Coulomb interaction of fermions on S^3

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In this paper we develop a formalism for inclusion of fermions in a discussion of QCD on a hypersphere. We apply this to the study of the Coulomb energy operator. We use operators formed from pairs of fermion operators to construct a set of states for use in variational calculations of the effect of the Coulomb energy operator. To examine the feasibility of numerical calculations based on this formalism, a calculation is undertaken which uses a simplified approximation for the effective Coulomb energy. The preliminary numerical results suggest that the Coulomb energy may be important for the chiral-symmetry properties of QCD.

I. INTRODUCTION

Quantum chromodynamics (QCD) is believed to describe the strong interaction of hadrons. However, in the strong-coupling regime, it is necessary to use numerical methods to investigate the properties of QCD. As an alternative to lattice formulations,[1] we use a continuum formulation in a hypersphere.[2] The hypersphere provides a quantization volume for the fields, and we impose a momentum cutoff. As a result, we have a system with a finite number of degrees of freedom. We use a Hamiltonian formulation together with variational methods to investigate directly the low-lying states of the system in the strong-coupling region. Since it is a continuum formulation, there are no fundamental problems with inclusion of fermions, as demonstrated by Sen.[3] Moreover, chiral symmetry, as well as rotational invariance, is preserved.

In calculations on Yang-Mills fields without quarks, we have found that the spectrum is primarily governed by properties of the Coulomb energy operator.[4] For strong coupling, there is a gap between the vacuum state and the first-excited state with vacuum quantum numbers, and a second gap of roughly the same size between the first- and higher-excited states.[5,6] This is consistent with confinement of the gluons and an interpretation of the first-excited state as a glueball. Several previous studies suggest that when light quarks are included in the theory, these quarks may become condensed in pairs as a consequence of the Coulomb interaction.[7,8] However, the association of this condensation with the phenomenon of confinement is not clear from the previous work. The condensation of fermions has also been considered in strong-coupling quantum electrodynamics (QED), using both continuum[9,10] and lattice methods.[11] To examine further the role of the Coulomb interaction in the condensation of light quarks and the chiral properties of the states, along with the relation of these phenomena to confinement, detailed numerical calculations will be required. These can be carried out by extending the methods we have used for the pure-glue models.

A second question about the relation between quark interactions and confinement is whether the quarks play a significant role in providing the mechanism for confinement of gluons. Our previous work suggested that behavior consistent with confinement does occur even in the absence of quarks. In other continuum-based studies, this question has not been examined, although it has been the focus of many lattice-based studies. The question addressed here is, does condensation of the quarks occur whenever the interaction is strong enough, or does it require that the interaction be confining? This question has been considered in previous studies,[7-10] but with conflicting results. We shall not attempt to settle this question here, but shall try to develop some new techniques which can be used for studying this question numerically.

In a previous paper, we have investigated the properties of electrons on the hypersphere under the influence of a uniform magnetic field.[12] In this paper, we will discuss second-quantized fermion fields on the hypersphere. First, we will give a brief review of fermions on the hypersphere. Then we will study the Coulomb energy operator using a simplified model. We will review briefly some general aspects of QCD which we wish to incorporate into our model, but we consider here only Coulomb interactions which have an externally prescribed form. We show how to construct states with vacuum quantum numbers by using operators constructed from pairs of fermion operators, and we evaluate the matrix elements of the Coulomb energy operator between such states. In order to get some insight into the technical numerical problems, some preliminary calculations are also undertaken using a simplified interaction. Results of these calculations are described. They suggest that the chiral-symmetry properties are influenced by the Coulomb interaction, but that more refined numerical techniques must be developed in order to study this question carefully.

II. FERMIONS ON THE HYPERSPHERE

The hypersphere S^3 is the surface of a sphere in four dimensions; in Cartesian coordinates, it is specified by

$$X_1^2+X_2^2+X_3^2+X_4^2=1 \ . \tag{1}$$

On the hypersphere, we have defined scalar harmonic functions $S_K^{\mu,\nu}$, and transverse vector harmonic functions $V_{K,\pm 1}^{\mu,\nu}$. In order to define spinors on the hypersphere, one needs to construct an orthonormal basis at each point on the sphere. One convenient choice is a set of vectors $\hat{e}_i$, with $i=1,2,3$, which are derived from the three-vector harmonic functions $\mathbf{V}_{1,+}^{\mu,\nu}$. This choice is not unique. For example, one could choose a similar set of vectors $\hat{c}_i$ based on the $\mathbf{V}_{1,-}^{\mu,\nu}$. The physics should not depend on the choice of the coordinate system, but for definiteness we choose the first basis.

In terms of this coordinate system, the general covariant derivative $\mathcal{D}_\mu$ can be expressed as

$$i\gamma^m e_m^\mu \mathcal{D}_\mu = i\gamma^m \partial_m - \tfrac{3}{2}\gamma^5\gamma^0 \ , \tag{2}$$

where $\partial_m=(\partial_0,\hat{e}_i\cdot\nabla)$, giving the Dirac equation as

$$(i\gamma^m\partial_m-\tfrac{3}{2}\gamma^5\gamma^0-M)\Psi=0 \ . \tag{3}$$

The $\gamma^5\gamma^0$ term does not violate parity. Under parity, $(t,X_i,X_4)\to(t,-X_i,X_4)$, the Dirac equation transforms to

$$(i\gamma^m\partial_m^c+\tfrac{3}{2}\gamma^5\gamma^0-M)\Psi=0 \ , \tag{4}$$

where $\partial_m^c=(\partial_0,\hat{\mathbf{c}}_i\cdot\nabla)$. However, if we choose the $\hat{c}_i$ coordinate system to start with, we end up with the covariant derivative

$$i\gamma^m c_m^\mu \mathcal{D}_\mu^c = i\gamma^m\partial_m^c+\tfrac{3}{2}\gamma^5\gamma^0 \ . \tag{5}$$

Thus the change of sign of the term $\gamma^5\gamma^0$ is consistent with the change in the coordinate system, and as a result, does not violate parity.

The general normal-mode solutions to the Dirac equation are easy to obtain. For $\kappa=1,2,\ldots$ and $h=\pm\frac{1}{2}$, a two-component spinor function χ is constructed in terms of the scalar harmonic functions as

$$\chi(\kappa,h,\nu,m)=\sum_{\mu,m_S}(2J+1)^{1/2}\begin{bmatrix} K & \frac{1}{2} & J \\ \mu & m_S & m\end{bmatrix} \times S_K^{\mu,\nu}\chi(m_S) \ , \tag{6}$$

with

$$\chi(\tfrac{1}{2})=\begin{bmatrix}1\\0\end{bmatrix}, \quad \chi(-\tfrac{1}{2})=\begin{bmatrix}0\\1\end{bmatrix} , \tag{7}$$

$2K=\kappa-h-\frac{1}{2}$, and $2J=\kappa+h-\frac{1}{2}$. For positive energy $E(\kappa)=[M^2+\theta(\kappa,h)^2]^{1/2}$, with $\theta(\kappa,h)=2h(\kappa+\frac{1}{2})$, the Dirac spinor is then

$$U(\kappa,h,\nu,m)=\begin{bmatrix} C_1(\kappa,h)\chi(\kappa,h,\nu,m) \\ -C_2(\kappa,h)\chi(\kappa,h,\nu,m)\end{bmatrix} , \tag{8}$$

where

$$C_1(\kappa,h)=[E(\kappa)+M]/D(\kappa) \ ,$$
$$C_2(\kappa,h)=\theta(\kappa,h)/D(\kappa) \ ,$$

and

$$D(\kappa)^2=2E(\kappa)[E(\kappa)+M] \ .$$

The negative-energy spinor is

$$V(\kappa,h,\nu,m)=-i\gamma^2U(\kappa,h,\nu,m)^* \ . \tag{9}$$

For each value of K and J, there are $2K+1$ values of ν and $2J+1$ values of m, giving a combined degeneracy $\kappa(\kappa+1)$ for each value of h. They form a complete set of normal-mode functions. In contrast to the situation in a periodic cube, there are no zero modes, and all spinors have a well-defined helicity h.

