Chiral-symmetry breaking in the Schwinger model with Wilson fermions

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We develop a new analytic approach to the Schwinger model in the Hamiltonian lattice gauge theory with Wilson fermions. The vacuum structure is examined by means of a unitary transformation and the variational method. The chiral order parameter $\langle \bar{\psi}\psi \rangle$ is calculated for any coupling constant. Chiral symmetry is shown to be broken in the massless limit and good scaling behavior is observed. Our result is consistent with the exactly calculable value.

I. INTRODUCTION

The Schwinger model[1] describes $(1+1)$-dimensional QED which is superrenormalizable and exactly solvable. Previous investigations of the model[2] revealed some important properties of QCD such as confinement, chiral-symmetry breaking, and the U(1) problem.

Lattice techniques offer the possibility of evaluating physical quantities nonperturbatively from first principles. As a test of more realistic theories, the Schwinger model on the lattice has been formulated and studied extensively using Monte Carlo simulations and analytic methods.[3-14] It is necessary to explore analytic methods in order to get wave functions of the vacuum and excited states which allow us to understand the quantitative pictures of the theory in the lattice Hamiltonian formalism. Among the early analytic methods, finite-lattice techniques[5,10,15-17] have been the most successful ones, giving many encouraging results such as continuum string tension and the chiral order parameter. It remains to be seen whether the finite-lattice techniques will prove effective for gauge theories in higher dimensions.

Recently, we developed a different analytic approach of treating naive lattice fermions[18] which consists of a unitary transformation and the variational method in the Hamiltonian formalism. The fermion condensate $\langle \bar{\psi}\psi \rangle$ was calculated for any coupling constant and fermion mass. A nice scaling behavior of $\langle \bar{\psi}\psi \rangle /g=\text{const}$ was observed. However, the value (-0.35 ± 0.05) for $\langle \bar{\psi}\psi \rangle /g$ is larger than the exactly calculable one (-0.16). Because the species of naive fermions are doubled, the extra fermions would probably give a nonzero contribution to the chiral order parameter.

Three major methods have been proposed to get rid of the extra fermions. They are the Wilson formalism, Kogut-Susskind formalism, and SLAC formalism.

In this paper, we develop our approach further and solve the species doubling problem by adopting the Wilson method. The reasons that we prefer to use Wilson fermions are that the flavor and spin quantum numbers are well defined, and $\pi^0\to 2\gamma$ can be well explained. However, because of the Wilson term, chiral symmetry is explicitly broken and the order parameter receives a nonvanishing contribution even in the weak-coupling limit, which should be subtracted before comparing with the continuum value. After subtraction, a nice scaling behavior is observed and the result is close to the exactly calculable one.

This paper is organized as follows. In Sec. II, we review the unitary transformation and variational method, and present our previous results. In Sec. III, we discuss chiral-symmetry breaking using the Hamiltonian with free Wilson fermions. The fermion condensate $\langle \bar{\psi}\psi \rangle$ in the Schwinger model with Wilson fermions is calculated in Sec. IV, and conclusions are summarized in Sec. V.

II. UNITARY TRANSFORMATION AND VARIATIONAL METHOD

We review the unitary transformation and variational method before applying them to the Schwinger model with Wilson fermions. The usual lattice Hamiltonian in $1+1$ dimensions with naive fermions is[18]

$$H=\frac{g^2}{2a}\sum_{x,j} E_j^2(x)+\frac{1}{2a}\sum_{x,k}\bar{\psi}(x)\sigma_k U(x,k)\psi(x+k) +m\sum_x \bar{\psi}(x)\psi(x) , \tag{2.1}$$

where $U(x,k)$ are the U(1) gauge link variables at sites x in the directions k, σ_k are the Pauli matrices with $\sigma_{-k}=-\sigma_k$, $k=\pm1$, $j=1$, a is the lattice spacing, and g is the bare coupling constant related to the charge by $g=ea$. H becomes the usual Schwinger Hamiltonian in the continuum limit. Introducing the two-component spinors $\psi(x)$ with

