General parametrization of the $V \to P\gamma$ meson decays

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We apply the general parametrization method [Phys. Rev. D **40**, 2997 (1989)] to the radiative decays of the light vector mesons. The ratios $\Gamma(\omega\to\pi\gamma)/\Gamma(\rho\to\pi\gamma)$ and $\Gamma(\eta'\to\rho\gamma)/\Gamma(\eta'\to\omega\gamma)$, corrected for the difference in momenta, are predicted to be 9 (to all orders in flavor breaking) plus possible contributions from gluon annihilation diagrams. So far there is no evidence (inside $\pm 15\%$ errors) for gluon effects in the above ratios; $\phi\to\pi^0\gamma$ leads to the same conclusion. We show that if the gluon diagrams are indeed negligible, the parametrization of the $V\to P\gamma$ decays (which is exact and thus includes automatically configuration mixing and all the complexities of the Fock expansion of the hadron states) coincides with the results of the nonrelativistic quark model (NRQM); this clarifies again why the NRQM works, independently of the internal v/c of the quarks. Another NRQM result shown here to be exact (to second order in flavor breaking) is $A(K^{*0}\to K^0\gamma)/A(K^{*+}\to K^+\gamma) = -(1+x)/(2-x)$ (with $x=\mu_\lambda/\mu_{\mathcal{N}}$) for the γ=decay amplitudes of K^{*0} and K^{*+} (in this case the gluon diagrams are absent).

I. INTRODUCTION

We extend the general parametrization of Ref. 1 to the $V\to P\gamma$ transitions (V and P are a vector and a pseudoscalar meson of the lowest nonets). Once more the aim is to separate the features specific of the nonrelativistic quark model[2] (NRQM), which has been used repeatedly[3-5] to treat the $V\to P\gamma$ decays, from those following from "first principles" or, more precisely, from a relativistic field theory satisfying two general conditions: (1) that the electromagnetic current is carried only by the quarks, and (2) that the only λ SU_3 (flavor) matrix in the strong-interaction Lagrangian is λ_8, from the flavor-breaking mass term. In QCD these conditions are satisfied because gluons are flavorless and neutral and no λ SU_3 (flavor) matrix enters in their coupling to quarks. The main results of this paper are listed in the summary and in Sec. XI.

II. SOME NOTATION

In the following A_i will be used always to indicate a vector meson (V) of the lowest nonet and B_j a pseudoscalar meson (P):

$$A_i=\rho,\omega,\phi,K^{*0},\bar{K}^{*0},K^{*\pm},\quad B_j=\pi,\eta,\eta',K^0,\bar{K}^0,K^\pm . \tag{1}$$

The matrix element for the transition $A_i\to B_j+\gamma$ in the rest system of A_i is

$$M_{ji}=\frac{1}{\sqrt{2k}}\frac{1}{\sqrt{2E_j(P)}}\int dt\,\exp(-ikt)\left\langle B_j(\mathbf{P})\left|\int d^3\mathbf{r}\,\exp(i\mathbf{k}\cdot\mathbf{r})\mathbf{j}(\mathbf{r},t)\right|A_i(0)\right\rangle\cdot\boldsymbol{\epsilon}\ , \tag{2}$$

where $\mathbf{j}(\mathbf{r},t)$ is the quark electromagnetic current and $\boldsymbol{\epsilon}$, $\mathbf{k}$, and k the photon polarization, momentum, and energy; $|B_j(\mathbf{P})\rangle$ and $|A_i(0)\rangle$ are, respectively, the true states of the pseudoscalar meson with momentum $\mathbf{P}$ and of the vector meson at rest; $E_j(P)$ is the energy of the P meson (until further notice we assume V to be heavier than P); $[2E_j(P)]^{-1/2}$ in (2) is required by Lorentz invariance if, as we do, we normalize both $|B_j(\mathbf{P})\rangle$ and $|A_i(0)\rangle$ to one meson per unit volume in the rest system of A_i.

We express the exact states $|B_j(\mathbf{P})\rangle$ and $|A_i(0)\rangle$ (each of which should be thought as a superposition of an infinite number of Fock states with quarks, antiquarks, and gluons) as

$$|A_i(0)\rangle=V|\phi_{A_i}0)\rangle,\quad |B_j(\mathbf{P})\rangle=V|\phi_{B_j}(\mathbf{P})\rangle\ , \tag{3}$$

where V is a unitary operator transforming the $1q$-$1\bar{q}$ model states $|\phi_{A_i}\rangle,|\phi_{B_j}\rangle$ into the exact states $|A_i\rangle,|B_j\rangle$. (Note: the same V is used for the operator V and the vector meson V, but there should be no risk of confusion.) Thus (2) becomes

$$M_{ji}=\frac{1}{\sqrt{2k}}\frac{1}{\sqrt{2E_j(P)}}\int dt\,\exp(-ikt)\left\langle\phi_{B_j}(\mathbf{P})\left|V^\dagger\int d^3\mathbf{r}\,\exp(i\mathbf{k}\cdot\mathbf{r})\mathbf{j}(\mathbf{r},t)V\right|\phi_{A_i}(0)\right\rangle\cdot\boldsymbol{\epsilon}\ . \tag{4}$$

We have written Eqs. (2) and (4) in some detail because, at variance with the cases dealt so far in Ref. 1, here the difference in the three-momenta of the initial and final states, 0 and **P**, respectively, must be considered. The model Hamiltonian $\mathcal{H}$ operating in the one-quark–one-antiquark sector can be taken in the form

$$\mathcal{H}=\mathcal{E}(P)+K(p)+X(r) , \tag{5}$$

where 1 and 2 are, respectively, q and $\bar{q}$; $\mathcal{E}$, K, X are three functions of the arguments indicated ($P=|\mathbf{p}_1+\mathbf{p}_2|$, $p=\frac{1}{2}|\mathbf{p}_1-\mathbf{p}_2|$, $\mathbf{r}=|\mathbf{r}_1-\mathbf{r}_2|$) that we do not need to specify here; they are assumed to be the same for all mesons of the P and V nonets; the common mass M_0 of the model mesons is

$$M_0=\mathcal{E}(0)+\epsilon_0 , \tag{6}$$

where ϵ_0 is the lowest eigenvalue of $K(p)+X(r)$.

In (4) it is

$$\mathbf{j}(\mathbf{r},t)=\exp[i(Ht-\mathbf{G}\cdot\mathbf{r})]\mathbf{j}(0)\exp[-i(Ht-\mathbf{G}\cdot\mathbf{r})] . \tag{7}$$

In (7) H and **G** are the exact (strong) Hamiltonian and momentum. Note that **G** commutes with V, whereas H of course does not. This is evident from the construction of V [Ref. 1(a)] and corresponds to the fact that the momentum of the model state can be the same as that of the exact state, but its mass or energy, obviously, is not the same. Indeed the *model states* of the nine P and V mesons at rest are all degenerate eigenstates of the model Hamiltonian $\mathcal{H}$ at the common mass value M_0 (6),

$$\mathcal{H}|\phi_{A_i}(0)\rangle=M_0|\phi_{A_i}(0)\rangle,\quad \mathcal{H}|\phi_{B_j}(\mathbf{P})\rangle=[M_0+\mathcal{E}(P)-\mathcal{E}(0)]|\phi_{B_j}(\mathbf{P})\rangle , \tag{8}$$

whereas the exact states at rest, related to the model states by (3), satisfy

$$H|A_i(0)\rangle=M(A_i)|A_i(0)\rangle,\quad H|B_j(0)\rangle=M(B_j)|B_j(0)\rangle . \tag{9}$$

Here $M(A_i)$ and $M(B_j)$ are the exact masses of V and P mesons, respectively. For the states with momentum **P** it is, of course,

$$H|B_j(\mathbf{P})\rangle=[P^2+M^2(B_j)]^{1/2}|B_j(\mathbf{P})\rangle\equiv E_j(P)|B_j(\mathbf{P})\rangle . \tag{10}$$

Inserting (5) into (4) and using Eqs. (7) and (8) we have

$$M_{ji}=\frac{1}{\sqrt{2k}}\frac{1}{\sqrt{2E_j(P)}}(2\pi)^4\delta^{(3)}(\mathbf{P}+\mathbf{k})\delta(M_i-k-E_j(P))\langle\phi_{B_j}(\mathbf{P})|V^\dagger\mathbf{j}(0)V|\phi_{A_i}(0)\rangle\cdot\boldsymbol{\epsilon} . \tag{11}$$

In the following, whenever this does not introduce ambiguities, we write M_i for $M(A_i)$ and M_j for $M(B_j)$; more generally the index i will refer to a vector meson and j to a pseudoscalar one.

