Supersymmetry, Foldy-Wouthuysen transformations, and relativistic oscillators

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Dirac Hamiltonians expressed in terms of supercharges are analyzed in connection with supersymmetric quantum mechanics. Unitary Foldy-Wouthuysen transformations lead to equivalent diagonalized Hamiltonians, functions of supersymmetric (nonrelativistic) ones. Such considerations permit the definition of relativistic oscillators described by Dirac equations.

I. INTRODUCTION

The observation[1] that the square of a Dirac Hamiltonian is intimately connected with a *nonrelativistic* supersymmetric Hamiltonian has already been exploited[2-4] in different contexts dealing, on the one hand, with *relativistic* (massless) particles in interaction with (constant) magnetic fields *and* relating, on the other hand, such subjects with recent results in supersymmetric quantum mechanics. Supersymmetric quantum mechanics[5] is particularly rich in information about nonrelativistic harmonic oscillators[6-12] in (n) arbitrary spatial dimensions (and specifically when n equals 1, 2, or 3) as well as about interactions with (constant) magnetic fields (in the $n=2$ case, for example[12,13]). Particle physics has also studied oscillatorlike quantum systems[14-22] but relativistic ones in connection with characteristics of hadronic mass spectra, linear Regge trajectories, quantum chromodynamical forces, relativistic strings, etc.

Through the above considerations relativistic and nonrelativistic physics can both profit from each other. In this direction the main purpose of this work is to show how the class of unitary Foldy-Wouthuysen (FW) transformations[23,24] has to play an interesting role by exploiting their original motivations. Let us mention that Hughes *et al.*[3] have already used an FW transformation[25] in order to discuss a realization of supersymmetric quantum mechanics in the standard first-order Dirac equation describing a (massless) Dirac particle in a magnetic field.

Here we recall after FW that an arbitrary Dirac Hamiltonian in the standard representation[23] is a sum of odd and even parts associated, respectively, with block-off-diagonal odd (block-diagonal even) matrices which (do not) couple the so-called large and small components. Then we observe that the even character of a supersymmetric Hamiltonian as well as the odd character of all the supercharges in supersymmetric quantum mechanics can be put in correspondence with the FW even and odd matrices showing that, in addition to their own interest (in studying nonrelativistic limits of Dirac descriptions), the FW ideas are well adapted in the language of superalgebra[26] underlying even and odd generators. Such a remark evidently absorbs Jackiw's observation[1] and extends it to nonzero-rest-mass particles.

In the following we want to point out some properties connecting supersymmetric quantum mechanics and Dirac theory (Sec. II). Restricting the context to $N=2$ supercharges, we discuss the free case and the harmonic oscillator within specific supersymmetrization procedures (Sec. III) and more particularly in the spin-orbit coupling considerations. In this way we get an *ad hoc* definition of Dirac Hamiltonians associated with *relativistic* quantum oscillators. Comments in connection with specific spatial dimensions are then presented as well as some information on (relativistic) symmetries admitted by the new Dirac equation(s) (Sec. IV). We finally (Sec. V) summarize the results and draw some conclusions.

II. SUPERSYMMETRY AND THE DIRAC-FW EQUIVALENCE

The description of a supersymmetric quantum-mechanical system[5] is subtended by *odd* supercharges Q^a ($a=1,2,\dots,N$) generating with the *even* Hamiltonian H_{SS} a superalgebra sqm(N) characterized by the structure relations

$$\{Q^a,Q^b\}=2\delta^{ab}H_{SS}\ ,\quad [Q^a,H_{SS}]=0\ ,\quad a,b=1,2,\dots,N\ . \tag{2.1}$$

The supersymmetric Hamiltonian appears as the square of an *arbitrary* supercharge among the N ones. Let us then define Dirac-like Hamiltonians as sums of odd and even parts where the odd part is given by *one* of the supercharges [up to a factor $\sqrt{2}$ introduced for convenience; see the definitions (9)] while the even part contains the mass term. We write

$$H_D=\sqrt{2}Q+\beta m \tag{2.2}$$

and choose the standard representation so that β is an even matrix classifying[23] the even $\mathcal{E}$ and odd $\mathcal{O}$ matrices of the corresponding Dirac (Clifford) algebra according to

$$[\mathcal{E},\beta]=0,\quad \{\mathcal{O},\beta\}=0\ . \tag{2.3}$$

Because of the odd character of the supercharges, we ask for

$$\{Q,\beta\}=0\ , \tag{2.4}$$

so that, through this condition, we immediately get

$$(H_D)^2=2[Q]^2+m^2=2H_{SS}+m^2\ . \tag{2.5}$$

Such a result corresponds to a meaningful change of representations in the first-order Dirac theory via a unitary FW transformation as

$$\begin{aligned} &H_{FW}=UH_DU^{-1}=\beta(2H_{SS}+m^2)^{1/2}\ ,\\ &[H_{FW}]^2=[H_D]^2\ . \end{aligned} \tag{2.6}$$