In the massless limit, we find $C_1(\kappa,h)\to 1/\sqrt{2}$ and $C_2(\kappa,h)\to -h\sqrt{2}$. Thus, for positive energy $E(\kappa)=\kappa+\frac{1}{2}$, we have

$$U_0(\kappa,h,\nu,m)=\frac{1}{\sqrt{2}}\begin{bmatrix}\chi(\kappa,h,\nu,m)\\ -2h\chi(\kappa,h,\nu,m)\end{bmatrix} . \tag{10}$$

These spinors are eigenstates of γ^5. The positive-energy spinors and the negative-energy spinors have opposite chirality.

For the second quantization of the fermionic operators on the hypersphere, we expand an arbitrary spinor in terms of the spinor normal-mode functions as

$$\Psi_a=\sum_{\kappa,h,\nu,m}[b_a(\kappa,h,\nu,m)U(\kappa,h,\nu,m) + d_a^\dagger(\kappa,h,\nu,m)V(\kappa,h,\nu,m)] \ . \tag{11}$$

The index a denotes all the internal degrees of freedom of the fermion. The expansion coefficients b_a and $d_a^\dagger$ are the destruction operator for a positive-energy fermion and the creation operator for a negative-energy fermion, respectively. They satisfy the standard anticommutator relations.

Under charge conjugation, one can show that these fermion operators transform as

$$\begin{aligned} C_{\rm op}b_a(\kappa,h,\nu,m)C_{\rm op}^\dagger &= d_a(\kappa,h,\nu,m) \ , \\ C_{\rm op}d_a^\dagger(\kappa,h,\nu,m)C_{\rm op}^\dagger &= b_a^\dagger(\kappa,h,\nu,m) \ . \end{aligned} \tag{12}$$

Under time reversal, with $T_{\rm op}=U_{\rm op}K_{\rm op}$ and $K_{\rm op}$ being the complex-conjugation operator, the fermion operators transform as

$$\begin{aligned} U_{\rm op}b_a(\kappa,h,\nu,m)U_{\rm op}^\dagger &= -i\eta(h,\nu,m)b_a(\kappa,h,-\nu,-m) \ , \\ U_{\rm op}d_a^\dagger(\kappa,h,\nu,m)U_{\rm op}^\dagger &= -i\eta(h,\nu,m)d_a^\dagger(\kappa,h,-\nu,-m) \ , \end{aligned} \tag{13}$$

where the phase factor is $\eta(h,\nu,m)=(-1)^{\nu+m-h}$.

III. THE MODEL HAMILTONIAN

For the construction of our model of fermion interactions, we are guided by the Hamiltonian formulation of QCD. However, we shall not try to determine in detail here from QCD how the quarks and gluons influence each other. In order to eliminate all the nonphysical degrees of freedom from QCD, we choose the Coulomb gauge, $\nabla\cdot\mathbf{A}_a=0$.[13] On the hypersphere, the full quantum Hamiltonian for SU(n) gauge theory is

$$H_c=\int[\tfrac{1}{2}(J^{-1}E_i^{a\,\mathrm{tr}}JE_i^{a\,\mathrm{tr}}+B_i^aB_i^a)+\bar{\Psi}_a(X)(-i\gamma_iD_i+\tfrac{3}{2}\gamma^5\gamma^0+M)\Psi_a(X)]d\Omega$$
$$+\tfrac{1}{2}g^2\int\int J^{-1}\sigma^a(X)\langle a,X|(\partial_iD_i)^{-1}(-\partial^2)(\partial_iD_i)^{-1}|b,X'\rangle J\sigma^b(X')d\Omega\,d\Omega' , \tag{14}$$

where X and X' are four-dimensional unit vectors. The covariant derivative is

$$D_i=\partial_i+ig\mathbf{T}\cdot\mathbf{A}_i \tag{15}$$

and the charge density is

$$\sigma^a=f^{abc}E_i^{b\,\mathrm{tr}}A_i^c+\Psi_i^\dagger t_{ij}^a\Psi_j . \tag{16}$$

The t_{ij}^a are the generators of SU(n) in the fundamental representation. The Faddeev-Popov determinant is defined to be

$$J=\det(\partial_iD_i) . \tag{17}$$

In our simplified model, we wish to consider only the fermion degrees of freedom. There may be important effects left out of such a model, but these are left for future study. To define this model, we neglect the fluctuations in the gluon field, and the effects of the quarks upon the gluon field. We are then left with a fermion-fermion interaction which is supposed to incorporate the important effects arising from the gluon field. In other words, to obtain the model Hamiltonian from H_c as given by Eq. (14), we replace all the vector fields by their average values over the inner Gribov region, and treat these vector fields as a fixed background. In this process, all factors in H_c which do not contain the fermion fields are replaced by constants. The Faddeev-Popov determinant, therefore, disappears from the Hamiltonian, and the vector potential term in the covariant derivative of fermion fields vanishes. In the Coulomb energy, the interference terms between the fermionic charges and the gluon charges also vanish, and we need to consider only the fermion charge density

$$\sigma^a=\Psi_i^\dagger t_{ij}^a\Psi_j . \tag{18}$$

The operator $(\partial_iD_i)^{-1}(-\partial^2)(\partial_iD_i)^{-1}$ is replaced by the average over gluon configurations, which we denote by G. However, we do not attempt to calculate G here from QCD, but just use an assumed functional form. As a result, the model Hamiltonian becomes

$$H_c'=\int[\bar{\Psi}_a(X)(-i\gamma_iD_i+\tfrac{3}{2}\gamma^5\gamma^0+M)\Psi_a(X)]d\Omega$$
$$+\tfrac{1}{2}g^2\int\int\sigma^a(X)\langle a,X|G|b,X'\rangle\sigma^b(X')d\Omega\,d\Omega' . \tag{19}$$

We shall consider a single quark flavor. The free part of the Hamiltonian (19) is

$$H_c^f=\sum_{\kappa,h}E(\kappa)[N_+(\kappa,h)+N_-(\kappa,h)] , \tag{20}$$

where the $N_\pm$ are the number operators for fermions and antifermions

$$N_+(\kappa,h)=\sum_{\nu,m}b^\dagger(\kappa,h,\nu,m)b(\kappa,h,\nu,m) , \tag{21}$$

and

$$N_-(\kappa,h)=\sum_{\nu,m}d^\dagger(\kappa,h,\nu,m)d(\kappa,h,\nu,m) . \tag{22}$$