III. THE MODEL STATES

We record, for clarity, the model states of the P and V mesons; with the model Hamiltonian (5) they are indeed (compare Ref. 1) the simplest possible ones compatible with the good quantum numbers. Suppressing the color factor, that here does not intervene, $|\phi_{A_i}(0)\rangle$ and $|\phi_{B_j}(\mathbf{P})\rangle$ are taken as

$$|\phi_{A_i}(0)\rangle=|\chi(A_i)\varphi(r)\rangle=\sum_{\mathbf{p}}\sum_{\rho_1\rho_2}\varphi(p)\chi_{\rho_1\rho_2}(A_i)a^\dagger_{\mathbf{p},\rho_1}b^\dagger_{-\mathbf{p},\rho_2}|0\rangle , \tag{12}$$

$$\begin{aligned}|\phi_{B_j}(\mathbf{P})\rangle&=|\chi(B_j)\varphi(r)\exp(i\mathbf{P}\cdot\mathbf{R})\rangle\\&=\sum_{\mathbf{p}}\sum_{\rho_1\rho_2}\varphi(p)\chi_{\rho_1\rho_2}(B_j)a^\dagger_{\mathbf{p}+(\mathbf{P}/2),\rho_1}b^\dagger_{-\mathbf{p}+(\mathbf{P}/2),\rho_2}|0\rangle .\end{aligned} \tag{13}$$

In the second form of $|\phi_{A_i}\rangle,|\phi_{B_j}\rangle$ in (12) and (13), $|0\rangle\equiv|0(q),0(\bar{q}),0(\text{gluons})\rangle$ is the bare vacuum of quarks, antiquarks, and gluons [the quarks and antiquarks $\mathcal{P}$, $\mathcal{N}$, λ and $\bar{\mathcal{P}}$, $\bar{\mathcal{N}}$, $\bar{\lambda}$ have renormalized "constituent" masses—Ref. 1(a)]; $a^\dagger_{\mathbf{p},\rho}$ and $b^\dagger_{\mathbf{p},\rho}$ are creation operators of a quark and antiquark, respectively, with masses as stated above, momentum **p**, and in the spin-flavor state ρ; **P** is the momentum of the pseudoscalar meson: for a transition $A_i\to B_j\gamma$ it is (writing $P=|\mathbf{P}|$)

$$P\equiv P_{ij}=(M_i^2-M_j^2)/(2M_i) . \tag{14}$$

In (12) and (13) ρ_1 and ρ_2 are indices referring to the spin-flavor state of the quark (1) and antiquark (2); *here, and in the following, 1 always refers to the quark (q) and 2 to the antiquark ($\bar{q}$)*; $\chi_{\rho_1\rho_2}$ are the spin-flavor functions [with spin one for the V mesons (A_i) and spin zero for the P mesons (B_j)]; in spite of the fact that our procedure is relativistic, the spin of the model states is described by Pauli two-component spinors [V operating on the model states also leads from Pauli to Dirac spinors; compare Ref. 1(a)]; $\varphi(r)$ and its Fourier transform $\varphi(p)$, equal for all states, that is, independent of the indices i and j, is the rotation invariant ($L=0$) space (or momentum) part of the model wave function, that of the lowest level of $\mathcal{H}$ (5).

There is a constraint on the flavor part of the spin-flavor wave functions $\chi_{\rho_1\rho_2}$ for the mesons of isospin $I=0$; this constraint is obvious in the nonrelativistic quark model (NRQM); but our parametrization is model independent; thus here we must show explicitly how it arises. The argument is this: As stated, there is freedom in choosing the model Hamiltonian and, therefore, the model wave functions; *but, once chosen, they must be the same in any calculation referring to the same hadrons.* In particular, for the mesons of the nonet, the model wave functions are restricted by the requirement that

$$\begin{aligned} \langle \phi_{A_i}(0)|V^\dagger H V|\phi_{A_i}(0)\rangle &= M(A_i)\ , \\ \langle \phi_{B_j}(0)|V^\dagger H V|\phi_{B_j}(0)\rangle &= M(B_j)\ , \end{aligned} \tag{15}$$

where H is the exact Hamiltonian of the strong interactions. The problem of parametrizing the masses of the vector and pseudoscalar mesons of the nonet was treated in Ref. 1(c); we showed that, correct to first order in flavor breaking, the flavor part of the model wave functions for the η and η' is

$$\eta = 0.603(\mathcal{P}\bar{\mathcal{P}}+\mathcal{N}\bar{\mathcal{N}}) - 0.522\lambda\bar{\lambda}\ , \tag{16}$$

$$\eta' = 0.367(\mathcal{P}\bar{\mathcal{P}}+\mathcal{N}\bar{\mathcal{N}}) + 0.854\lambda\bar{\lambda}\ . \tag{17}$$

We also noted that with the usual definition of the pseudoscalar mixing angle θ_P, $\eta=-\eta_1\sin\theta_P+\eta_8\cos\theta_P$ and $\eta'=\eta_1\cos\theta_P+\eta_8\sin\theta_P$, where $\eta_1=(1/\sqrt{3})(\mathcal{P}\bar{\mathcal{P}}+\mathcal{N}\bar{\mathcal{N}}+\lambda\bar{\lambda})$ and $\eta_8=(1/\sqrt{6})(\mathcal{P}\bar{\mathcal{P}}+\mathcal{N}\bar{\mathcal{N}}-2\lambda\bar{\lambda})$, the wave functions (16) and (17) correspond to

$$\sin\theta_P \cong -0.39 \quad (\text{that is, } \theta_P \cong -23^\circ)\ . \tag{18}$$

Thus we must select the flavor factor for η and η' as given by Eqs. (16) and (17). As to the $I=0$ vector mesons, their masses are rather well represented by an "ideal" mixing angle θ_V, corresponding to

$$\omega = (1/\sqrt{2})(\mathcal{P}\bar{\mathcal{P}}+\mathcal{N}\bar{\mathcal{N}})\ , \tag{19}$$

$$\phi = \lambda\bar{\lambda}\ , \tag{20}$$

and given by

$$\theta_V = 35.3^\circ\ . \tag{21}$$

We will always use in what follows this "ideal" θ_V, except, of course, for $\phi\to\pi^0\gamma$ which depends just on the small deviation $(\theta_V^*-35.3^\circ)$ of the exact angle θ_V^* from the ideal one. In the linear case L it is

$$\theta_V^{*L} = 37^\circ \pm 1.2^\circ\ . \tag{22}$$

IV. THE GENERAL EXPRESSION OF THE MATRIX ELEMENT

Using (12) and (13), the matrix element $\langle \phi_{B_j}(\mathbf{P})|V^\dagger \mathbf{j}(0) V|\phi_{A_i}(0)\rangle$ in (11) is

$$\langle \phi_{B_j}(\mathbf{P})|V^\dagger \mathbf{j}(0) V|\phi_{A_i}(0)\rangle = \sum_{\mathbf{p}\rho_1\rho_2}\sum_{\mathbf{p}'\rho_1'\rho_2'} \varphi^*(p')\chi^*_{\rho_1'\rho_2'}(B_j)\mathbf{F}_{\rho_1'\rho_2',\rho_1\rho_2}(\mathbf{p}',\mathbf{p},\mathbf{P})\chi_{\rho_1\rho_2}(A_i)\varphi(p)\ , \tag{23}$$

where the operator $\mathbf{F}_{\rho_1'\rho_2',\rho_1\rho_2}(\mathbf{p}',\mathbf{p},\mathbf{P})$ is

$$\mathbf{F}_{\rho_1'\rho_2',\rho_1\rho_2}(\mathbf{p}',\mathbf{p},\mathbf{P}) = \langle 0|b_{-\mathbf{p}'+(\mathbf{P}/2),\rho_2'}a_{\mathbf{p}'+(\mathbf{P}/2),\rho_1'}V^\dagger \mathbf{j}(0) V a^\dagger_{\mathbf{p},\rho_1}b^\dagger_{-\mathbf{p},\rho_2}|0\rangle\ . \tag{24}$$

We now evaluate the matrix element (23). Because $\varphi(p)$ has been chosen rotational invariant, only the projection of $\mathbf{F}_{\rho_1'\rho_2',\rho_1\rho_2}(\mathbf{p}',\mathbf{p},\mathbf{P})$ on the states with $L=0$ in the $\mathbf{p},\mathbf{p}'$ space contributes after performing in (23) the sums over $\mathbf{p}$ and $\mathbf{p}'$ (that is, the integrations over $d^3\mathbf{p}$ and $d^3\mathbf{p}'$). On defining

$$\mathcal{F}_{\rho_1'\rho_2',\rho_1\rho_2}(\mathbf{P}) = \sum_{\mathbf{p}'}\sum_{\mathbf{p}} \varphi^*(p')\mathbf{F}_{\rho_1'\rho_2',\rho_1\rho_2}(\mathbf{p}',\mathbf{p},\mathbf{P})\varphi(p)\ , \tag{25}$$

we have

$$\langle \phi_{B_j}(\mathbf{P})|V^\dagger \mathbf{j}(0) V|\phi_{A_i}(0)\rangle = \sum_{\rho_1'\rho_2'}\sum_{\rho_1\rho_2} \chi^*_{\rho_1'\rho_2'}(B_j)\mathcal{F}_{\rho_1'\rho_2',\rho_1\rho_2}(\mathbf{P})\chi_{\rho_1\rho_2}(A_i)\ . \tag{26}$$

The fact that only the $L=0$ part of $\mathbf{F}_{\rho_1'\rho_2',\rho_1\rho_2}(\mathbf{p}',\mathbf{p},\mathbf{P})$ contributes to (23), corresponds to the omission in the NRQM of the part of the magnetic current due to the orbital angular momentum. But one important point must be added. The model states have a $\varphi(p)$ with $L=0$, but the states obtained applying V to the model states are exact; they include, in particular, all kinds of configuration mixing. Thus, whereas in the NRQM treatment the orbital angular momentum part of the transition current is omitted invoking the long-wavelength limit and neglecting configuration mixing, here we do not introduce such assumptions.