$$\psi(x)=\begin{bmatrix}\xi(x)\\ \eta^\dagger(x)\end{bmatrix} , \tag{2.2}$$

the bare vacuum is determined by

$$\xi(x)|0\rangle=\eta(x)|0\rangle=E_j^2(x)|0\rangle=0 . \tag{2.3}$$

The Hamiltonian is partly diagonalized by the unitary transformation

$$H'=\exp(-i\theta_1 S_{f_1})H\exp(i\theta_1 S_{f_1}) , \tag{2.4}$$

where θ_1 is the variational parameter and

$$S_{f_1}=\frac{i}{\sqrt{2}}\sum_{x,k}\psi^\dagger(x)\sigma_k U(x,k)\psi(x+k) . \tag{2.5}$$

The physical vacuum state is assumed to be

$$|\Omega\rangle=\exp(i\theta_1 S_{f_1})|0\rangle , \tag{2.6}$$

corresponding to a "filled Dirac sea" on the lattice. The vacuum energy is given by

$$E_\Omega = \frac{\langle \Omega | H | \Omega \rangle}{\langle \Omega | \Omega \rangle} = \langle 0 | H' | 0 \rangle \ . \tag{2.7}$$

After successive commutation with S_{f_1}, Eq. (2.7) becomes

$$\begin{aligned}\mathcal{E}_\Omega &= \frac{aE_\Omega}{N_l} \\ &= -maJ_0(2\sqrt{2}\theta_1) - J_1(2\sqrt{2}\theta_1) \\ &\quad + \frac{g^2}{2}\int_0^{2\sqrt{2}\theta_1} dx_2 \int_0^{x_2} dx_1 J_0^2(x_1) \\ &\quad + \frac{g^2}{128}\int_0^{4\sqrt{2}\theta_1} dx_2 \int_0^{x_2} dx_1 J_0(x_1) - \frac{g^2\theta_1^2}{4} \ , \end{aligned} \tag{2.8}$$

where N_l is the total number of lattice sites, and J_0 and J_1 are the Bessel functions of the first kind. The value of θ_1 for any coupling constant and fermion mass can be determined by the condition of the lowest vacuum energy satisfying $\partial \mathcal{E}_\Omega / \partial \theta_1 = 0$. We can obtain the relation between θ_1 and $1/g^2$ by substituting $\theta_1(1/g^2)$ into the formula

$$\frac{\left\langle \Omega \left| \sum_x \bar{\psi}(x)\psi(x) \right| \Omega \right\rangle}{-N_l} = J_0(2\sqrt{2}\theta_1) \ . \tag{2.9}$$

Because the Schwinger model is superrenormalizable, the charge e does not vary with the cutoff. Let ψ_l and ψ_c stand for the fermion fields on the lattice and in the continuum, respectively; by dimensional analysis, $\langle \bar{\psi}\psi \rangle_l$ should scale as

$$\frac{\langle \bar{\psi}\psi \rangle_l}{N_l} = a\langle \bar{\psi}\psi \rangle_c \propto a \propto g \ . \tag{2.10}$$

In Ref. 18 a good scaling behavior was obtained, and the value for $\langle \bar{\psi}\psi \rangle_l / g$ is (-0.40 ± 0.05), which is higher than the exactly calculable one:[10]

$$\frac{\langle \bar{\psi}\psi \rangle_c}{e} = -\frac{e^\gamma}{2\pi^{3/2}} = -0.16 \ . \tag{2.11}$$

A three-link term in addition to the one-link term in S_{f_1} was considered in Ref. 18, which resulted in an improved value (-0.35 ± 0.05). However, more link terms change the value subtly. We then conjectured that the extra fermions would probably give a nonzero contribution to the fermion condensate because the species of naive fermions are doubled. We will solve this problem in the following sections.