By the completeness of the scalar harmonic functions and rotational invariance, we can write

$$\langle a,X|G|b,X'\rangle=\sum_{K,\mu,\nu}G_K\delta_{ab}S_K^{\mu,\nu}(X)^*S_K^{\mu,\nu}(X') . \tag{23}$$

The Coulomb energy operator for a general SU(n) gauge theory is, therefore,

$$H_c^c=\tfrac{1}{2}g^2\sum_{K,\mu,\nu}\sum_{a=1}^{n^2-1}G_K\sigma_1^a(K,\mu,\nu)\sigma_2^a(K,\mu,\nu) , \tag{24}$$

where

$$\sigma_1^a(K,\mu,\nu)=t_{ij}^a\int\Psi_i^\dagger S_K^{\mu,\nu*}\Psi_j d\Omega ,$$
$$\sigma_2^a(K,\mu,\nu)=t_{ij}^a\int\Psi_i^\dagger S_K^{\mu,\nu}\Psi_j d\Omega' . \tag{25}$$

Since we consider only colorless states, the matrix elements of σ vanish for $K=0$. The Coulomb Hamiltonian for fermions in QED [U(1) gauge theory] has a similar form, with $G_K=[K(K+2)]^{-1}$. In previous calculations, we observed that in strongly coupled QCD, the Coulomb matrix elements with $K=1$ are much bigger than the free-field Laplacian and dominate the higher-K values.[14] Therefore, we shall use

$$G_K=G_1\delta_{K,1} \tag{26}$$

in some numerical calculations described later. More generally, we would wish to consider an interaction of the form $G_K\sim K^{-\alpha}$ for large K, where $\alpha=3$ would correspond to the boundary between confining and nonconfining interactions. The "superconfining" interaction (26) is a limiting case, and since it leads to fewer nonvanishing matrix elements, it slightly simplifies our exploratory numerical work. We shall use the following abbreviated notations: (i) $\equiv(\kappa_i,h_i,\nu_i,m_i)$ for $i=1,3,4,6$ and (ii) $\equiv(K,\mu,\nu)$. Expanding the charge density in

terms of the creation and destruction operators, we have

$$\begin{aligned}\sigma_1^a(K,\mu,\nu)=t_{ij}^a \sum_{(13)} [\, & b_i^\dagger(1)b_j(3)T_{++}(321)^\dagger \\ & +b_i^\dagger(1)d_j^\dagger(3)T_{-+}(321)^\dagger \\ & +d_i(1)b_j(3)T_{+-}(321)^\dagger \\ & -d_j^\dagger(3)d_i(1)T_{--}(321)^\dagger] \,, \\ \sigma_2^a(K,\mu,\nu)=t_{kl}^a \sum_{(46)} [\, & b_k^\dagger(4)b_l(6)T_{++}(426) \\ & +b_k^\dagger(4)d_l^\dagger(6)T_{+-}(426) \\ & +d_k(4)b_l(6)T_{-+}(426) \\ & -d_l^\dagger(6)d_k(4)T_{--}(426)] \,,\end{aligned} \tag{27}$$

where $T_{\alpha\beta}$, with $\alpha,\beta=\pm$, denotes the integral of a product involving spinor harmonic functions and a scalar harmonic function. The subscripts $\pm$ denote the positive- and negative-energy spinors. For example, we have

$$T_{+-}(321)=\int U^\dagger(3)S_K^{\mu,\nu}V(1)\,d\Omega \,. \tag{28}$$

In each $T_{\alpha\beta}$, the integral over three scalar harmonic functions gives a reduced matrix element multiplied by two 3-j symbols and other factors from Eq. (6). Three of the 3-j symbols can be summed over internal magnetic quantum numbers to give a 6-j symbol and a new 3-j symbol. The results, listed below, can be interpreted as a reduced matrix element for the spinor harmonic functions multiplied by a pair of 3-j symbols

$$\begin{aligned}T_{++}(123)=\,& \Omega^{-1/2}(-1)^{K/2+1/2}(-1)^{\nu_1+m_3}[C_1(\kappa_1,h_1)C_1(\kappa_3,h_3)+C_2(\kappa_1,h_1)C_2(\kappa_3,h_3)] \\ & \times[(2K_1+1)(2J_1+1)(K+1)(2K_3+1)(2J_3+1)]^{1/2}\begin{pmatrix}J_3 & J_1 & \frac{1}{2}K\\ m_3 & -m_1 & -\mu\end{pmatrix}\begin{pmatrix}K_3 & K_1 & \frac{1}{2}K\\ \nu_3 & -\nu_1 & \nu\end{pmatrix}\begin{Bmatrix}J_3 & J_1 & \frac{1}{2}K\\ K_1 & K_3 & \frac{1}{2}\end{Bmatrix}, \\ T_{--}(123)=\,& T_{++}(321) \,, \\ T_{+-}(123)=\,& \Omega^{-1/2}(-1)^{-K_3-J_1-1/2}(-1)^{\nu_1+\nu_3}[C_1(\kappa_3,h_3)C_2(\kappa_1,h_1)-C_1(\kappa_1,h_1)C_2(\kappa_3,h_3)] \\ & \times[(2K_1+1)(2J_1+1)(K+1)(2K_3+1)(2J_3+1)]^{1/2}\begin{pmatrix}J_3 & J_1 & \frac{1}{2}K\\ m_3 & m_1 & \mu\end{pmatrix}\begin{pmatrix}K_3 & K_1 & \frac{1}{2}K\\ -\nu_3 & -\nu_1 & \nu\end{pmatrix}\begin{Bmatrix}J_3 & J_1 & \frac{1}{2}K\\ K_1 & K_3 & \frac{1}{2}\end{Bmatrix}, \\ T_{-+}(123)=\,& \Omega^{-1/2}(-1)^{K_1-J_1+K/2-1/2}(-1)^{m_1+m_3}[C_1(\kappa_3,h_3)C_2(\kappa_1,h_1)-C_1(\kappa_1,h_1)C_2(\kappa_3,h_3)] \\ & \times[(2K_1+1)(2J_1+1)(K+1)(2K_3+1)(2J_3+1)]^{1/2}\begin{pmatrix}J_3 & J_1 & \frac{1}{2}K\\ m_3 & m_1 & -\mu\end{pmatrix}\begin{pmatrix}K_3 & K_1 & \frac{1}{2}K\\ \nu_3 & \nu_1 & \nu\end{pmatrix}\begin{Bmatrix}J_3 & J_1 & \frac{1}{2}K\\ K_1 & K_3 & \frac{1}{2}\end{Bmatrix},\end{aligned} \tag{29}$$

where $\Omega=2\pi^2$ is the volume of the hypersphere and $2K_i=\kappa_i-h_i-\frac{1}{2}$, $2J_i=\kappa_i+h_i-\frac{1}{2}$.