The unitary transformation leading to this physically equivalent representation is readily obtained on the form

$$\begin{aligned} &U=\exp(iS),\quad S=S^\dagger,\quad S=-\frac{i}{2}\beta QH^{-1}\theta\ ,\\ &\tan\theta=\sqrt{2}\frac{H}{m},\quad [\theta,\beta]=0,\quad \{H_D,S\}=0\ , \end{aligned} \tag{2.7}$$

where H is even and defined as the positive square root of the supersymmetric Hamiltonian. This transformation can also be written

$$U=\frac{E+\sqrt{2}\beta Q+m}{[2E(E+m)]^{1/2}},\quad E\equiv(2H_{SS}+m^2)^{1/2}\ . \tag{2.8}$$

III. THE $N=2$ CASE AND THE SPIN-ORBIT COUPLING PROCEDURE

Let us choose the $N=2$ context subtending an already large set of physical applications containing all the ones mentioned above and the harmonic-oscillator case in particular. Let us then recall some specifications of the $N=2$ supersymmetric quantum mechanics by considering the two Hermitian supercharges[11]

$$Q^1=\frac{1}{\sqrt{2}}(\mathbf{p}\cdot\boldsymbol{\varphi}^1+\nabla W\cdot\varphi^2),\quad Q^2=\frac{1}{\sqrt{2}}(\mathbf{p}\cdot\varphi^2-\nabla W\cdot\boldsymbol{\varphi}^1)\ , \tag{3.1}$$

where we refer to the superpotential $W(\mathbf{x})$ expressed as usual in terms of bosonic "variables" x_j canonically conjugated to the "momenta" p_k according to

$$[p_k,x_j]=-i\delta_{kj},k,j=1,\dots,n\ . \tag{3.2}$$

By defining the fermionic quantities

$$\xi^\pm=\tfrac{1}{2}(\boldsymbol{\varphi}^{-1}\pm i\boldsymbol{\varphi}^2),\quad (\xi^\pm)^\dagger=\xi^\mp\ , \tag{3.3a}$$

we can mainly distinguish[11] two particular procedures of supersymmetrization, i.e., the so-called *standard* procedure[5] characterized by

$$\{\varphi_j^a,\varphi_k^b\}=2\delta^{ab}\delta_{jk}\quad (a,b=1,2) \tag{3.3b}$$

and the so-called *spin-orbit coupling* procedure[27–30] characterized[11] by

$$\{\varphi_j^a,\varphi_k^a\}=2\delta_{jk},\quad \{\varphi_j^1,\varphi_k^2\}=2\Xi_{jk},\quad \Xi_{jk}=-\Xi_{kj}\ . \tag{3.3c}$$

In correspondence with Eqs. (3.3) we are led in sqm(2) to two types of supersymmetric Hamiltonians: with Eqs. (3.3a) and (3.3b) we get the standard one

$$H_{SS}^{ST}=\tfrac{1}{2}[\mathbf{p}^2+(\nabla W)^2]+\tfrac{1}{2}(\partial_j\partial_k W)[\xi_j^+,\xi_k^-] \tag{3.4a}$$

while, with Eqs. (3.3a) and (3.3c), we obtain the spin-orbit one

$$\begin{aligned} H_{SS}^{SO}=&\tfrac{1}{2}[\mathbf{p}^2+(\nabla W)^2]+\tfrac{1}{2}(\partial_j\partial_k W)[\xi_j^+,\xi_k^-]\\ &-\tfrac{1}{2}[(\partial_j W)p_k-(\partial_k W)p_j]\Xi_{jk}\ . \end{aligned} \tag{3.4b}$$

Let us now restrict the spatial context to *three* dimensions ($n=3$, $j,k=1,2,3$) as the more natural case in connection with relativistic wave equations such as the Dirac one. Then let us consider two specific applications: the free case and the harmonic oscillator.

A. The free case

The *free* case evidently corresponds to a null superpotential in both contexts and to an *identical* supersymmetric Hamiltonian obtained from supercharges of the following type:

$$Q^{(0)}\equiv\frac{1}{\sqrt{2}}\boldsymbol{\alpha}\cdot\mathbf{p}\ . \tag{3.5}$$

The Hamiltonian (2.2) becomes identical to the free Dirac Hamiltonian and the transformation (2.7) or (2.8) identical to the original (free) FW transformation leading to the FW Hamiltonian (2.6):

$$H_{FW}^{(0)}=\beta E_p,\quad E_p=(\mathbf{p}^2+m^2)^{1/2}\ . \tag{3.6}$$

Such a result is the interesting Hamiltonian leading to the (free) nonrelativistic Schrödinger Hamiltonian $\mathbf{p}^2/2m$ (if $|\mathbf{p}|\ll m$), a purely bosonic Hamiltonian which is supersymmetric and issued from a Dirac Hamiltonian.