Consider now $\mathcal{F}(\mathbf{P})\equiv\mathcal{F}_{\rho_1'\rho_2',\rho_1\rho_2}(\mathbf{P})$ in (26); $\mathcal{F}(\mathbf{P})$ leads from a spin-one to a spin-zero state; in the spin space of 1 and 2, $\mathcal{F}(\mathbf{P})$ must thus be a vector, constructed in terms of the Pauli spin matrices $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$; call $\mathbf{S}_\mu$ the set of such possible spin vectors and Γ_ν the set of the possible flavor operators that act in the flavor space of the quark and antiquark; both will be listed in Sec. VII where also the multiplicative constant present both in $\mathbf{S}_\mu$ and $\Gamma_\nu(f)$ will be fixed; the set of all operators $\boldsymbol{\Omega}$ in spin-flavor space is

$$\boldsymbol{\Omega}_{\mu\nu}=\mathbf{S}_\mu\cdot\Gamma_\nu \ . \tag{27}$$

We anticipate that (Sec. VII) the spin part $\mathbf{S}_\mu$ is the same in all $\boldsymbol{\Omega}_{\mu\nu}$'s, whereas for the flavor operators Γ_ν there are various (seven) choices; thus the index μ can be omitted and (27) is rewritten

$$\boldsymbol{\Omega}_\nu=\mathbf{S}\cdot\Gamma_\nu \ . \tag{28}$$

In this notation the most general $\mathcal{F}(\mathbf{P})$ is

$$\mathcal{F}(\mathbf{P})=\sum_\nu [h_\nu(P)\boldsymbol{\Omega}_\nu+g_\nu(P)\mathbf{P}\times\boldsymbol{\Omega}_\nu] \ . \tag{29}$$

Here $h_\nu(P)$ and $g_\nu(P)$ are coefficients depending on P; the sum over ν in (29) extends to all possible spin-flavor operators $\boldsymbol{\Omega}_\nu$. For a transition between two mesons A_i and B_j with the same parity (magnetic transition) the first (h_ν) term in (29) must vanish and $\mathcal{F}(\mathbf{P})$ becomes

$$\mathcal{F}(\mathbf{P})=\sum_\nu g_\nu(P)\boldsymbol{\Omega}_\nu\times\mathbf{k} \ , \tag{30}$$

where $\mathbf{P}$ has been replaced by $-\mathbf{k}$ [Eq. (9)]; recall that $\boldsymbol{\epsilon}\times\mathbf{k}$ is proportional to the magnetic field of the photon. We rewrite (11) after insertion of (30) in (26) and of the latter in (11); putting

$$\mathcal{M}(B_j A_i)=\sum_\nu g_\nu(P)\langle\chi(B_j)|\boldsymbol{\Omega}_\nu|\chi(A_i)\rangle=\sum_\nu g_\nu(P)\sum_{\rho_1\rho_2}\sum_{\rho_1'\rho_2'}\chi^*_{\rho_1'\rho_2'}(B_j)(\boldsymbol{\Omega}_\nu)_{\rho_1'\rho_2',\rho_1\rho_2}\chi_{\rho_1\rho_2}(A_i) \ , \tag{31}$$

we get

$$M_{ji}=(2\pi)^4\frac{1}{\sqrt{2k}}\frac{1}{\sqrt{2E_j(P)}}\delta^{(3)}(\mathbf{P}+\mathbf{k})\delta(M_i-k-E_j(P))\mathcal{M}(B_j A_i)\cdot\mathbf{k}\times\boldsymbol{\epsilon} \ . \tag{32}$$

Introducing the abbreviation

$$\boldsymbol{\Omega}_\nu(B_j A_i)\equiv\langle\chi(B_j)|\boldsymbol{\Omega}_\nu|\chi(A_i)\rangle \ , \tag{33}$$

(31) becomes

$$\mathcal{M}(B_j A_i)=\sum_\nu g_\nu(P)\boldsymbol{\Omega}_\nu(B_j A_i) \ . \tag{34}$$

Because the spin-flavor functions χ can be factorized in a spin factor κ times a flavor factor ξ, each $\boldsymbol{\Omega}_\nu(B_j A_i)$ (33) is the product of a spin matrix element $\langle\kappa(B_j)|\mathbf{S}|\kappa(A_i)\rangle$ times a flavor matrix element $\langle\xi(B_j)|\Gamma_\nu|\xi(A_i)\rangle$:

$$\boldsymbol{\Omega}_\nu(B_j A_i)=\langle\kappa(B_j)|\mathbf{S}|\kappa(A_i)\rangle\langle\xi(B_j)|\Gamma_\nu|\xi(A_i)\rangle \ . \tag{35}$$

In Sec. VII we will see that the spin operator $\mathbf{S}$ in (28) is

$$\mathbf{S}=\tfrac{1}{2}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2) \ . \tag{36}$$

The factor $\frac{1}{2}$ in (36) is a convenient normalizations factor; we now introduce the abbreviations

$$\mathbf{e}=\langle\kappa(B_j)|\mathbf{S}|\kappa(A_i)\rangle \ , \tag{37a}$$

$$\Gamma_\nu(B_j, A_i)=\langle\xi(B_j)|\Gamma_\nu|\xi(A_i)\rangle \ . \tag{37b}$$

Thus,

$$\boldsymbol{\Omega}_\nu(B_j A_i)=\mathbf{e}\Gamma_\nu(B_j, A_i) \ , \tag{38}$$

so that $\mathcal{M}(B_j A_i)$ in (32) is

$$\mathcal{M}(B_j A_i)=\mathbf{e}\sum_\nu g_\nu(P)\Gamma_\nu(B_j, A_i) \ . \tag{39}$$

V. THE GENERAL COVARIANT MATRIX ELEMENT AND THE PARAMETRIZED ONE

We now compare the decay matrix element (32) [with $\mathcal{M}(B_j A_i)$ given by Eq. (39)] with that derived uniquely from the requirement of relativistic invariance (our treatment is noncovariant, but relativistic). For a $V_i\to P_j\gamma$ transition there exists only one possible vertex:[3] namely,

$$G_{ij}\partial_\alpha A_\beta\partial_\mu V_\nu P\epsilon_{\alpha\beta\mu\nu} \ , \tag{40}$$

where A_α, V_ν, and P are the electromagnetic, vector, and pseudoscalar fields, $\epsilon_{\alpha\beta\mu\nu}$ is the Levi-Civita symbol, and G_{ij} is a real constant, with the dimensions of a magnetic moment, depending on the i,j pair; $G_{ij}=G_{ij}(p_1^2,p_2^2,p_3^2)$ is a Lorentz invariant that can depend only on invariants constructed with the four-momenta p_1, p_2, p_3 of the three external "legs" of the $V\to P\gamma$ diagram; because it is $p_1+p_2=p_3$, only two such invariants exist, the masses of the vector meson and of the P meson, so that

$$G_{ij}\equiv G_{ij}(M_i^2,M_j^2) \ . \tag{41}$$

Calculate now the matrix element (call it M_{ij}') from (40) *in the frame where the vector meson is at rest;* normalize the vector field V of the ith meson in (40) so that the matrix element for destruction of a V meson at rest (one per unit volume) with spin up is $(1/\sqrt{2})\langle 0|(V_x-iV_y)|A_i\uparrow\rangle=(1/\sqrt{2M_i})$. As easily checked (Appendix), it is

$$M'_{ji}=(2\pi)^4\frac{1}{\sqrt{2k}}\frac{1}{\sqrt{2E_j(P)}}\delta^{(3)}(\mathbf{P}+\mathbf{k})\times\delta(M_i-k-E_j(P))\mathcal{M}'(B_j\,A_i)\cdot\mathbf{k}\times\epsilon\ , \quad (42)$$

with

$$\mathcal{M}'(B_j\,A_i)=\mathbf{e}'G_{ij}(M_i^2,M_j^2)(M_i/2)^{1/2}\ . \quad (43)$$

Here as above, the indices i and j refer, respectively, to the V and P mesons; $\mathbf{e}'$ is a unit vector specifying the polarization of the V meson; it is equal to $\mathbf{e}$ [defined by Eq. (37a)] if $\mathbf{S}$ is defined by Eq. (36). Identifying M'_{ij} with M_{ij} and thus $\mathbf{M}'(B_j\,A_i)$ (43) with $\mathbf{M}(B_j\,A_i)$ (36) we obtain

$$(M_i/2)^{1/2}G_{ij}(M_i^2,M_j^2)\mathbf{e}'=\sum_\nu g_\nu(P)\mathbf{\Omega}_\nu(B_j\,A_i)\ , \quad (44)$$

or, simplifying,

$$(M_i/2)^{1/2}G_{ij}(M_i^2,M_j^2)=\sum_\nu g_\nu(P)\Gamma_\nu(B_j,A_i)\ . \quad (45)$$

VI. AN APPARENT PARADOX AND ITS SOLUTION

The equation (45) is our main equation; it expresses the coefficient G_{ij} of the relativistic vertex (40) in the parametrized form; it is an exact result. Before proceeding to list the flavor operators Γ_ν on the right-hand side of Eq. (45) [and show that (36) is the only possible $\mathbf{S}$ in (27)] we must, however, clarify a point in Eq. (45) that, at first sight, is puzzling.

Indeed we would have expected $G_{ij}(M_i^2,M_j^2)$ [as well as $M_i^{1/2}G_{ij}(M_i^2,M_j^2)$] to be a general function of the two masses M_j and M_i; but the right-hand side of (45) does not appear to confirm this expectation; it seems that $M_i^{1/2}G_{ij}(M_i^2,M_j^2)$ depends only on $P\equiv P_{ji}=(M_j^2-M_i^2)/2M_i$. We will now prove that there is, in fact, no inconsistency. Here we summarize the proof, referring to the Appendix for more details.

The masses M_j (or M_i) of a P (or V) meson of the nonet are [as shown in Ref. 1(c)] the eigenvalues of a linear combination of spin-flavor operators; these operators belong to a set (call it SF) in the one-quark–one-antiquark spin-flavor space. In the space of the spin-flavor functions, SF is a closed set: $(\mathrm{SF})^n$ consists again of operators belonging to SF; this means that an arbitrary function of the masses can be written as a linear combination of operators belonging to SF. Now consider Eq. (44): If we multiply one of the $\mathbf{\Omega}_\mu$'s in (44) by any operator in SF, we produce only a combination of the same $\mathbf{\Omega}_\mu$'s (insofar as the calculation of a $V\rightarrow P\gamma$ matrix element is concerned). Thus no contradiction is present in (44) [and therefore in (45)]; indeed each operator $\mathbf{\Omega}_\mu$ in (44), multiplied by an arbitrary function of the masses of the P and V mesons, becomes again a linear combination of $\mathbf{\Omega}_\nu$'s; under such multiplication Eq. (44) continues to have the same form [with different coefficients $g_\nu(P)$]. We conclude that we can multiply the right-hand side of (44) by an arbitrary function of M_i,M_j [changing the $g_\nu(P)$'s but] without changing its general structure; thus the paradox has disappeared.