By using the Hermiticity of the Coulomb energy operator and charge-conjugation invariance, we can simplify the form of the Coulomb energy. Identifying terms with different combinations of creation and destruction operators, we can write the Coulomb energy as

$$H_c^c=H_1+H_2+H_3+H_2^\dagger+H_1^\dagger \,. \tag{30}$$

The first term is

$$\begin{aligned}H_1=\sum{}^* \, & b_i^\dagger(1)d_j^\dagger(3)b_\kappa^\dagger(4)d_l^\dagger(6)T_{-+}(321)^\dagger \\ & \times T_{+-}(426) \,,\end{aligned} \tag{31}$$

where we use the abbreviated notation

$$\sum{}^* \equiv \sum_{aijkl} \tfrac{1}{2}g^2 t_{ij}^a t_{kl}^a \sum_{(12346)} G_K \,. \tag{32}$$

The second term is

$$H_2=H_{2a}+H_{2b} \,, \tag{33}$$

with

$$\begin{aligned}H_{2a}=\sum{}^* [\, & 2b_i^\dagger(1)b_k^\dagger(4)d_l^\dagger(6)b_j(3)T_{++}(321) \\ & \times T_{+-}(426) \\ & -2b_i^\dagger(1)b_k^\dagger(4)d_j^\dagger(3)b_l(6)T_{-+}(321) \\ & \times T_{++}(426)] \,,\end{aligned} \tag{34}$$

$$\begin{aligned}H_{2b}=\sum{}^* \, & 2b_i^\dagger(1)d_l^\dagger(6)\delta_{jk}\delta_p(34)\delta_z(34) \\ & \times T_{++}(321)T_{+-}(426) \,.\end{aligned} \tag{35}$$

The third term is

$$H_3=H_{3a}+H_{3b}+H_{3c} \,, \tag{36}$$

with

$$\begin{aligned}H_{3a}=\sum{}^* [\, & -2b_i^\dagger(1)b_k^\dagger(4)b_j(3)b_l(6)T_{++}(321)T_{++}(426)+2b_i^\dagger(1)d_l^\dagger(6)b_j(3)d_k(4)T_{++}(321)T_{--}(426) \\ & -2b_k^\dagger(4)d_l^\dagger(6)b_j(3)d_i(1)T_{+-}(321)T_{+-}(426)] \,,\end{aligned} \tag{37}$$

$$H_{3b}=\sum{}^{*}[+2b_i^{\dagger}(1)b_l(6)\delta_{jk}\delta_p(34)\delta_z(34)T_{++}(321)T_{++}(426)-2b_k^{\dagger}(4)b_j(3)\delta_{il}\delta_p(16)\delta_z(16)T_{+-}(321)T_{+-}(426)] , \tag{38}$$

$$H_{3c}=\sum{}^{*}\delta_{il}\delta_{jk}\delta_p(16)\delta_p(34)\delta_z(16)\delta_z(34)T_{+-}(321)^{\dagger}T_{+-}(426) . \tag{39}$$

Since $H_{3c}=E_0$ is just a constant, it could be dropped. We will keep it as part of our unit of energy for some numerical calculations. The symbols δ_p and δ_z are composite delta functions defined as $\delta_p(i,j)\equiv\delta_{\kappa_i,\kappa_j}\delta_{h_i,h_j}$ and $\delta_z(i,j)\equiv\delta_{\nu_i,\nu_j}\delta_{m_i,m_j}$, respectively.

IV. PAIR STATES

For our variational calculations, we want to construct a set of vacuum states containing many fermion-antifermion pairs. We expect that the creation operator (or destruction operator) for a pair should be related to the integrals of local field operators

$$\Phi_S=\sum_{i=1}^{n}\int\bar{\Psi}_i\Psi_i d\Omega \quad\text{and}\quad \Phi_P=\sum_{i=1}^{n}\int\bar{\Psi}_i\gamma_5\Psi_i d\Omega . \tag{40}$$

We expand $\bar{\Psi}$ and Ψ in terms of creation and destruction operators and keep just those terms which contain a product of two creation operators. These terms contain sums over κ and h of the following operators, which create a pair in which the two particles having a definite magnitude of momentum κ and helicity h have been combined to give a total momentum of zero and no net color:

$$M(\kappa,h)^{\dagger}=N(\kappa)\sum_i\sum_{\nu,m}\eta(h,\nu,m)b_i^{\dagger}(\kappa,h,-\nu,-m)\times d_i^{\dagger}(\kappa,h,\nu,m) . \tag{41}$$

Here $N(\kappa)=[n\kappa(\kappa+1)]^{-1/2}$ is a normalization constant obtained by setting

$$\langle 0|M(\kappa',h')M(\kappa,h)^{\dagger}|0\rangle=\delta_{\kappa,\kappa'}\delta_{h,h'} , \tag{42}$$

where $|0\rangle$ is the perturbative vacuum. These pair operators are invariant under charge conjugation. Note that in the pair-creation parts of the operators $\Phi_{\pm}=\Phi_P\pm\Phi_S$, the sums only include terms with helicity $h=\pm\frac{1}{2}$.

The only nonvanishing commutator of the pair operators is

$$[M(\kappa',h'),M(\kappa,h)^{\dagger}]=\delta_{h,h'}\delta_{\kappa,\kappa'}+\delta_{h,h'}\delta_{\kappa,\kappa'}O(\kappa,h) \tag{43}$$

which is more complicated than the normal bosonic commutator. The correction term contains an operator $O(\kappa,h)$ given by

$$O(\kappa,h)=-\sum_i\sum_{\nu,m}N^2(\kappa)d_i^{\dagger}(\kappa,h,\nu,m)d_i(\kappa,h,\nu,m)-\sum_i\sum_{\nu,m}N^2(\kappa)b_i^{\dagger}(\kappa,h,-\nu,-m)\times b_i(\kappa,h,-\nu,-m) . \tag{44}$$

This operator $O(\kappa,h)$ is Hermitian, and satisfies $O(\kappa,h)|0\rangle=0$. In addition, we find

$$[M(\kappa',h'),O(\kappa,h)]=-2N^2(\kappa)M(\kappa,h)\delta_{h,h'}\delta_{\kappa,\kappa'} . \tag{45}$$

We will consider general linear combinations of the states

$$|l_1,l_2,\ldots,l_m\rangle=N_{[l]}[M(\kappa_1,h_1)^{\dagger}]^{l_1}\times[M(\kappa_2,h_2)^{\dagger}]^{l_2}\cdots[M(\kappa_m,h_m)^{\dagger}]^{l_m}|0\rangle , \tag{46}$$

where l_i are positive integers and each pair (κ_i,h_i) is distinct. The overall normalization constant is $N_{[l]}$. These states are all colorless and have a total momentum of zero, but do not have a definite parity. To obtain an expression for $N_{[l]}$, we first consider the norm $K(l,\kappa)\equiv\langle l|l\rangle$ of the state

$$|l\rangle=[M(\kappa,h)^{\dagger}]^l|0\rangle . \tag{47}$$

We commute one of the $M(\kappa,h)$ to the right and obtain the relation

$$\langle l|l\rangle=l\langle 0|[M(\kappa,h)]^{l-1}[M(\kappa,h)^{\dagger}]^{l-1}|0\rangle+\langle 0|[M(\kappa,h)]^{l-1}O(\kappa,h)[M(\kappa,h)^{\dagger}]^{l-1}|0\rangle+\cdots+\langle 0|[M(\kappa,h)]^{l-1}[M(\kappa,h)^{\dagger}]^{l-2}\times O(\kappa,h)M(\kappa,h)^{\dagger}|0\rangle . \tag{48}$$

