant shift to the interquark potential in order to better fit the center of gravity of hadronic masses would not do. We have shown that such potential shifts cancel against corresponding one-quark energies.

In a quark model with a chirally invariant confining potential, we find that the vacuum is condensed and is not chirally invariant. In the limit of massless quarks, the Hamiltonian is fully chirally invariant, and the breaking of the chiral symmetry by the vacuum, implies that all ground-state pseudoscalars are massless (their number is n_f^2). In this limit the only scale that we have is the scale of the confining potential. This scale appears in angular and radial splittings in hadronic spectroscopy. The mechanism of π-ρ mass splitting is illustrated in Fig. 11. In Fig. 11(a) we show the expectation values (in S-wave and in P-wave harmonic-oscillator wave functions of running parameter α) of the sum of one-particle energies of quark and antiquark and the potential between those two particles. In Fig. 11(b) we show the total expectation values of the single particle and potential energies. The result of the dynamics is that, while mesons without space excitations are massless, all the others show the scale of the potential. We obtain correctly the bound-state masses with the Bethe-Salpeter equation,[20] and it turns out that all bound states (except pseudoscalars like the π) are coupled to radial or angular excitations. This is an explicit example of PCAC at work.

In nonrelativistic quark models two more scales appear, the so-called constituent quark masses and the hyperfine splittings. The constitutent quark mass is typically $m_\rho/2$ and the hyperfine splitting scale is $m_\rho-m_\pi\cong m_\rho$. Thus in the chirally invariant Hamiltonian model, these two extra scales are superfluous because they are absorbed into the potential scale.

The current-quark masses explicitly break the chiral symmetry of the Hamiltonian. The masses of pseudoscalar mesons are, in this case, nonvanishing but they show only indirectly the scale of the current-quark masses. The mass increase of these mesons (in fact their own masses, the increase being from zero mass) is of the order of the corresponding increase in quark and antiquark single-particle energies $E(k_m)$ from the massless quark case (k_m is the mean value of k in the meson in question). This energy shift is, in general, different from the current masses m_q and $m_{\bar{q}}$.

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