We clarify, at this stage, the meaning of (45). Assume, for instance, that we were able to perform a QCD calculation (and that QCD is the correct theory); after fixing once forever the Γ_ν's (as listed in Sec. VII), the calculation would lead to a definite set of $g_\nu(P)$'s on the right-hand side of (45). It may happen that some $g_\nu(P)$'s contribute negligibly compared to others; then the neglect in (45) of such "small" $g_\nu(P)$'s may lead to relationships between decays that are approximate predictions of QCD. If, as is the case, we are unable to perform the QCD calculation, but we can make a reasonable guess on which $g_\nu(P)$'s are negligible, we will find again such relationships. This will be essentially the use of Eq. (45), by which (we shall see) many results of the NRQM will be reproduced.

VII. THE LIST OF OPERATORS $\mathbf{\Omega}_\nu$ IN THE SPIN-FLAVOR SPACE

A. Flavor structure

Call Ψ the mass renormalized [constituent—compare Refs. 1(a)–1(c)] quark fields,

$$\Psi=\begin{vmatrix}\mathcal{P}\\ \mathcal{N}\\ \lambda\end{vmatrix}\ ,$$

and, in terms of the Gell-Mann SU_3 (flavor) matrices λ_3 and λ_8 [$\lambda_3=\mathrm{Diag}(1,-1,0)$ and $\lambda_8=\mathrm{Diag}(1,1,-2)$], introduce the projection operators $\Pi^{\mathcal{P}},\Pi^{\mathcal{N}},\Pi^{\lambda}$ on the $\mathcal{P},\mathcal{N},\lambda$ fields given, of course, by $\Pi^{\mathcal{P}}=\frac{1}{6}(2+3\lambda_3+\lambda_8)$, $\Pi^{\mathcal{N}}=\frac{1}{6}(2-3\lambda_3+\lambda_8)$, $\Pi^{\lambda}=\frac{1}{3}(1-\lambda_8)$. Introduce also the "charge" combination Q of the above projection operators:[6]

$$Q=\tfrac{2}{3}\Pi^{\mathcal{P}}-\tfrac{1}{3}\Pi^{\mathcal{N}}-\tfrac{1}{3}\Pi^{\lambda}\ . \quad (46)$$

In terms of Q the electromagnetic current $j_\mu(x)$ in (2) is

$$j_\mu(x)=:\bar{\Psi}(x)\gamma_\mu Q\Psi(x): \quad (47)$$

The operators $\Pi^{\mathcal{P}},\Pi^{\mathcal{N}},\Pi^{\lambda}$ defined above, give one when applied either to a state containing a quark or an antiquark of the appropriate flavor:

$$\Pi^{\mathcal{P}}|\mathcal{P}\rangle=|\mathcal{P}\rangle,\quad \Pi^{\mathcal{P}}|\bar{\mathcal{P}}\rangle=|\bar{\mathcal{P}}\rangle\ ,$$
$$\Pi^{\mathcal{N}}|\mathcal{N}\rangle=|\mathcal{N}\rangle,\quad \Pi^{\mathcal{N}}|\bar{\mathcal{N}}\rangle=|\bar{\mathcal{N}}\rangle\ , \quad (48)$$
$$\Pi^{\lambda}|\lambda\rangle=|\lambda\rangle,\quad \Pi^{\lambda}|\bar{\lambda}\rangle=|\bar{\lambda}\rangle\ ,$$

and zero otherwise; *with the above definition of Q the charge of an antiquark is* $(-Q)$. The flavor-breaking part of the mass term in the Hamiltonian is

$$\Delta m\int d^3\mathbf{r}:\bar{\Psi}(\mathbf{r})\Pi^{\lambda}\Psi(\mathbf{r}):\ . \quad (49)$$

As noted in Ref. 1, the only flavor operators in the Lagrangian are Q and Π^{λ} and they commute. Because of this (and because we calculate only to first order in electromagnetism), the end result for $V^\dagger \mathbf{j}(0)V$ and thus for each $\mathbf{\Omega}_\nu$ in (34) must be linear in Q, no matter how complicated the calculation of $V^\dagger \mathbf{j}(0)V$ in (11). Therefore the flavor structure of each $\mathbf{\Omega}_\nu$ will be a linear combination of terms of the form

$$Q_i; \quad Q_i \cdot \Pi_k^\lambda; \quad Q_i \cdot \Pi_1^\lambda \cdot \Pi_2^\lambda , \tag{50}$$

where the indices i,k have the values 1 (quark) and 2 (antiquark).

The list (50) of the flavor operators is complete if both the mesons V and P in the $V \to P\gamma$ transition have isospin $I=1$; if either V or P, or both, have $I=0$, additional flavor structures can intervene. In writing them we follow the same line of Ref. 1(c) (Sec. III) expanding a little on some points.

As an example take the $\omega \to \eta\gamma$ transition; the flavor factors of ω and η are $\omega = (1/\sqrt{2})(\bar{\mathcal{P}}\mathcal{P} + \bar{\mathcal{N}}\mathcal{N})$ and $\eta = r(1/\sqrt{2})(\bar{\mathcal{P}}\mathcal{P} + \bar{\mathcal{N}}\mathcal{N}) + t\bar{\lambda}\lambda$ with r and t two coefficients related to the pseudoscalar mixing angle θ_P. The flavor part of the matrix element (34) for the $\omega \to \eta\gamma$ transition is thus

$$\langle r(1/\sqrt{2})(\bar{\mathcal{P}}\mathcal{P} + \bar{\mathcal{N}}\mathcal{N}) + t\bar{\lambda}\lambda | V^\dagger \mathbf{j}(0) V | (1/\sqrt{2})(\bar{\mathcal{P}}\mathcal{P} + \bar{\mathcal{N}}\mathcal{N}) . \tag{51}$$

Assume now, for a moment, that the only flavor operators in $V^\dagger \mathbf{j}(0)V$ were those listed in (50); consider first the part

$$\langle (1/\sqrt{2})(\bar{\mathcal{P}}\mathcal{P} + \bar{\mathcal{N}}\mathcal{N}) | V^\dagger \mathbf{j}(0) V | (1/\sqrt{2})(\bar{\mathcal{P}}\mathcal{P} + \bar{\mathcal{N}}\mathcal{N}) \rangle$$

of (51); the operators (50) connect only $\bar{\mathcal{P}}\mathcal{P}$ with $\bar{\mathcal{P}}\mathcal{P}$ and $\bar{\mathcal{N}}\mathcal{N}$ with $\bar{\mathcal{N}}\mathcal{N}$; no one of the operators (50) has a matrix element between $\bar{\mathcal{P}}\mathcal{P}$ and $\bar{\mathcal{N}}\mathcal{N}$; in the complete expression (51) there is no matrix element connecting $\bar{\mathcal{P}}\mathcal{P} + \bar{\mathcal{N}}\mathcal{N}$ to $\bar{\lambda}\lambda$. But in QCD there exist diagrams that annihilate $\bar{q}q$ $I=0$ states into gluons and then give rise, from these gluons, to other $I=0$ states. Because of the emission of the photon (that can change the isospin by one unit) the above gluonic diagrams can play a role both in transitions between two mesons with $I=0$ and in transitions $I=0$ to $I=1$ (respecting, of course, charge conjugation). Examples of such processes are (we indicate a gluon with g)

$$\phi \to 3g \to \pi^0 + \gamma, \quad \phi \to 2g + \gamma \to \eta + \gamma ,$$

$$\omega \to 3g \to \pi^0 + \gamma, \quad \omega \to 2g + \gamma \to \eta + \gamma .$$

In other words, these annihilation processes are of course contained in the exact transition operator $V^\dagger \mathbf{j}(0)V$ but are not represented by the flavor structures (50). To which flavor structures do they correspond? To answer, let us first introduce the $I=0$ flavor kets:

$$\begin{aligned} |z'\rangle &= \frac{1}{\sqrt{3}}|z\rangle = \frac{1}{\sqrt{3}}|\mathcal{P}_1\bar{\mathcal{P}}_2 + \mathcal{N}_1\bar{\mathcal{N}}_2 + \lambda_1\bar{\lambda}_2\rangle , \\ |w\rangle &= |\lambda_1\bar{\lambda}_2\rangle \end{aligned} \tag{52}$$

[$|z'\rangle$ is normalized to 1 ($\langle z'|z'\rangle = 1$), whereas $|z\rangle$, used in Ref. 1(c), was normalized to 3 ($\langle z|z\rangle = 3$)].

Then to the flavor operators (50) we should add

$$(Q_1+Q_2)|z'\rangle\langle z'| + |z'\rangle\langle z'|(Q_1+Q_2) , \tag{53a}$$

$$(Q_1+Q_2)|z'\rangle\langle w| + |w\rangle\langle z'|(Q_1+Q_2) , \tag{53b}$$

$$|z'\rangle\langle w|(Q_1+Q_2) + (Q_1+Q_2)|w\rangle\langle z'| , \tag{53c}$$

$$(Q_1+Q_2)|w\rangle\langle |w| + |w\rangle\langle w|(Q_1+Q_2) . \tag{53d}$$

Note the following. (a) With the definitions (46), (48) the total charge of the quark 1 and antiquark 2 is $Q_1 - Q_2$ (not $Q_1 + Q_2$); for neutral states, as are those with $I=0$, the total charge is zero; this is why we did not list operators with $Q_1 - Q_2$. (b) It is $|w\rangle\langle w| \equiv \Pi_1^\lambda \cdot \Pi_2^\lambda$ so that the last operator (53d) is already included in (50); thus in the list only the first three operators (53a)–(53c) survive.