Again, we can commute the operator $O(\kappa,h)$ to the right and get

$$\langle 0|[M(\kappa,h)]^{l-1}[M(\kappa,h)^{\dagger}]^iO(\kappa,h)[M(\kappa,h)^{\dagger}]^{l-i}|0\rangle=-2N(\kappa)^2(l-i)\langle 0|[M(\kappa,h)]^{l-1}\times[M(\kappa,h)^{\dagger}]^{l-1}|0\rangle . \tag{49}$$

Since $|l-1\rangle=[M(\kappa,h)^{\dagger}]^{l-1}|0\rangle$, we obtain

$$\langle l|l\rangle=\left[l-2N(\kappa)^2\sum_{m=1}^{l-1}m\right]\langle l-1|l-1\rangle=F(l,\kappa)\langle l-1|l-1\rangle , \tag{50}$$

where

$$F(k,\kappa)=k[1-(k-1)N^2(\kappa)] . \tag{51}$$

Apart from the normalization factor, the matrix elements of $M^{\dagger}$ equal those of an angular-momentum-raising operator for spin $j=n\kappa(\kappa+1)/2$. The norm $\langle l|l\rangle$ is

$$K(l,\kappa)=\prod_{k=1}^{l}F(k,\kappa) , \tag{52}$$

and the overall normalization constant is

$$N_{[l]}=\prod_i K(l_i,\kappa_i)^{-1/2} . \tag{53}$$

V. EFFECTIVE HAMILTONIAN FOR MASSLESS FERMIONS

Since the general states are expressed in terms of pair operators $M(\kappa,h)^\dagger$ instead of quark operators $b_i(\kappa,h,\nu,m)$ and $d_i(\kappa,h,\nu,m)$, it is useful to rewrite the Coulomb Hamiltonian in terms of the pair operators and the pair number operators:

$$N_{\kappa,h}=\tfrac{1}{2}[N_+(\kappa,h)+N_-(\kappa,h)] . \tag{54}$$

We first write down the general form, then we will specialize to massless fermions. The Coulomb energy operator can be written as

$$\tilde{H}_c=E_0+\sum_{i,s,\kappa,\kappa',K}\mathcal{H}_{is}(\kappa,\kappa',K)\mathcal{M}_{is}(\kappa,\kappa') , \tag{55}$$

where E_0 is given by (39). The operators $\mathcal{M}_{is}(\kappa,\kappa')$ are some specific combinations of the pair operators and the pair number operators and are listed below. The Hamiltonian coefficients $\mathcal{H}_{is}$ can be determined by evaluating the matrix elements explicitly in the context of quark operators. However, we only need to evaluate the simplest nonvanishing matrix elements to determine these coefficients. We will demonstrate this by example later, following Wick's theorem. Equation (55) gives the correct recursion formulas for matrix elements, so that all the matrix elements between states of various occupancy can be written down at once, given the form of the $\mathcal{M}_{is}$ and the expressions for the $\mathcal{H}_{is}$.

In general, we can write the Hamiltonian coefficients as a product of several factors

$$\mathcal{H}_{is}(\kappa,\kappa',K)=\gamma\bar{G}_K D_s\mathcal{R}_s/\Delta_i . \tag{56}$$

The common strength factor γ is given as

$$\gamma=\frac{g^2G_1n_g}{4\pi^2n} , \tag{57}$$

where n_g is the number of exchanged vector particles; $n_g=n^2-1$ for SU(n) and $n_g=1$ for U(1). The quantity G_1 is the Coulomb propagator for exchanged momentum $K=1$. For the simplified model, this is the only term. Also, we have $\bar{G}_K=G_K/G_1$. The Dirac factors D_s are made up of combinations of the spinor amplitudes C given in Eq. (29). For massless fermions, we must have at each vertex either no change in helicity (fermion scattering) or opposite helicities (pair creation). Corresponding helicities must occur at both vertices, so the Hamiltonian term H_2 does not contribute, and in the nonvanishing terms, the label s takes on only two values $\pm$, with

$$D_\pm=1 \quad \text{for } h'=\pm h . \tag{58}$$

For each value of s, there are only three values of i.

The degeneracy factors Δ_i, which are listed below, arise from the normalization factors $N(\kappa)$ in the $M^\dagger$ as given in Eq. (41)

$$\begin{aligned}\Delta_1&=\kappa(\kappa+1) ,\\ \Delta_2&=n\kappa(\kappa+1)\kappa'(\kappa'+1) ,\\ \Delta_3&=\sqrt{\kappa(\kappa+1)\kappa'(\kappa'+1)} .\end{aligned} \tag{59}$$

We have absorbed one factor of n^{-1} into γ from the normalization terms.

The kinematical factor $\mathcal{R}_s$ is the product of the two reduced matrix elements obtained from Eq. (29). For massless fermions, there are only two different combinations of 6-j symbols consistent with the helicity constraints

$$\begin{aligned}S_+(\kappa',K,\kappa)&=\begin{Bmatrix}\frac{1}{2}\kappa & \frac{1}{2}\kappa' & \frac{1}{2}K\\ \frac{1}{2}(\kappa'-1) & \frac{1}{2}(\kappa-1) & \frac{1}{2}\end{Bmatrix}^2 ,\\ S_-(\kappa',K,\kappa)&=\begin{Bmatrix}\frac{1}{2}\kappa & \frac{1}{2}(\kappa'-1) & \frac{1}{2}K\\ \frac{1}{2}\kappa' & \frac{1}{2}(\kappa-1) & \frac{1}{2}\end{Bmatrix}^2 .\end{aligned} \tag{60}$$

Using the explicit algebraic expressions for the $S_\pm$, we obtain

$$\begin{aligned}\mathcal{R}_+(\kappa',K,\kappa)&=\tfrac{1}{4}(K+\kappa+\kappa'+2)(\kappa+\kappa'-K)(K+1) ,\\ \mathcal{R}_-(\kappa',K,\kappa)&=\tfrac{1}{4}(K+\kappa-\kappa'+1)(K-\kappa+\kappa'+1)\\ &\quad\times(K+1) .\end{aligned} \tag{61}$$

Note that the volume factor $1/\Omega$ from the integrals has been absorbed into γ. The six $\mathcal{M}_{is}$ are listed here

$$\mathcal{M}_{1-}=-(N_{\kappa+}+N_{\kappa-}) , \tag{62}$$

$$\mathcal{M}_{2-}=2N_{\kappa+}N_{\kappa'-} , \tag{63}$$

$$\mathcal{M}_{3-}=M(\kappa,+)^\dagger M(\kappa',-)^\dagger , \tag{64}$$

$$\mathcal{M}_{1+}=N_{\kappa+}+N_{\kappa-} , \tag{65}$$

$$\mathcal{M}_{2+}=-:(N_{\kappa+}N_{\kappa'+}+N_{\kappa-}N_{\kappa'-}): , \tag{66}$$

$$\mathcal{M}_{3+}=-[M(\kappa',+)^\dagger M(\kappa,+)+M(\kappa',-)^\dagger M(\kappa,-)] . \tag{67}$$