III. LATTICE GAUGE THEORY WITH FREE WILSON FERMIONS

The lattice Hamiltonian with free Wilson fermions is

$$\begin{aligned} H &= \frac{1}{2a}\sum_{x,k} \bar{\psi}(x)\sigma_k \psi(x+k) + m\sum_x \bar{\psi}(x)\psi(x) \\ &\quad + \frac{r}{2a}\sum_{x,k} \bar{\psi}(x)[\psi(x) - \psi(x+k)] \ , \end{aligned} \tag{3.1}$$

where r is the Wilson parameter. The last term vanishes in the continuum limit so that chiral symmetry is restored when $m=0$. The physical vacuum of this Hamiltonian is defined in a similar way by

$$|\Omega\rangle = \exp\left[i\sum_p \theta_p S_p \right] |0\rangle \ , \tag{3.2}$$

where

$$\begin{aligned} S_p &= -\frac{1}{A_p}\sum_j \psi^\dagger(p)\sigma_j\psi(p)\frac{\sin(pja)}{a} \ , \\ A_p &= \left[\sum_j \left[\frac{\sin pja}{a} \right]^2 \right]^{1/2} \ , \end{aligned} \tag{3.3}$$

which is just the Foldy-Wouthuysen transformation. The transformed Hamiltonian in momentum space is

$$\begin{aligned} H'(p) &= \exp(-i\theta_p S_p) H(p) \exp(i\theta_p S_p) \\ &= \left[\left[m + \frac{ra}{2}\frac{4}{a^2}\sum_j \sin^2 pja/2 \right] \cos 2\theta_p + A_p \sin 2\theta_p \right] \bar{\psi}(p)\psi(p) \\ &\quad + \left[\cos 2\theta_p - \frac{1}{A_p}\left[m + \frac{ra}{2}\frac{4}{a^2}\sum_j \sin^2 pja/2 \right] \sin 2\theta_p \right] \sum_j \bar{\psi}(p) i\sigma_j \psi(p) \frac{\sin pja}{a} \ . \end{aligned} \tag{3.4}$$

The value of θ_p is determined by the condition of the lowest vacuum energy:

$$\tan 2\theta_p = \frac{A_p}{m + \frac{ra}{2}\frac{4}{a^2}\sum_j \sin^2 pja/2} \ , \tag{3.5}$$

which also eliminates the last term of Eq. (3.4). Therefore, the transformation Eq. (3.2) diagonalizes the Hamiltonian with free Wilson fermions exactly. The vacuum energy is

$$\langle \Omega | H | \Omega \rangle = -N_l \sum_p \left[\sum_j \left[\frac{\sin pja}{a} \right]^2 + \left[m + \frac{ra}{2}\frac{4}{a^2}\sum_j \sin^2 pja/2 \right]^2 \right]^{1/2} \ , \tag{3.6}$$

which is just the dispersion law and has low-frequency modes only near the origin of momentum space.

Now the free fermion condensate $\langle \bar{\psi}\psi \rangle$ can be calculated directly using Eqs. (3.4) and (3.5). In the massless limit, we have

$$\langle \bar{\psi}\psi \rangle_{\text{free}} = \left\langle 0 \left| \exp(-i\theta_p S_p) \sum_x \bar{\psi}\psi \exp(i\theta_p S_p) \right| 0 \right\rangle$$

$$= \sum_p \langle 0|\bar{\psi}(p)\psi(p)|0\rangle \cos 2\theta_p - \sum_p \frac{1}{A_p} \sin 2\theta_p \left\langle 0 \left| \sum_j \bar{\psi}(p) i\sigma_j \frac{\sin pja}{a} \psi(p) \right| 0 \right\rangle$$

$$= -\sum_p \cos 2\theta_p = -\frac{N_l}{2\pi} \int_{-\pi/a}^{\pi/a} d(pa) \frac{2r\sin^2 pa/2}{[(2r\sin^2 pa/2)^2 + \sin^2 pa]^{1/2}} . \tag{3.7}$$

For $r=1$, the integral can be evaluated exactly, and the free fermion condensate per site is

$$\frac{\langle \bar{\psi}\psi \rangle_{\text{free}}}{N_l} = -\frac{2}{\pi} , \tag{3.8}$$

which does not vanish (in fact for any $r \neq 0$) because of the Wilson term.