One more remark. Because any purely gluonic state is a flavor singlet, one might object that we should keep only operators constructed with $|z\rangle$, not with $|w\rangle$. This is not so; the flavor-breaking term due to the quark mass difference affects also the amplitudes of the gluon annihilation diagrams; thus the operators (53b) and (53c), which are of first order in flavor breaking, can intervene. [We did not include $i(|z\rangle\langle w| - |w\rangle\langle z|)$ in the list because its matrix elements between states with a real-flavor wave function vanish.]

B. Spin structure

Only three spin operators exist:

$$\boldsymbol{\sigma}_1, \quad \boldsymbol{\sigma}_2, \quad \boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2 . \tag{54}$$

However, we will show that [as already stated in Sec. V, Eq. (36)] only the difference of $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$,

$$\mathbf{S} = \tfrac{1}{2}(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) , \tag{36}$$

will enter in Ω_ν (28), if we recall that $\mathbf{\Omega}_\nu$ must then be inserted in (31). To see this use the fact that B_j is a singlet-spin state; therefore,

$$2\mathbf{J}|\chi(B_j)\rangle = (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2|\chi(B_j)\rangle = 0 . \tag{55}$$

As to $\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2$, it can be transformed as follows [in the formulas below $\Gamma(f)$ is any flavor operator in the variables 1 and 2]:

$$\begin{aligned}\langle\chi(B_j)|(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\Gamma(f)|\chi(A_i)\rangle &= \langle\chi(B_j)|2\mathbf{J}\times\boldsymbol{\sigma}_2\Gamma(f)|\chi(A_i)\rangle-\langle\chi(B_j)|\boldsymbol{\sigma}_2\times\boldsymbol{\sigma}_2\Gamma(f)|\chi(A_i)\rangle\\ &=-2i\langle\chi(B_j)|\boldsymbol{\sigma}_2\Gamma(f)|\chi(A_i)\rangle\\ &=-i\langle\chi(B_j)|(\boldsymbol{\sigma}_2-\boldsymbol{\sigma}_1)\Gamma(f)|\chi(A_i)\rangle\ . \end{aligned} \tag{56}$$

Therefore one is led again to $\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2$ and Eq. (36) is justified.

Another remark on the spin operators. The fact that the most general spin operator in (28), $\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2$, is linear in the quark and antiquark does not imply that at some stage of the simplification of the complete field-theoretical calculation we do not encounter expressions of the form

$$\boldsymbol{\sigma}_1(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) \quad\text{or}\quad \boldsymbol{\sigma}_2(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) \tag{57}$$

that contain simultaneously the variables of the quark and of the antiquark. Of course it is

$$\boldsymbol{\sigma}_1(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)=\boldsymbol{\sigma}_2+i(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\ , \tag{58}$$

and the $(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)$ term reduces to $\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2$ [Eq. (56)]; thus we are back to (54). But note that the overall coefficient in the final result multiplying $\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2$ can receive a contribution also from two quark terms such as those on the left of Eq. (58).

C. The spin-flavor operators

We now list the possible $\boldsymbol{\Omega}_\nu$'s. The result is simplified by the fact that $V^\dagger \mathbf{j}(0)V$ must be odd under charge conjugation C; indeed $\mathbf{j}(0)$ is odd; V, being constructed in terms of the strong-interaction Hamiltonian, is invariant under C; in the $1q$-$1\bar{q}$ sector, C amounts to the interchange of 1 and 2.

Because $\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2$ changes sign under C, the only combinations $\Gamma(f)$ of the flavor operators that multiply $(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)$ must be symmetric on exchange of 1 and 2; on the basis of (50) and (57) the complete list of the $\boldsymbol{\Omega}_\nu$'s is thus

$$\boldsymbol{\Omega}_\nu=\tfrac{1}{2}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\Gamma_\nu(f)\equiv\mathbf{S}\Gamma_\nu(f)\quad(\nu=\mathrm{I},\mathrm{II},3,\ldots,7)\ , \tag{59}$$

with

$$\Gamma_{\mathrm{I}}=(Q_1+Q_2)\quad[0]\ , \tag{60a}$$

$$\Gamma_{\mathrm{II}}=(Q_1+Q_2)(\Pi_1^\lambda+\Pi_2^\lambda)\quad[\Delta m/m_\lambda]\ , \tag{60b}$$

$$\Gamma_3=(Q_1-Q_2)(\Pi_2^\lambda-\Pi_1^\lambda)\equiv QS\quad[\Delta m/m_\lambda]\ , \tag{60c}$$

$$\Gamma_4=(Q_1+Q_2)\Pi_1^\lambda\Pi_2^\lambda\quad[(\Delta m/m_\lambda)^2]\ , \tag{60d}$$

$$\Gamma_5=[(Q_1+Q_2)|z'\rangle\langle z'|+|z'\rangle\langle z'|(Q_1+Q_2)]\quad[0]\ , \tag{60e}$$

$$\Gamma_6=[|z'\rangle\langle w|(Q_1+Q_2)+(Q_1+Q_2)|w\rangle\langle z'|]\ [\Delta m/m_\lambda]\ , \tag{60f}$$

$$\Gamma_7=[(Q_1+Q_2)|z'\rangle\langle w|+|w\rangle\langle z'|(Q_1+Q_2)]\ [\Delta m/m_\lambda]\ . \tag{60g}$$

In the last form of (60c), Q stands for total charge and S for strangeness. We have affixed the first two $\boldsymbol{\Omega}_\nu$'s and Γ_ν's in the above formulas by roman indices I and II to avoid confusion with the indices 1 and 2 that in this paper are used to indicate quark and antiquark. In square brackets we noted the order in flavor breaking (0 means flavor symmetric). The expressions $\boldsymbol{\Omega}_{\mathrm{I}}$ to $\boldsymbol{\Omega}_4$ are (or can be) present in the conventional NRQM description, while $\boldsymbol{\Omega}_5$ to $\boldsymbol{\Omega}_7$ correspond to gluonic diagrams; usually they are neglected in the NRQM treatment of the $V\to P\gamma$ decays.

Note, incidentally, that it is possible to display the $\boldsymbol{\Omega}_\nu$'s in an additive form similar to that of the NRQM. Indeed $\frac{1}{2}(Q_1+Q_2)(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)$ can be rewritten, when dealing with *its matrix elements between a spin-1 and a spin-0 state*, as

$$\begin{aligned}\tfrac{1}{2}\langle\chi(B_j)|(Q_1+Q_2)(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)|\chi(A_i)\rangle &=\tfrac{1}{2}\langle\chi(B_j)|Q_1\boldsymbol{\sigma}_1-Q_2\boldsymbol{\sigma}_2-Q_1\boldsymbol{\sigma}_2+Q_2\boldsymbol{\sigma}_1|\chi(A_i)\rangle\\ &=\langle\chi(B_j)|(Q_1\boldsymbol{\sigma}_1-Q_2\boldsymbol{\sigma}_2)|\chi(A_i)\rangle\ ,\end{aligned} \tag{61}$$

where in the last step we used $\langle\chi(B_j)|\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2|\chi(A_i)\rangle=0$.

With the seven $\boldsymbol{\Omega}_\nu$'s above (60a)–(60g) the calculation of any transition reduces to that of (39),

$$\boldsymbol{\mathcal{M}}(B_jA_i)=\mathbf{e}\sum_\nu g_\nu(P)\Gamma_\nu(B_jA_i)\ ,$$

to be inserted in (32); before doing this we write

$$g_\nu(P)=M^{1/2}\mu_\nu f_\nu(P)\quad(\nu=\mathrm{I},\mathrm{II},3,\ldots,7)\ . \tag{62}$$

Here M is a mass, to be fixed in a moment, the μ_ν's are the dimensions of a magnetic moment, and the $f_\nu(P)$'s are adimensional and normalized so that

$$f_\nu(0)=1\ . \tag{63}$$

The $f_\nu(P)$'s can be interpreted as form factors; although we work with a relativistic theory, they depend only on P and have thus a nonrelativistic structure, typical of the NRQM.

Changing M in (62), we change the scale of the magnetic moments μ_ν; we choose M to reproduce, as closely as possible, the results of the NRQM: $M=2\overline{M}_V$, where $\overline{M}_V$ is the average mass of the vector mesons. Thus $\boldsymbol{\mathcal{M}}(B_jA_i)$ becomes

$$\boldsymbol{\mathcal{M}}(B_jA_i)=(2\overline{M}_V)^{1/2}\mathbf{e}\sum_{\nu=I}^{7}\mu_\nu f_\nu(P)\Gamma_\nu(B_jA_i)\ . \tag{64}$$

Equivalently from (45) we get

$$G_{ij}(M_i^2,M_j^2)=2(\overline{M}_V/M_i)^{1/2}\sum_{\nu=I}^{7}\mu_\nu f_\nu(P)\Gamma_\nu(B_jA_i)\ . \tag{65}$$

If we replace $\overline{M}_V/M_i$ in (65) by one, and keep only the terms (60a) and (60b) ($\mu_{\rm I}$ and $\mu_{\rm II}$), Eq. (65) becomes identical to that of the NRQM treatment[7] of the $V\to P\gamma$ decays (then $\mu_{\rm I}$ was taken to be 2.79 proton magnetons, from fitting the proton and neutron magnetic moments).

VIII. SMALLNESS OF THE GLUONIC CONTRIBUTIONS: THE $(\omega\to\pi\gamma)/(\rho\to\pi\gamma)$ AND $(\eta'\to\rho\gamma)/(\eta'\to\omega\gamma)$ RATIOS

We recall two facts on the $\omega\to\pi\gamma$, $\rho\to\pi\gamma$ decays.