B. The harmonic oscillator

If we are interested in the harmonic-oscillator context, the superpotential is

$$W(\mathbf{x})=\tfrac{1}{2}m\omega x^2 \tag{3.7}$$

and the supercharges are constrained by the fermionic quantities $\boldsymbol{\varphi}^a$ ($a=1,2$) such that, for example, Eqs. (3.3a) and (3.3c) are satisfied. A corresponding convenient choice is given by

$$\begin{aligned} &\varphi_j^1\equiv\sigma_j\otimes\sigma_1=\alpha_j\ ,\\ &\varphi_j^2\equiv\sigma_j\otimes\sigma_2=i\varphi_j^1(\sigma_0\otimes\sigma_3)=i\alpha_j\beta\ , \end{aligned} \tag{3.8}$$

leading to

$$\Xi_{jk}=\frac{i}{2}[\sigma_j,\sigma_k]\otimes\sigma_3=-\Xi_{kj} \tag{3.9}$$

and to

$$\{\xi_j^+,\xi_k^-\}=\delta_{jk}I-i\Xi_{jk}\ . \tag{3.10}$$

The resulting Hamiltonian (3.4b) becomes

$$H_{SS(HO)}^{SO}=\tfrac{1}{2}(\mathbf{p}^2+m^2\omega^2\mathbf{x}^2)+\tfrac{1}{2}m\omega(3\sigma_0+2\mathbf{L}\cdot\boldsymbol{\sigma})\otimes\sigma_3\ , \tag{3.11}$$

and coincides with Balantekin's result.[28] Here it thus appears in the result (2.6) when the FW transformation (2.7) or (2.8) is expressed in terms of one of the supercharges (3.1) with the above characteristics.

For these reasons, we define the Dirac Hamiltonians for the *relativistic* quantum oscillator as the ones given by our Eq. (2.2) expressed in terms of a corresponding supercharge $Q\equiv(3.1)$ characterized by the relations (3.7) and (3.8). We thus get two equivalent proposals leading to the Hermitian Dirac Hamiltonians

$$H_{D,1}^{\mathrm{HO}}=\boldsymbol{\alpha}\cdot(\mathbf{p}+im\omega\mathbf{x}\beta)+\beta m \tag{3.12}$$

and

$$H_{D,2}^{\mathrm{HO}}=\boldsymbol{\alpha}\cdot(i\mathbf{p}\beta+m\omega\mathbf{x})+\beta m \ . \tag{3.13}$$

Both are linear in the bosonic operators (3.2) and lead to quadratic nonrelativistic terms (as expected for the harmonic oscillator).

The version (3.12) gives a Dirac equation already proposed a long time ago by Cook[17] and recently reactualized by Moshinsky and Szczepaniak[31] and collaborators.[32] In fact, it is easy to convince ourselves that our unitary FW transformation (2.7) or (2.8) expressed in terms of the supercharge Q^1 coincides, in this particular context, with the one obtained by Moreno and Zentella[32] in order to get the nonrelativistic limit given by Cook. The second version (3.13) could evidently be exploited in a completely parallel way.

Let us end this section by noticing that the *covariant* form of the corresponding Dirac equations can easily be obtained from the above characteristics. For example, we get, with the first supercharge (3.1) and the superpotential (3.7),

$$\begin{aligned} i\partial_t\Psi(x)&=(\sqrt{2}Q^1+m\beta)\Psi(x)\\ &=[\varphi_j^1p_j+\varphi_k^2(\partial_k W)+m\beta]\Psi(x)\ , \end{aligned}\tag{3.14}$$

$$i\partial_t\Psi(x)=(\boldsymbol{\alpha}\cdot\boldsymbol{\Pi}+\beta m)\Psi(X),\quad \boldsymbol{\Pi}\equiv\mathbf{p}+im\omega\mathbf{x}\beta\ . \tag{3.15}$$

This Hamiltonian form immediately leads inside the current relativistic conventions[24] to the *covariant* equation

$$\{i\gamma^\mu\partial_\mu[I-\gamma^0W(\mathbf{x})]-m\}\Psi(x)=0\ , \tag{3.16}$$

equivalent to the one already obtained by Cook[17,32] in the oscillator case.