IV. FERMION CONDENSATE IN THE SCHWINGER MODEL WITH WILSON FERMIONS

The lattice Hamiltonian with Wilson fermions in the presence of a gauge field can be obtained by adding the Wilson term H_r to Eq. (2.1):

$$H = H_g + H_f + H_r , \tag{4.1}$$

where

$$H_g = \frac{g^2}{2a} \sum_{x,j} E_j^2(x) ,$$

$$H_f = \frac{1}{2a} \sum_{x,k} \bar{\psi}(x) \sigma_k U(x,k) \psi(x+k) + m \sum_x \bar{\psi}(x)\psi(x) ,$$

$$H_r = \frac{r}{2a} \sum_{x,k} [\bar{\psi}(x)\psi(x) - \bar{\psi}(x) U(x,k) \psi(x+k)] . \tag{4.2}$$

To diagonalize this Hamiltonian, we add to Eq. (2.5) a two-link term, and the unitary transformation becomes

$$H'' = \exp(-i\theta_1 S_{f_1} - i\theta_2 S_{f_2}) H \exp(i\theta_1 S_{f_1} + i\theta_2 s_{f_2}) , \tag{4.3}$$

where θ_2 is the additional variational parameter and

$$S_{f_2} = \frac{i}{\sqrt{2}} \sum_{x,k} \psi^\dagger(x) \sigma_k U(x,2k) \psi(x+2k) . \tag{4.4}$$

It can be easily proven that S_{f_1} and S_{f_2} are Hermitian, and in 1+1 dimensions they commute so that the transformation can be applied separately. In Ref. 18 and Sec. II, $\exp(-i\theta_1 S_{f_1}) H_g \exp(i\theta_1 S_{f_1})$ and $\exp(-i\theta_1 S_{f_1}) H_f \times \exp(i\theta_1 S_{f_1})$ have been expanded in powers of θ_1, and the momentum-dependent part of the Wilson term is

$$\exp(-i\theta_1 S_{f_1}) \left[-\frac{r}{2a} \sum_{x,k} \bar{\psi}(x) U(x,k) \psi(x+k) \right] \exp(i\theta_1 S_{f_1}) \equiv H'_{r_2}$$

$$= -\frac{r}{2a} \sum_{n_1=0}^{\infty} \frac{1}{n_1!} \left[\frac{-2\theta_1}{\sqrt{2}} \right]^{n_1} \bar{\psi}(x) \sigma_{k_1} \cdots \sigma_{k_{n_1}} U(x,k_1,\ldots,k_{n_1-1},2k_{n_1}) \psi(x+k_1+\cdots+k_{n_1-1}+2k_{n_1}) . \tag{4.5}$$

After successive commutations with S_{f_2}, the transformed Hamiltonians become

$$H''_m = m \sum_{n_2=0}^{\infty} \frac{1}{n_2!} \left[\frac{-2\theta_2}{\sqrt{2}} \right]^{n_2} \sum_{n_1=0}^{\infty} \frac{1}{n_1!} \left[\frac{-2\theta_1}{\sqrt{2}} \right]^{n_1} \bar{\psi}(x) \sigma_{k_1} \cdots \sigma_{k_{n_1}}, \sigma_{p_1} \cdots \sigma_{p_{n_2}} U(x,k_1,\ldots,k_{n_1},2p_1,\ldots,2p_{n_2})$$
$$\times \psi(x+k_1+\cdots+k_{n_1}+2p_1+\cdots+2p_{n_2}) ,$$

$$H''_k = \frac{1}{2a} \sum_{n_2=0}^{\infty} \frac{1}{n_2!} \left[\frac{-2\theta_2}{\sqrt{2}} \right]^{n_2} \sum_{n_1=0}^{\infty} \frac{1}{n_1!} \left[\frac{-2\theta_1}{\sqrt{2}} \right]^{n_1} \bar{\psi}(x) \sigma_{k_1} \cdots \sigma_{k_{n_1+1}} \sigma_{p_1} \cdots \sigma_{p_{n_2}} U(x,k_1,\ldots,k_{n_1+1},2p_1,\ldots,2p_{n_2})$$
$$\times \psi(x+k_1+\cdots+k_{n_1+1}+2p_1+\cdots+2p_{n_2}) , \tag{4.6}$$