(a) Nonviolated SU_3 (flavor) implies that $\rho\to\pi\gamma$ is a pure octet→octet amplitude and $\omega\to\pi\gamma$ a combination of an octet→octet and octet→singlet amplitudes. Thus SU_3 (flavor) invariance alone does not relate the two amplitudes $\mathcal{M}(\omega\pi),\mathcal{M}(\rho\pi)$.

(b) The NRQM omits the gluonic part of the transition; it predicts $\Gamma(\omega\to\pi\gamma)/\Gamma(\rho\to\pi\gamma)=9$ (aside from the slight difference in momenta).

What does the present parametrization say? Only $\mathbf{\Omega}_{\rm I}$ and $\mathbf{\Omega}_5$ contribute to the $\omega\to\pi\gamma$, $\rho\to\pi\gamma$ decays; thus it is

$$\mathcal{M}(\omega,\pi)=e(2\bar{M}_V)^{1/2}[\mu_{\rm I}f_{\rm I}(P)+\tfrac{2}{3}\mu_5 f_5(P)] \;, \tag{66}$$

$$\mathcal{M}(\rho,\pi)=e(2\bar{M}_V)^{1/2}(1/3)\mu_{\rm I}f_{\rm I}(P) \;. \tag{67}$$

To write (66) and (67) we used for ω the ideal θ_V, that is, we wrote its flavor state as $(1/\sqrt{2})(\bar{\mathcal{P}}\mathcal{P}+\bar{\mathcal{N}}\mathcal{N})$. The main interest of Eqs. (66) and (67) lies in how large is the deviation from 9 of the ratio between the rates of $\Gamma(\omega\to\pi\gamma)$ and $\Gamma(\rho\to\pi\gamma)$; with a factor 1.06 for the difference of momentum in the two cases (when the data will improve this should be recalculated because ρ is very wide) it is

$$\frac{\Gamma(\omega\to\pi\gamma)}{\Gamma(\rho\to\pi\gamma)}=\left[3+\frac{2\mu_5 f_5(P)}{\mu_{\rm I}f_{\rm I}(P)}\right]^2\times 1.06 \;. \tag{68}$$

Any deviation from 9.5 of the right-hand side of (68) is a measure of the gluonic annihilation contribution to $\omega\to\pi\gamma$. With the present data,[8]

$$[\Gamma(\omega\to\pi\gamma)/\Gamma(\rho\to\pi\gamma)]_{\rm expt}=9.9\pm1.6 \;, \tag{69}$$

the gluonic contribution to the above ratio [the term $\mu_5 f_5$ in (68)] stays inside the error; one would like to know if QCD explains this smallness. Note that (68) is an exact consequence of any relativistic field theory that satisfies the assumptions stated in Sec. I; it is correct, in particular, to all orders in flavor breaking.

The smallness of the gluonic effects appears also in the ratio of the $\eta'\to\rho\gamma$ and $\eta'\to\omega\gamma$ decays (these are $P\to V\gamma$ instead of $V\to P\gamma$ but, as we shall see, the above treatment applies also to them); from the third and fourth rows of Table I (Appendix), these decays are seen to depend on $\mathbf{\Omega}_{\rm I}$ and $\mathbf{\Omega}_5$, $\mathbf{\Omega}_6$, $\mathbf{\Omega}_7$; the ratio $\Gamma(\eta'\to\rho\gamma)/\Gamma(\eta'\to\omega\gamma)$ is experimentally $(30\pm1.6)/(2.7\pm0.5)\cong 11.1\pm2$. The error in the denominator is still large but again the ratio is compatible with the value $9\times1.22\cong 11$ obtained neglecting the gluonic contributions (1.22 is the analogue of 1.06 above; it comes from the ratio of the third powers of the momenta).

One final remark. Our original NRQM calculation[3] of the $\omega\to\pi\gamma$ predicted also the width $\Gamma(\omega\to\pi\gamma)$ itself [not just the ratio $\Gamma(\omega\to\pi\gamma)/\Gamma(\rho\to\pi\gamma)$]. As stated at the end of the preceding section, that calculation used for $\mu_{\rm I}$ in (67) the value that fits the proton; certainly $\mu_{\rm I}$ has that order of magnitude, but at present we cannot say more because we do not have enough information on the momentum dependence of $f_{\rm I}(P)$ (that is, we ignore to the required precision the "radius" of this transition form factor).

IX. THE $\phi\to\pi^0\gamma$ BRANCHING RATIO

A confirmation of the smallness of the gluonic annihilation diagrams comes from the $\phi\to\pi^0\gamma$ decay. It is known, indeed, that the order of magnitude of this decay can be reproduced by the small deviation of θ_V with respect to its ideal value [compare (22) and (21)]. The theoretical uncertainty (particularly from the form factor and also from θ_V^*) is comparatively large; still, because a

TABLE I. The values of $\Gamma_\nu(B_j A_i)$ appearing in Eq. (65). The abbreviations used are indicated at the bottom of the table; the flavor wave function of each meson is assumed to be normalized to one; θ_V is taken to have its ideal value, except in the calculation of $\Gamma_{\rm I}$ for $\phi\to\pi^0\gamma$. The example given in the Appendix, in Eq. (A7), illustrates the use of the table.

	$\Gamma_{\rm I}$	$\Gamma_{\rm II}$	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7
$\rho\pi\gamma$	1/3	0	0	0	0	0	0
$\omega\pi\gamma$	1	0	0	0	2/3	0	0
$\rho\eta\gamma$	K	0	0	0	$-s\sqrt{2/3}$	0	$-H\sqrt{2/3}$
$\omega\eta\gamma$	$K/3$	0	0	0	$L\sqrt{2/9}$	$H\sqrt{8/27}$	$-H\sqrt{2/27}$
$\rho\eta'\gamma$	H	0	0	0	$c\sqrt{2/3}$	0	$K\sqrt{2/3}$
$\omega\eta'\gamma$	$H/3$	0	0	0	$N\sqrt{2/9}$	$-K\sqrt{8/27}$	$K\sqrt{2/27}$
$\phi\eta\gamma$	$2H/3$	$4H/3$	0	$2H/3$	$N\sqrt{2/9}$	R	W
$\phi\eta'\gamma$	$-2K/3$	$-4K/3$	0	$-2K/3$	$-L\sqrt{2/9}$	Z	T
$\phi\pi^0\gamma$	$\sin(\theta_V^*-\theta_V)$	0	0	0	$\sqrt{2/9}$	0	$\sqrt{2/3}$
$K^{*0}K^0\gamma$	$-2/3$	$-2/3$	0	0	0	0	0
$K^{*+}K^+\gamma$	1/3	1/3	1	0	0	0	0

$s=\sin\theta_P$; $c=\cos\theta_P$; $K=1/\sqrt{3}(c-s\sqrt{2})$; $H=1/\sqrt{3}(s+c\sqrt{2})$; $N=1/\sqrt{3}(c+s\sqrt{2})$;
$L=1/\sqrt{3}(c\sqrt{2}-s)$; $R=\frac{2}{9}(4s+c\sqrt{2})$; $T=\sqrt{2}/9(5s-c\sqrt{2})$; $W=\sqrt{2}/9(+5c+s\sqrt{2})$; $Z=\frac{2}{9}(-4c+s\sqrt{2})$.

sensible order of magnitude of the rate is obtained[9] without invoking the gluon annihilation diagrams, it can be at least asserted that there is no evidence, inside the errors, for the additional (three-gluon annihilation) terms appearing in the ninth row of Table I that, *a priori,* might be important in this case. It would be of interest to have a QCD explanation of why the three-gluon annihilation diagrams are so small at these low Q's.

X. THE $K^{*0} \to K^0\gamma$ AND $K^{*+} \to K^+\gamma$ DECAYS

According to the NRQM the ratio between the rates $K^{*0} \to K^0\gamma$ and $K^{*+} \to K^+\gamma$ is a function only of the ratio $x \equiv \mu_\lambda/\mu_\mathcal{N}$ between the magnetic moments of the λ quark and $\mathcal{N}$ quark inside the above mesons; at present the experiment gives $x_{\text{expt}} = 0.80 \pm 0.09$. It might be of interest to reduce the error to compare the above x_{expt} with that (0.65 ± 0.02) obtained from the baryons. But first one should know if the NRQM formula giving x continues to be true in the exact parametrization. The NRQM formula for the ratio of the decay amplitudes (the widths Γ are proportional to $|A|^2$) is

$$\frac{A(K^{*0} \to K^0\gamma)}{A(K^{*+} \to K^+\gamma)} = -\frac{1+x}{2-x} . \tag{70}$$