These operators arise as follows. The first term of H_{3b} in Eq. (38) is a self-energy contribution for the pair κ and provides the term $\mathcal{H}_{1+}\mathcal{M}_{1+}$ in (55), in which the factor $\mathcal{R}_+$ is to be summed over κ'. The second term in H_{3b} gives the self-energy contribution involving virtual pairs, $\mathcal{H}_{1-}\mathcal{M}_{1-}$. The first term of H_{3a} in Eq. (37) and the third term give rise to $\mathcal{H}_{2\pm}\mathcal{M}_{2\pm}$, respectively, and have the effect of reducing the self-energy when the intermediate states are partially filled. In the term $\mathcal{H}_{2+}\mathcal{M}_{2+}$, for $\kappa=\kappa'$, we have $:N^2:=N(N-1)$. The term H_1 of Eq. (31) creates two pairs containing opposite helicities and gives the term $\mathcal{H}_{3-}\mathcal{M}_{3-}$. (There is also the Hermitian-conjugate term.) Finally, the fermion-antifermion interaction term, the second term of H_{3a}, provides scattering of a pair with internal momentum κ into a pair with momentum κ', giving $\mathcal{H}_{3+}\mathcal{M}_{3+}$. Here, for $\kappa=\kappa'$, we replace $M_\kappa^\dagger M_\kappa$ by N_κ.

Although the structure of the effective Hamiltonian in Eq. (55) has a natural and expected general form, our algebraic derivation is not very simple or transparent. However, there are some obvious checks on the coefficients. The leading divergences in the $\mathcal{H}_{1s}$ cancel, and particle-hole symmetry relates the $\mathcal{H}_{1s}$ and $\mathcal{H}_{2s}$.

As examples, to demonstrate that Eqs. (63) and (64) give the correct operators for use in (55), we shall evalu-

ate the corresponding matrix elements of the Hamiltonian explicitly. Considering the forms of the operators $\mathcal{M}_{3-}$ and H_1, we can rewrite these two matrix elements as

$$I_1=\langle l_1',l_2',\ldots,l_m'|H_1|l_1,l_2,\ldots,l_m\rangle$$
$$=N_{[l']}N_{[l]}\langle 0|[M(\kappa_m',h_m')]^{l_m'}\cdots[M(\kappa_1',h_1')]^{l_1'}$$
$$\times[M(\kappa_1,h_1)^\dagger]^{l_1}\cdots[M(\kappa_m,h_m)^\dagger]^{l_m}H_1|0\rangle \quad (68)$$

and

$$\tilde{I}_1=\langle l_1',l_2',\ldots,l_m'|\tilde{H}_1|l_1,l_2,\ldots,l_m\rangle$$
$$=N_{[l']}N_{[l]}\langle 0|[M(\kappa_m',h_m')]^{l_m'}\cdots[M(\kappa_1',h_1')]^{l_1'}$$
$$\times[M(\kappa_1,h_1)^\dagger]^{l_1}\cdots[M(\kappa_m,h_m)^\dagger]^{l_m}\tilde{H}_1|0\rangle\ , \quad (69)$$

where the operator $\tilde{H}_1$ is

$$\tilde{H}_1=\sum_{\kappa,\kappa',K}\mathcal{H}_{3-}(\kappa,\kappa',K)\mathcal{M}_{3-}(\kappa,\kappa')\ . \quad (70)$$

Let us define $\lambda=\sum l_i'=2+\sum l_i$. If we commute $M(\kappa_i,h_i)^\dagger$ to the left in the expression I_1, we obtain

$$I_1=\lambda \text{ terms of the form } \delta_{\kappa_i,\kappa_j'}\delta_{h_i,h_j'}\langle 0|[M(\kappa_m',h_m')]^{l_m'}\cdots[M(\kappa_j',h_j')]^{l_j'-1}\cdots[M(\kappa_1',h_1')]^{l_1'}$$
$$\times[M(\kappa_1,h_1)^\dagger]^{l_1}\cdots[M(\kappa_i,h_i)^\dagger]^{l_i-1}\cdots[M(\kappa_m,h_m)^\dagger]^{l_m}H_1|0\rangle N_{[l]}N_{[l']}$$
$$+\tfrac{1}{2}\lambda(\lambda-1)\text{ terms of the form } -2N^2(\kappa_i,h_i)\delta_{\kappa_i,\kappa_j'}\delta_{\kappa_i,\kappa_l'}\delta_{h_i,h_j'}\delta_{h_i,h_l'}$$
$$\times\langle 0|M(\kappa_i,h_i)[M(\kappa_m',h_m')]^{l_m'}\cdots[M(\kappa_l',h_l')]^{l_l'-1}\cdots[M(\kappa_j',h_j')]^{l_j'-1}\cdots[M(\kappa_1',h_1')]^{l_1'}$$
$$\times[M(\kappa_1,h_1)^\dagger]^{l_1}\cdots[M(\kappa_i,h_i)^\dagger]^{l_i-1}\cdots[M(\kappa_m,h_m)^\dagger]^{l_m}H_1|0\rangle N_{[l]}N_{[l']}\ . \quad (71)$$

By commuting $M^\dagger$ and M, the first set of terms is obtained, along with terms containing an operator O. The second set of terms is obtained by commuting O and M. Thus Eq. (71) provides a recursion formula for the matrix element I_1. Since I_1 and $\tilde{I}_1$ have similar structures, $\tilde{I}_1$ will satisfy a similar recursion formula. By applying these recursion relations repeatedly, we see that it is sufficient to examine the two simple matrix elements

$$I_b=\langle 0|M(\kappa_1,h_1)M(\kappa_2,h_2)H_1|0\rangle,\quad \tilde{I}_b=\langle 0|M(\kappa_1,h_1)M(\kappa_2,h_2)\tilde{H}_1|0\rangle\ . \quad (72)$$

In the matrix element I_b, the color sum gives a numerical factor

$$\sum_{ijkl}\sum_{a=1}^{n^2-1}t_{ij}^at_{kl}^a\delta_{il}\delta_{jk}=\frac{n^2-1}{2}\ ; \quad (73)$$

in addition, using the delta functions $\delta_p(16)\delta_z(1,-6)\delta_p(3,4)\delta_z(-3,4)$ as well as phase factors in the pair operators, we obtain sums over the remaining 3-j symbols from Eq. (29) of the form

$$\sum_{\lambda\mu\nu}\begin{bmatrix}K_3 & \frac{1}{2}K & K_1\\ \lambda & \mu & \nu\end{bmatrix}^2=1\ . \quad (74)$$

Residual phase factors cancel, giving

$$I_b=\frac{g^2(n^2-1)}{8\pi^2}\sum_{\kappa,h}\sum_K\sum_{\kappa',h'}G_KN(\kappa)N(\kappa')\mathcal{R}_-(\kappa',K,\kappa)[C_1(\kappa',h')C_2(\kappa,h)-C_2(\kappa',h')C_1(\kappa,h)]^2$$
$$\times(\delta_{\kappa_2,\kappa}\delta_{h_2,h}\delta_{\kappa_1,\kappa'}\delta_{h_1,h'}+\delta_{\kappa_1,\kappa}\delta_{h_1,h}\delta_{\kappa_2,\kappa'}\delta_{h_2,h'})\ , \quad (75)$$

where $\mathcal{R}_-$ is given by Eq. (61). In the massless limit, we also have

$$[C_1(\kappa',h')C_2(\kappa,h)-C_2(\kappa',h')C_1(\kappa,h)]=\delta_{-h,h'}\ . \quad (76)$$