$$H''_{r_2} = -\frac{r}{2a} \sum_{n_2=0}^{\infty} \frac{1}{n_2!} \left[\frac{-2\theta_2}{\sqrt{2}} \right]^{n_2} \sum_{n_1=0}^{\infty} \frac{1}{n_1!} \left[\frac{-2\theta_1}{\sqrt{2}} \right]^{n_1} \bar{\psi}(x) \sigma_{k_1} \cdots \sigma_{k_{n_1}} \sigma_{p_1} \cdots \sigma_{p_{n_2}}$$
$$\times U(x,k_1,\ldots,k_{n_1-1},2k_{n_1},2p_1,\ldots,2p_{n_2})$$
$$\times \psi(x+k_1+\cdots+k_{n_1-1}+2k_{n_1}+2p_1+\cdots+2p_{n_2}) .$$

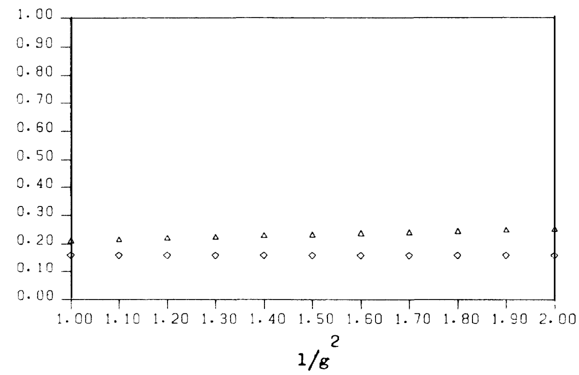

FIG. 1. $-\langle \bar{\psi}\psi \rangle / g$ as a function of $1/g^2$ for $r=1$ and $m=0$, where the triangles represent our calculated result and the diamonds stand for the exactly calculable value.

The transformed electric field can be calculated in a similar way.

We assume the physical vacuum state with Wilson fermions to be

$$|\Omega\rangle = \exp(i\theta_1 S_{f_1} + i\theta_2 S_{f_2})|0\rangle \ . \tag{4.7}$$

By solving the equations $\partial\mathcal{E}_\Omega/\partial\theta_1=0$ and $\partial\mathcal{E}_\Omega/\partial\theta_2=0$ we can determine the values of θ_1 and θ_2 for any coupling constant, fermion mass, and Wilson parameter. For $r=0$ (naive fermions), the chiral order parameter has been calculated in Sec. II, and θ_2 automatically vanishes.

For $r \neq 0$ (Wilson fermions), because H_r breaks chiral symmetry explicitly, $\bar{\psi}\psi$ can mix with the identity operator;[19–21] the Wilson term gives rise to a nonvanishing contribution even in the weak-coupling limit as discussed in Sec. III, which should be subtracted before comparing with the scaling behavior[19–21]

$$\frac{\langle \bar{\psi}\psi \rangle_c}{e} = \frac{\langle \bar{\psi}\psi \rangle_l - \langle \bar{\psi}\psi \rangle_{\rm free}}{g N_l} \ . \tag{4.8}$$

The right-hand side of Eq. (4.8) as a function of $1/g^2$ in the massless limit and for $r=1$ is shown in Fig. 1. As one sees, our result is very close to the exactly calculable value and shows the graceful scaling behavior predicted by Eq. (2.10) after subtraction.

V. SUMMARY AND DISCUSSION

In the preceding sections, the unitary transformation and variational method have been used to study the vacuum structure and chiral-symmetry breaking in the Schwinger model with Wilson fermions. The chiral order parameter $\langle \bar{\psi}\psi \rangle$ has been calculated as a function of $1/g^2$ and r. For $r=1$ and $m=0$, the result shows a nice scaling behavior and agrees within error with the exactly calculable value, Eq. (2.10).