Among the $\boldsymbol{\Omega}_\nu$'s listed in (59) and (60a)–(60e) only $\boldsymbol{\Omega}_{\text{I}}, \boldsymbol{\Omega}_{\text{II}}, \boldsymbol{\Omega}_3$ enter in the decays of K^{*0} and K^{*+} ($\boldsymbol{\Omega}_3$ does not contribute to K^{*0}). Putting

$$D = \mu_{\text{I}} f_{\text{I}}(P) + \mu_{\text{II}} f_{\text{II}}(P), \quad C = \mu_3 f_3(P) ,$$

one finds easily

$$\frac{A(K^{*0} \to K^0\gamma)}{A(K^{*+} \to K^+\gamma)} = \frac{-2D}{D+3C} \tag{71}$$

that corresponds to (70) with

$$x = \frac{1-(C/D)}{1+(C/D)} . \tag{72}$$

Neglecting terms of second order in flavor breaking (note that μ_3/μ_{I} and $\mu_{\text{II}}/\mu_{\text{I}}$ are both of first order in flavor breaking),

$$x \cong \frac{1-(\mu_3 f_3/\mu_{\text{I}} f_{\text{I}})}{1+(\mu_3 f_3/\mu_{\text{I}} f_{\text{I}})} \cong 1 - 2\frac{\mu_3 f_3}{\mu_{\text{I}} f_{\text{I}}} , \tag{73}$$

or $\mu_3 f_3/\mu_{\text{I}} f_{\text{I}} \simeq (1-x)/2$. If the P dependence of f_3 and f_{I} is the same, this simplifies into

$$\mu_3/\mu_{\text{I}} \cong (1-x)/2 . \tag{74}$$

The Eqs. (73) or (74) show that the x of the NRQM has a meaning more general than one might have thought; Eq. (73) is true independently of the NRQM, provided that we neglect terms of second order in flavor breaking. There is, however, a difference between these general results and those of the NRQM: the NRQM is formulated in terms only of two parameters: the magnetic moments of the nonstrange and λ quarks; here, even omitting the gluonic diagrams and neglecting second-order flavor breaking, there are three parameters: μ_{I}, μ_{II}, and μ_3; it is easy to see that, for a λ quark belonging to a meson of zero strangeness, (64) leads to a ratio $x' = 1 + 2\mu_{\text{II}}/\mu_{\text{I}}$ between the magnetic moments of the λ and $\mathcal{N}$ quark, whereas, as we just saw [Eq. (74)], the ratio $\mu_\lambda/\mu_\mathcal{N}$ for a λ belonging to a strange meson is $x = 1 - (2\mu_3/\mu_{\text{I}})$; x' and x are equal only if

$$\mu_{\text{II}} \cong -\mu_3 . \tag{75}$$

The present data do not allow us to say how nearly the above equality (75) is satisfied.

XI. CONCLUSION: THE COMPLETE PARAMETRIZATION AND THE NRQM

If the gluonic amplitudes $\boldsymbol{\Omega}_5$ to $\boldsymbol{\Omega}_7$ are negligible, all the zero strangeness decays, except $\phi\eta\gamma$ and $\phi\eta'\gamma$, are governed (Table I) only by one parameter (one magnetic moment μ_{I}), the same in all cases, exactly as in the NRQM; that parameter has the order of magnitude assumed for it in the original calculation with the NRQM; $\phi\eta\gamma$ and $\phi\eta'\gamma$ are governed (excluding second-order flavor breaking) by two parameters μ_{I} and μ_{II} again as in the NRQM (although now there can be a difference between $\mu_\lambda/\mu_\mathcal{N}$ deduced from transitions in the strangeness zero sector or in the strangeness ± 1 sector). One sees now the reason why the NRQM treatment is essentially correct; the reason is that, if the gluonic terms are negligible, the form of the exact parametrization and its number of parameters are very near to those of the NRQM (independently of the quark velocity inside the mesons); we found in particular that the exact formula for the ratio between the rates $K^{*0} \to K^0\gamma$ and $K^{*+} \to K^+\gamma$ is identical (barring terms of second order in flavor breaking) to the formula given by the NRQM; in this case there are no gluonic terms.

We conclude with three remarks.

(1) We hope that this analysis may stimulate precision measurements of the $V \to P\gamma$ decays, especially those discussed in Secs. VIII, IX, and X; but also for the other $V \to P\gamma$ decays the discussion on θ_P in Ref. 1(c) and recent results[10] on θ_P add perhaps interest to improvements in the data.

(2) The results of Ref. 1(a) (magnetic moments, electromagnetic transition amplitudes, and masses of the baryons), of Ref. 1(b) (semileptonic baryon decays), and of Ref. 1(c) (meson masses), together with those of this paper, clarify why the NRQM can be fairly successful quantitatively (even if the internal velocity of the quarks is not $\ll 1$). As a matter of fact the quantitative successes of the NRQM in the above problems have been rather mysterious for many years. One now sees that the merit of the NRQM is to provide a parametrization in terms of a few parameters; this parametrization is either completely equivalent to the exact one (charged-meson masses and—neglecting second-order flavor breaking—also neutral-meson masses), or, if not, selects the dominant terms among those of the exact parametrization. There remains an open question that probably must wait for a better understanding of QCD (or whatever field theory is the basic one): Why, in the basic parametrization, are the terms that the NRQM does not include (e.g., in the present paper, the gluon terms) in fact really negli-

gible? Why do the selected terms constitute such a good approximation?

(3) This paper (together with those of Ref. 1) illustrates, we hope, the meaning of the NRQM and how it provides a sensible parametrization. Clearly one may ask if other models of hadronic structure, for instance, bag models of various kinds, or "relativistic" quark models (either two-body Dirac equations or some *ad hoc* prescription), can do a similar job, namely, parametrize the results of the basic field theory in a sensible way. We hope to discuss this in the future. Our present view is as follows: Once agreed that the model state is in all cases (in the NRQM, in a "relativistic quark model," or in a bag model) rather far from the exact state (simply because it is just one term of the Fock expansion), there does not seem to be much advantage (and in fact there are several disadvantages) in taking as a model state for the parametrization a "relativistic" state or, say, a bag state instead of the most naive NRQM model state; indeed such states imply, with respect to the NRQM state, an unnecessary increase in the number of parameters, and therefore a decrease in predictive power; we have stressed at length in Ref. 1(a) that the basic advantage of the NRQM is that of having very simple model states *endowed with the maximum symmetry compatible with the quantum numbers of the system that we describe.* It is this feature that reduces the number of intervening parameters, a feature that is lost in the above-mentioned "relativistic" models, unless arbitrary assumptions are made.[11] But, as already stated, we hope to come back to these points in the future.

APPENDIX

Some details of the derivation of the Eq. (65)

The terms in Eq. (40) that contain the destruction of a V meson A_i at rest polarized up and the creation of a P meson B_j are

$$G_{ij}P[(\partial_2 A_3-\partial_3 A_2)\partial_4 V_1\epsilon_{2341}+(\partial_2 A_3-\partial_3 A_2)\partial_4 V_2\epsilon_{1342}]$$
$$=G_{ij}P(H_x\partial_4 V_1+H_y\partial_4 V_2)$$
$$=\tfrac{1}{2}G_{ij}P(H_x+iH_y)\partial_4(V_x-iV_y)+\tfrac{1}{2}G_{ij}P(H_x-iH_y)\partial_4(V_x+iV_y)\ . \tag{A1}$$

Only the term with V_x-iV_y intervenes in the destruction of a V polarized up [polarization state $(V_x-iV_y)/\sqrt{2}$]; aside from a phase and leaving out the factor $(2\pi)^4\delta^{(3)}(\mathbf{P}+\mathbf{k})\delta(M_i-k-E_j(P))$, the matrix element M'_{ji} in Eq. (42) for $A_j(0,\uparrow)\to B_j(P)\gamma$ is

$$G_{ij}(\boldsymbol{\epsilon}\times\mathbf{k})_{x+iy}\frac{1}{\sqrt{2}}\frac{M_i}{\sqrt{2M_i}}\frac{1}{\sqrt{2E_j(P)}}\frac{1}{\sqrt{2k}}\ . \tag{A2}$$

Equation (A2) coincides with (42), (43). The same matrix element calculated from (32) and (64) between a V spin state $=\alpha_1\alpha_2$ and a P spin state $=(1/\sqrt{2})(\alpha_1\beta_2-\alpha_2\beta_1)$ using the expression (36) for $\mathbf{S}$ and (37a) of $\mathbf{e}$ is

$$(\overline{M}_V)^{1/2}\left[\sum_{\nu=I}^{7}\mu_\nu f_\nu(P)\Gamma_\nu(B_j A_i)\right](\boldsymbol{\epsilon}\times\mathbf{k})_{x+iy}\frac{1}{\sqrt{2E_j(P)}}\frac{1}{\sqrt{2k}}\ . \tag{A3}$$

Equating (A2) and (A3) we obtain Eq. (65) of the text.

The formulas for the rates and the calculation of $\Gamma_\nu(B_j A_i)$ in Eq. (65)

In terms of the G_{ij}'s appearing in (40) and parametrized in Eq. (65) the rate of a $V\to P\gamma$ and of a $P\to V\gamma$ decay are

$$(V\to P\gamma)\quad \Gamma(A_i\to B_j\gamma)=G_{ij}^2k^3/(12\pi)\ , \tag{A4}$$

$$(P\to V\gamma)\quad \Gamma(B_j\to A_i\gamma)=G_{ij}^2k^3/(4\pi)\ . \tag{A5}$$

For the $P\to V\gamma$ decays (in practice $\phi\to\omega\gamma$ and $\phi\to\rho\gamma$) the following remark is appropriate: In deriving the parametrization in the text we assumed that the V meson is at rest; in a $P\to V\gamma$ decay it is not. However, because the vertex (40) is Lorentz invariant one can, in a $P\to V\gamma$ decay, calculate the matrix element of (40) in the reference frame in which the final V is at rest, thus using the results in the text; this matrix element is invariant for a transformation to the frame where P is at rest; of course to have it in that frame one must duly transform the kinematic quantities (including the polarizations of V and γ); but it is unnecessary to perform in practice this transformation of polarizations because we sum over the polarizations in calculating the decay rate. In conclusion we can insert G_{ij} parametrized according to (65) also in Eq. (A5) [of course i and j must be interchanged in P, Eq. (14)].

To derive the formulas given previously for some decays and in view of other applications of the parametrization (65) we gave in Table I the values of $\Gamma_\nu(B_j A_i)$ appearing in (65) [defined by Eq. (37) with the Γ_ν's listed in (60)] for the various transitions; the flavor wave functions $\xi(B_j)$, $\xi(A_i)$ in (37) are, of course, assumed to be normalized to one. For the vector angle θ_V we took the ideal value (21) except in Γ_I for $\phi\to\pi^0\gamma$ decay. To simplify Table I we used the abbreviations $c=\cos\theta_P$, $s=\sin\theta_P$

plus the other listed at the bottom of the table.