The matrix element I_b is invariant under interchange of κ with κ' and of h with h'. We can sum over the two helicities, and obtain

$$I_b=\sum_K\frac{g^2(n^2-1)G_K}{4n\pi^2}\sum_{\kappa,\kappa'}\frac{\mathcal{R}_-(\kappa',K,\kappa)}{\sqrt{\kappa(\kappa+1)\kappa'(\kappa'+1)}}(\delta_{\kappa_2,\kappa}\delta_{h_2,+}\delta_{\kappa_1,\kappa'}\delta_{h_1,-}+\delta_{\kappa_1,\kappa}\delta_{h_1,+}\delta_{\kappa_2,\kappa'}\delta_{h_2,-})\ . \quad (77)$$

For the matrix element $\tilde{I}_b$, we find

$$\tilde{I}_b=\sum_{\kappa,\kappa',K}\mathcal{H}_{3-}(\kappa,\kappa',K)\langle 0|M(\kappa_1,h_1)M(\kappa_2,h_2)\mathcal{M}_{3-}(\kappa,\kappa')|0\rangle$$
$$=\sum_{\kappa,\kappa',K}\mathcal{H}_{3-}(\kappa,\kappa',K)[\delta_{\kappa_2,\kappa}\delta_{h_2,+}\delta_{\kappa_1,\kappa'}\delta_{h_1,-}+\delta_{\kappa_1,\kappa}\delta_{h_1,+}\delta_{\kappa_2,\kappa'}\delta_{h_2,-}]\ , \quad (78)$$

which has the same form.

Similarly, to demonstrate that Eq. (63) is correct, we examine the matrix element of $H_{3a}(3)$. First, $\mathcal{M}_{2-}$ acting on a general state gives

$$
\begin{aligned}
|\tilde{S}\rangle &= 2N_{\kappa+}N_{\kappa'-}|l_1,l_2,\ldots,l_m\rangle \\
&= 2N_{[l]}l_1l_2[M(\kappa_1,h_1)^\dagger]^{l_1-1}[M(\kappa_2,h_2)^\dagger]^{l_2-1}[M(\kappa_3,h_3)^\dagger]^{l_3}\cdots[M(\kappa_m,h_m)^\dagger]^{l_m} \\
&\quad\times\{[\delta_{\kappa_2,\kappa}\delta_{h_2,+}\delta_{\kappa_1,\kappa'}\delta_{h_1,-}+\delta_{\kappa_1,\kappa}\delta_{h_1,+}\delta_{\kappa_2,\kappa'}\delta_{h_2,-}]M(\kappa_1,h_1)^\dagger M(\kappa_2,h_2)^\dagger\}|0\rangle \\
&\quad+2N_{[l]}l_1l_3[M(\kappa_1,h_1)^\dagger]^{l_1-1}[M(\kappa_2,h_2)^\dagger]^{l_2}[M(\kappa_3,h_3)^\dagger]^{l_3-1}\cdots[M(\kappa_m,h_m)^\dagger]^{l_m} \\
&\quad\times\{[\delta_{\kappa_3,\kappa}\delta_{h_3,+}\delta_{\kappa_1,\kappa'}\delta_{h_1,-}+\delta_{\kappa_1,\kappa}\delta_{h_1,+}\delta_{\kappa_3,\kappa'}\delta_{h_3,-}]M(\kappa_1,h_1)^\dagger M(\kappa_3,h_3)^\dagger\}|0\rangle+\cdots\}|0\rangle\ . \qquad (79)
\end{aligned}
$$

To find a similar relation for $H_{3a}(3)$, we need to consider the effect of $2b_j(3)d_i(1)$ only, which gives

$$
\begin{aligned}
|S\rangle &= 2b_j(3')d_i(1')|l_1,l_2,\ldots,l_m\rangle \\
&= -2N_{[l]}N_p(\kappa_1',h_1',\nu_1',m_1')\delta_{i,j}\delta_{\kappa_1',\kappa_3'}\delta_{h_1',h_3'}\delta_{-\nu_1',\nu_3'}\delta_{-m_1',m_3'} \\
&\quad\times\{l_1\delta_{\kappa_1',\kappa_1}\delta_{h_1',h_1}[M(\kappa_1,h_1)^\dagger]^{l_1-1}\cdots[M(\kappa_m,h_m)^\dagger]^{l_m}|0\rangle+\cdots\} \\
&\quad+2N_{[l]}N_p(\kappa_1',h_1',\nu_1',m_1')N_p(\kappa_3',h_3',-\nu_3',-m_3')b_i^\dagger(\kappa_1',h_1',-\nu_1',-m_1')d_j^\dagger(\kappa_3',h_3',-\nu_3',-m_3') \\
&\quad\times\{l_1(l_1-1)\delta_{\kappa_1',\kappa_1}\delta_{h_1',h_1}\delta_{\kappa_3',\kappa_1}\delta_{h_3',h_1}[M(\kappa_1,h_1)^\dagger]^{l_1-2}\cdots[M(\kappa_m,h_m)^\dagger]^{l_m}|0\rangle+\cdots\} \\
&\quad+2N_{[l]}N_p(\kappa_1',h_1',\nu_1',m_1')N_p(\kappa_3',h_3',-\nu_3',-m_3')b_i^\dagger(\kappa_1',h_1',-\nu_1',-m_1')d_j^\dagger(\kappa_3',h_3',-\nu_3',-m_3') \\
&\quad\times\{l_1l_2[\delta_{\kappa_2,\kappa_3'}\delta_{h_2,h_3'}\delta_{\kappa_1,\kappa_1'}\delta_{h_1,h_1'}+\delta_{\kappa_1,\kappa_3'}\delta_{h_1,h_3'}\delta_{\kappa_2,\kappa_1'}\delta_{h_2,h_1'}] \\
&\qquad\times[M(\kappa_1,h_1)^\dagger]^{l_1-1}[M(\kappa_2,h_2)^\dagger]^{l_2-1}[M(\kappa_3,h_3)^\dagger]^{l_3}\cdots[M(\kappa_m,h_m)^\dagger]^{l_m}|0\rangle+\cdots \\
&\qquad+l_{m-1}l_m[\delta_{\kappa_m,\kappa_3'}\delta_{h_m,h_3'}\delta_{\kappa_{m-1},\kappa_1'}\delta_{h_{m-1},h_1'}+\delta_{\kappa_{m-1},\kappa_3'}\delta_{h_{m-1},h_3'}\delta_{\kappa_m,\kappa_1'}\delta_{h_m,h_1'}] \\
&\qquad\times[M(\kappa_1,h_1)^\dagger]^{l_1}\cdots[M(\kappa_{m-1},h_{m-1})^\dagger]^{l_{m-1}-1}[M(\kappa_m,h_m)^\dagger]^{l_m-1}|0\rangle\}\ , \qquad (80)
\end{aligned}
$$

where

$$N_p(\kappa,h,\nu,m)=\eta(h,\nu,m)N(\kappa,h,\nu,m)\ . \qquad (81)$$

The first set of terms yields zero, because t_{ij}^a is traceless. The second set of terms will also give zero, because the integral $T_{+-}(321)$ has a factor

$$C_1(\kappa_3',h_3')C_2(\kappa_1',h_1')-C_2(\kappa_3',h_3')C_1(\kappa_1',h_1')\ . \qquad (82)$$

For massless fermions, this will vanish when $h_1'=h_3'$. Thus only the last set of terms will contribute.