Therefore our approach to the Schwinger model with Wilson fermions provides a feasible analytic method of studying the properties of lattice gauge theory for any coupling constant, fermion mass, and Wilson parameter. It has been seen that our method is very effective for investigating the scaling behavior of physical quantities. This method has been directly generalized to lattice gauge theory in higher dimensions.[22–25] To obtain the mass spectrum of the Schwinger model, one has to find the eigenstates and to determine the eigenvalues of the transformed Hamiltonian. There has been previous work[26–30] on variational approaches (the coefficients of the eigenstates are variational parameters) for calculating the mass spectrum in lattice Hamiltonian field theories and the results were excellent. In a future work, we hope to apply these methods to calculate the mass spectrum of the Schwinger model.

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[1]J. Schwinger, Phys. Rev. **128**, 2425 (1962).

[2]W. Marciano and H. Pagels, Phys. Rep. **36C**, 137 (1978), and references therein.

[3]T. Banks, J. Kogut, and L. Susskind, Phys. Rev. D **13**, 1043 (1976).

[4]A. Carroll, J. Kogut, D. K. Sinclair, and L. Susskind, Phys. Rev. D **13**, 2270 (1976).

[5]D. P. Crewther and C. J. Hamer, Nucl. Phys. **B170**, 353 (1980).

[6]E. Marinari, G. Parisi, and C. Rebbi, Nucl. Phys. **B190**, 734 (1981).

[7]A. Duncan and M. Furman, Nucl. Phys. **B190**, 767 (1981).

[8]D. J. Scalapino and R. L. Sugar, Phys. Rev. Lett. **46**, 519 (1981).

[9]O. Martin and S. Otto, Nucl. Phys. **B203**, 297 (1982).

[10]C. J. Hamer, J. Kogut, D. P. Crewther, and M. M. Mazzolini, Nucl. Phys. **B208**, 413 (1982).

[11]J. Ranft and A. Schiller, Nucl. Phys. **B225**, 204 (1983).

[12]H. Gausterer and J. B. Klauder, Phys. Lett. **164B**, 127 (1983).

[13]J. Bartholomew and J. Sloan, Phys. Lett. B **172**, 407 (1986).

[14]M. Wiltgen, Z. Phys. C **41**, 95 (1988).

[15]C. J. Hamer and M. N. Barber, J. Phys. A **13**, L169 (1980).

[16]C. J. Hamer and M. N. Barber, J. Phys. A **14**, 241 (1981).

[17]C. J. Hamer and M. N. Barber, J. Phys. A **14**, 2009 (1981).

[18]X. Q. Luo and Q. Z. Chen, J. Phys. G (to be published).

[19]M. Fukugita, T. Kaneko, and A. Ukawa, Phys. Lett. **130B**, 199 (1983).

[20]M. Fukugita, T. Kaneko, and A. Ukawa, Nucl. Phys. **B230**, 62 (1984).

[21]R. H. Tsuchida, Ph.D. thesis, University of California at Los Angeles, 1987.

[22]S. H. Guo, Q. Z. Chen, J. M. Liu, and L. Hu, Commun. Theor. Phys. **3**, 481 (1984).

[23]X. Q. Luo, Q. Z. Chen, and S. H. Guo, High Energy Phys. Nucl. Phys. **13**, 328 (1989).

[24]Q. Z. Chen, X. Q. Luo, and S. H. Guo, Acta Sci. Nat. Univ. Sun. **28**, 96 (1989).

[25]X. Q. Luo, Q. Z. Chen, and S. H. Guo, Z. Phys. C (to be published).

[26]L. Hu, Ph.D. thesis, Zhongshan University, 1984.

[27]S. A. Chin *et al.*, Phys. Rev. D **37**, 3006 (1988).

[28]A. Duncan and R. Roskies, Phys. Rev. D **31**, 364 (1985).

[29]A. Duncan, Nucl. Phys. **B258**, 125 (1985).

[30]A. Duncan and R. Roskies, Phys. Rev. D **37**, 472 (1988).