To calculate, for instance, the $\omega \to \eta\gamma$ rate compute, K, L, H in terms of $\cos\theta_P$ and $\sin\theta_P$ and use Eq. (65). Neglecting the gluon annihilation the result is

$$G_{ij} = G_{\omega\eta} = 2(\overline{M}_V / M_\omega)^{1/2} \mu_1 f_1(P) K/3 \; . \tag{A6}$$

Equation (A6) reproduces the NRQM. Keeping the gluon diagrams, we have

$$\begin{aligned} G_{\omega\eta} = 2(\overline{M}_V/M_\omega)^{1/2}[&\mu_1 f_1(P)K/3 + \mu_5 f_5(P) L\sqrt{2/9} \\ &+ \mu_6 f_6(P) H\sqrt{8/27} \\ &- \mu_7 f_7(P) H \sqrt{2/27}] \; . \end{aligned} \tag{A7}$$

The above $G_{\omega\eta}$'s must be inserted in (A4) to obtain the rate $\omega \to \eta\gamma$.

Some additional remarks on the closed algebra of Sec. VI

The masses of the mesons of the P or V nonets can be written [Ref. 1(c)] in spin-flavor space as the eigenvalues of an operator of the form

$$\begin{aligned} M = A &+ B\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + C(\Pi_1^\lambda + \Pi_2^\lambda) + D\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2(\Pi_1^\lambda+\Pi_2^\lambda) \\ &+ (E + F\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)|z\rangle\langle z| \\ &+ (H + G\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(|z\rangle\langle w| + |w\rangle\langle z|) \\ &+ (N + T\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\Pi_1^\lambda\Pi_2^\lambda \; . \end{aligned} \tag{A8}$$

It is clear (recall that $\Pi_1^\lambda \Pi_2^\lambda \equiv |w\rangle\langle w|$) that any power of M, and therefore any function of M, has the same structure as (A8); in other words it is again a combination with different coefficients of the same spin-flavor operators (called in the text the set SF) appearing in (A8). Now multiply any one of the spin-flavor operators in (A8) by one of the operators $\boldsymbol{\Omega}_\mu$ that all have the structure $(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\Gamma_\nu$ where the Γ_ν's are listed in Eq. (60). It is easy to verify that, taking into account Eqs. (58) and (56), one reproduces (insofar as the calculation of a $V \to P\gamma$ transition matrix element is concerned) the same set of operators $\boldsymbol{\Omega}_\mu$. This is the "closed algebra" argument sketched in the text.

An independent consistency check of Eq. (45)

An independent consistency check of Eq. (45) comes from the following argument: By the same procedure of Sec. IV we might have parametrized, instead of (23), the matrix element

$$\langle B_j(\mathbf{P})|H^n \mathbf{j}(0) H^k | A_i(0)\rangle = \langle \phi_{B_j}(\mathbf{P})| V^\dagger H^n \mathbf{j}(0) H^k V |\phi_{A_i}(0)\rangle \; , \tag{A9}$$

where H is the exact Hamiltonian and n, k two arbitrary integers; because the transformation properties of $H^n \mathbf{j}(0) H^k$ are the same as those of $\mathbf{j}(0)$, the same $\boldsymbol{\Omega}_\nu$'s appearing in Eq. (30) appear also in the equation analogous to (30) written for (A9); calling $\hat{\mathcal{F}}(\mathbf{P})$ the analogue of $\mathcal{F}(\mathbf{P})$ in the parameterization of the right-hand side of (A9), the analogue of (30) for (A9) is

$$\hat{\mathcal{F}}(\mathbf{P}) = \sum_\nu g_\nu^{(n,k)}(P) \boldsymbol{\Omega}_\nu \times \mathbf{k} \; , \tag{A10}$$

where now $g_\nu^{(n,k)}(P)$ is the coefficient replacing $g_\nu(P)$ in (30). Because it is $H|A_i(0)\rangle = M_i |A_i(0)\rangle$ and $H|B_j(P)\rangle = E_j(P)|B_j(P)\rangle$ the left-hand side of (A9) is

$$E_j^n(P) M_i^k \langle B_j(\mathbf{P})|\mathbf{j}(0)|A_i(0)\rangle \equiv [(M_i^2 + M_j^2)/(2M_i)]^n M_i^k \sum_\nu g_\nu(P) \boldsymbol{\Omega}_\nu(B_j, A_i) \times \mathbf{k}$$

having used the expression (14) of P; clearly the following identity must be true:

$$[(M_i^2+M_j^2)/(2M_i)]^n M_i^k \sum_\nu g_\nu(P)\boldsymbol{\Omega}_\nu(B_j,A_i) = \sum_\nu g_\nu^{(n,k)}(P)\boldsymbol{\Omega}_\nu(B_j,A_i) \; . \tag{A11}$$

Multiply both members of (A11) by a set of arbitrary coefficients $a_{n,k}$ and sum over n, k. We get

$$\sum_{n,k} a_{n,k}[(M_i^2+M_j^2)/(2M_i)]^n M_i^k \sum_\nu g_\nu(P)\boldsymbol{\Omega}_\nu(B_j,A_i) = \sum_\nu \sum_{n,k} a_{n,k} g_\nu^{(n,k)}(P)\boldsymbol{\Omega}_\nu(B_j,A_i) \; . \tag{A12}$$

Because the $a_{n,k}$'s are arbitrary, the sum over n, k on the left-hand side of (40) is an arbitrary function Y of M_j and M_i:

$$Y(M_j, M_i) \equiv \sum_{n,k} a_{n,k}[(M_i^2+M_j^2)/(2M_i)]^n M_i^k \; .$$

On the other hand $\sum_{n,k} a_{n,k} g_\nu^{(n,k)}(P)$ is some function of P, call it $W_\nu(P)$; Eq. (A12) can thus be rewritten more transparently as

$$\sum_\nu g_\nu(P)\boldsymbol{\Omega}_\nu(B_j,A_i) = Y^{-1}(M_j,M_i) \cdot \sum_\mu W_\mu(P)\boldsymbol{\Omega}_\mu(B_j,A_i) \; . \tag{A13}$$

Equation (A13) confirms our previous argument (based on the closed algebra) that there is no inconsistency in an equation of the form (45).

[1]G. Morpurgo, (a) Phys. Rev. D **40**, 2997 (1989); (b) **40**, 3111 (1989); (c) **41**, 2865 (1990). In (a) the method is formulated and applied to the magnetic moments, electromagnetic transition matrix elements, and masses of the baryons; in (b) it is applied to the semileptonic baryon decays, and in (c) to the meson masses.

[2]G. Morpurgo, (a) Phys. (N.Y.) **2**, 95 (1965) [also reproduced in J. J. Kokkedee, *The Quark Model* (Benjamin, New York, 1969), p. 132]; (b) in *Theory and Phenomenology in Particle Physics,* proceedings of the International School of Physics "Ettore Majorana," Erice, Italy, 1968, edited by A. Zichichi (Academic, New York, 1969), pp. 83–217 [the right-hand side of Eq. (5.2b) of this reference must be multiplied by 2; the subsequent equations are correct]; (c) Annu. Rev. Nucl. Sci. **20**, 105 (1970).

[3]C. Becchi and G. Morpurgo, Phys. Rev. **140B**, 687 (1965).

[4]P. J. O'Donnell, Rev. Mod. Phys. **53**, 673 (1981), and references quoted therein.

[5]D. P. Stanley and D. Robson, Phys. Rev. D **21**, 3180 (1980); S. Godfrey and N. Isgur, *ibid.* **32**, 189 (1985), and references quoted therein.

[6]In Ref. 1(a) the projection operators called here $\Pi^{\mathcal{P}}$, $\Pi^{\mathcal{N}}$, Π^{λ} were indicated with $P^{\mathcal{P}}$, $P^{\mathcal{N}}$, P^{λ}; the quantity Q (46) was called P^q (we changed notation to avoid an excessive proliferation in this paper of the symbol P).

[7]Compare the NRQM Eq. (34) of Ref. 2(c); there f is the same quantity called here G_{ij}. For $\omega \to \pi\gamma$ Eq. (65) of this paper leads to $G_{ij}=2(\overline{M}_V/M_\omega)^{1/2}\mu_I f_I(P)$. Putting $f_I(P)=1$, $\overline{M}_V=M_\omega$, and $\mu_I=\mu_P$ Eq. (65) reproduces Eq. (34) of Ref. 2(c).

[8]The experimental data are from the Particle Data Group, G. P. Yost *et al.*, Phys. Lett. B **204**, 1 (1988).

[9]N. Isgur, Phys. Rev. Lett. **36**, 1262 (1976).

[10]N. A. Roe *et al.*, Phys. Rev. D **41**, 17 (1990) ($\theta_P=-19.8°\pm2.2°$); J. Jousset *et al.*, *ibid.* **41**, 1389 (1990) ($\theta_P=-19.1\pm1.4°$); also F. J. Gilman and R. Kauffman, *ibid.* **36**, 2761 (1987) ($\theta_P\cong-20°$). These data and their analysis point to a θ_P rather near the linear value ($-23°+$ terms of second order in flavor breaking) as indicated in Ref. 1(c). An older analysis pointing to a linear θ_P was G. Morpurgo, in *Properties of the Fundamental Interactions,* proceedings of the International School of Physics "Ettore Majorana," Erice, Italy, 1971, edited by A. Zichichi (Editrice Compositori, Bologna, 1973), pp. 432–477.

[11]As to the parton model, this is a totally different matter. The starting point of the parton model is, essentially, an approximation to the complete Fock expansion of the true, exact, states (those that, in our notation, are $V|\phi\rangle$) in terms of quarks with bare (that is, unrenormalized) mass; needless to say the parton model parametrizes correctly with a minimum number of parameters the high-momentum-transfer phenomena to which it is addressed.