For the second step in the discussion of the recursion relations, we consider the following operators, which arise from typical terms of Eqs. (79) and (80):

$$\tilde{H}_3(u,v)=\sum_{\kappa,\kappa',K}\mathcal{H}_{2-}(\kappa,\kappa',K)(\delta_{\kappa_v,\kappa}\delta_{h_v,+}+\delta_{\kappa_u,\kappa'}\delta_{h_u,-}+\delta_{\kappa_u,\kappa}\delta_{h_u,+}+\delta_{\kappa_v,\kappa'}\delta_{h_v,-})2M(\kappa_u,h_u)^\dagger M(\kappa_v,h_v)^\dagger \qquad (83)$$

and

$$
\begin{aligned}
H_{3a}(u,v)=-\sum{}^{*}\,&2N_p(\kappa_1',h_1',\nu_1',m_1')N_p(\kappa_3',h_3',-\nu_3',-m_3') \\
&\times(\delta_{\kappa_v,\kappa_3'}\delta_{h_v,h_3'}\delta_{\kappa_u,\kappa_1'}\delta_{h_u,h_1'}+\delta_{\kappa_u,\kappa_3'}\delta_{h_u,h_3'}\delta_{\kappa_v,\kappa_1'}\delta_{h_v,h_1'})b_k^\dagger(4')d_l^\dagger(6') \\
&\times b_i^\dagger(\kappa_1',h_1',-\nu_1',-m_1')d_j^\dagger(\kappa_3',h_3',-\nu_3',-m_3')T_{+-}(321)T_{+-}(426)\ , \qquad (84)
\end{aligned}
$$

where $u\equiv(\kappa_u,h_u)$. These operators are similar in form to the operators $\tilde{H}_1$ and H_1, which were already shown to satisfy similar recursion relations. It is, therefore, again sufficient to consider only the simplest matrix elements

$$
\begin{aligned}
\tilde{L}_b &= \langle 0|M(\kappa_3,h_3)M(\kappa_4,h_4)\tilde{H}_3(u,v)|0\rangle\ , \\
L_b &= \langle 0|M(\kappa_3,h_3)M(\kappa_4,h_4)H_{3a}(u,v)|0\rangle\ . \qquad (85)
\end{aligned}
$$

and the demonstration follows exactly the same lines as in Eqs. (72)–(78).

VI. PARITY

As we discussed earlier, the free Dirac equation is invariant under parity, and the Hamiltonian (14) is also invariant. It is not easy to see directly how this translates into the transformation properties of the Dirac spinors, because the spinor basis is also changed in a complicated way. However, to motivate our construction of the pair operators $M^\dagger$, we made use of the expected form of scalar and pseudoscalar operators. It is easy to see that we can define an effective parity operator P such that the perturbative vacuum is invariant under its action $P|0\rangle=|0\rangle$ and whose action on the pair-creation operators is, as required,

$$PM(\kappa,h)^\dagger P=-M(\kappa,-h)^\dagger . \tag{86}$$

The effective Hamiltonian (55) is invariant under the action of this operator. For a state which has a definite parity $\mathcal{P}$ and is represented in the form

$$\Psi=\sum_n B(n_{1+},n_{1-},\ldots) \times|n_{1+},n_{1-},\ldots,n_{\kappa+},n_{\kappa-},\ldots\rangle , \tag{87}$$

we have

$$B(n_{1+},n_{1-},\ldots)=\mathcal{P}(-1)^N B(n_{1-},n_{1+},\ldots) .$$

Four of the operators listed in Eqs. (62)–(67) are diagonal in the occupation numbers $n_{\kappa\pm}$. The other two operators have the property that the difference $Z=\sum(n_{\kappa+}-n_{\kappa-})$ remains constant, so Z is a good quantum number. The sum $N=\sum(n_{\kappa+}+n_{\kappa-})$ will remain either even or odd, depending on whether Z is even or odd. The parity operation gives $M(\kappa,\pm)\rightarrow -M(\kappa,\mp)$, and hence $Z\rightarrow -Z$. The energy of eigenstates will depend only on $|Z|$, so for $Z\neq 0$ there are pairs of degenerate states which form parity doublets. For $Z=0$, there are both scalar and pseudoscalar states. These are not degenerate, and there are also more scalar states; for example, the state with no pairs, and also the state with the maximum possible number of pairs, are scalar.

If we allow the mass of the quark to have a finite value, the helicity selection rules no longer hold. Then Z is no longer a good quantum number, and there are, in general, no parity doublets.

VII. NUMERICAL RESULTS

In this section, we present some numerical results for SU(3) obtained applying the analytical formulation derived above. We consider only the term in the Hamiltonian with $K=1$, as discussed earlier, and ignore the constant factor G_1. We assume that the coupling constant g is so large that we can ignore the free Hamiltonian, and concentrate on the Coulomb energy matrix.

TABLE I. Ratios of doublet energies to singlet energies.

Λ	Two pairs	Three pairs	Four pairs
1	1.385	1.329	1.326
2	1.100	1.059	1.064
3	1.045	1.015	1.019
4	1.030	1.002	1.011
5	1.020	0.998	

Since we consider only massless fermions, the Hamiltonian is invariant under the chiral transformation.

We evaluate the matrix elements of the Coulomb operator in several different sets of states. The Nth set of states includes states with zero, one, . . . , up to a maximum of N pairs. The largest value of N considered is $N=4$. In order to have a finite matrix, we introduce a cutoff Λ such that $\kappa\leq\Lambda$. Then we diagonalize the matrix to obtain energy levels.

We define E_S to be the lowest energy for $Z=0$. This state was found to be nondegenerate and to have even parity. It resembles the usual perturbative vacuum. The numerical calculations indicate that the lowest pseudoscalar state is a state with $|Z|\neq 0$. Therefore, there is a corresponding scalar state, with the same energy. We define the energy of this lowest doublet state to be E_D. In Table I we list values of the ratio E_D/E_S for different values of N and Λ. Notice that for $N=3$, when $\Lambda=5$, this ratio is less than one. In this case, one can interpret E_D as the vacuum energy, and the vacuum state is degenerate. The vacuum will not have a definite parity, which is a signal that the chiral symmetry is broken spontaneously. This kind of level crossing also occurs when $N=1$, but with $\Lambda=10$. It is apparent that when N and Λ are increased, the singlet and doublet energies become more similar. It is plausible that if either N or the cutoff Λ is big enough, a level crossing may occur.

As a result, we have some indication that the Coulomb energy may provide the underlying dynamical reason for the spontaneous chiral-symmetry breaking of QCD. However, much further work needs to be done. Along with Capstick and Joensen, we have carried out numerical calculations using larger sets of states, and also including the kinetic energy.[15] More general forms for the assumed Coulomb interactions were also considered, to study whether condensation depends on confinement. We should also include more flavors of quarks in the formulation. It will be interesting to study the effect of the quark mass. Eventually, we need to use a better model for the Hamiltonian, taking the gluons into account ex-

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