IV. COMMENTS

The discussion in the preceding section has been developed essentially for three spatial dimensions in the spin-orbit coupling supersymmetrization procedure. Let us *first* notice that it is possible to show from an algebraic point of view[33] that the typical requirements (3.3a) *and* (3.3c) on the *fermionic* quantities, realized by the matrices (3.8) and (3.9), lead to the unitary Lie superalgebra su(2|2) admitting a 4×4 fundamental representation just convenient for studying the simplest Dirac theory. Such a discussion is well adapted in order to understand that the parallel consideration of the *standard* procedure of supersymmetrization—remember Eqs. (3.3a) and (3.3b) and the Hamiltonian (3.4a)—leads to a possible eight-dimensional matrix realization when three spatial dimensions are still required. Indeed, it is clear that Eq. (3.3b) and the matrix β inform us we are dealing with a Clifford algebra Cl_{2n+1} which only gives us with $n=3$ an eight-dimensional nontrivial representation.[34,11] The corresponding relativistic Hamiltonian would be an 8-×-8 Dirac-like Hamiltonian, but all our developments can still apply. The number of spatial dimensions and the supersymmetrization procedure have thus to play an important role in connection with matrix dimensions. If $n=2$, the standard procedure is subtended by a Clifford algebra Cl_4 and its 4-×-4 matrices, so that the fundamental Dirac representation appears one more time here while the spin-orbit procedure picks out 2-×-2 Pauli matrices. These last properties justify *a posteriori* the Hughes *et al.*[3] and Jackiw[1] realizations, respectively. Let us also stress on a physical interest of this $n=2$ case. In fact, it is well known[35,12,13] that there is a one-to-one correspondence between the $n=2$ harmonic oscillator and a particle confined to a plane in a constant magnetic field, this last system being relevant[3] to the study of the quantized Hall effect when the mass is taken to be zero. Our results could thus, in principle, be applied to all these three contexts with their specificities. Finally, if $n=1$, the procedures are identical and the usual Clifford algebra Cl_2 is responsible for the matrix dimension.

As a *second* comment let us remember that the spin-orbit coupling procedure,[27,30] initially applied to three spatial dimensions, has been generalized to arbitrary n dimensions by Kostelecky *et al.*[36] Its connection with the standard procedure in the oscillator case (and, in particular, through the Kostelecky *et al.* contribution) has been already analyzed (see Secs. 2.3 and 3.4 in Ref. 12), explaining different matrix dimensions recovered here as discussed above. Let us also remark that the other procedure given by Kostelecky *et al.*[36] consists to supersymmetrize the radial equation[37] associated with an n-dimensional harmonic oscillator (or any other separable system), so that only one variable is still significant. Consequently, this method essentially reduces to the $n=1$ standard treatment and does not need to be more developed here.

A final comment concerns the (super)symmetries admitted by these equations for *relativistic* oscillators. Because of the interaction term, the Poincaré invariance is evidently broken and we have to determine what are the effective symmetries of Eq. (3.15), for example. By asking for a general operator X such that[38]

$$[i\partial_t-(\boldsymbol{\alpha}\cdot\boldsymbol{\Pi}+\beta m),X]=0\ , \tag{4.1}$$

we find, in addition to the identity operator, *five* nontrivial generators corresponding to the (expected) rotational and time-translational invariances supplemented by a new generator which can be put in the form

$$Y\equiv(\mathbf{L}\cdot\boldsymbol{\sigma}+\tfrac{3}{2}\sigma_0)\otimes\sigma_3-\tfrac{1}{2}\sigma_0\otimes\sigma_3\ . \tag{4.2}$$

It corresponds to an explicit supersymmetry contained in the largest invariance superalgebra obtained by Balantekin[28] for the supersymmetric equation associated with the

Hamiltonian (3.11). This superalgebra[28,11] is $osp(2|2) \oplus so(3)$ and the operator Y has to generate the subalgebra so(2) of the even part of osp(2|2). The invariance structure is thus the direct sum $\{J_1, J_2, J_3\} \oplus \{H_D, Y\} \oplus I$ seen as a substructure of $so(3) \oplus osp(2|2) \oplus I$. As already mentioned in Durand *et al.*[4] but for *massless* particles, we recover constants of motion for nonrelativistic (supersymmetric) problems from the symmetries of Dirac equations. In fact, as in our case, we are dealing with nonzero rest mass particles, it is easy to show that condition (4.1) expressed in terms of one supercharge, let us call it Q, requires the constraint (when $m \neq 0$)

$$\{Q, [\beta, X]\} = 0 . \tag{4.3}$$

Then all the symmetries of the Dirac Hamiltonian are symmetries of the supersymmetric Hamiltonian H_{SS} related to H_D by Eq. (2.6), for example. Our five previously determined operators trivially satisfy Eq. (4.3).

V. SUMMARY AND CONCLUSIONS

Relativistic nonzero-rest-mass particle descriptions such as Dirac wave equations are considered and studied in connection with nonrelativistic information issued from supersymmetric quantum mechanics applied to harmonic-oscillatorlike systems. The main discussion concerns Hamiltonian operators in both contexts when three spatial dimensions are concerned in direct connection with Minkowski space-time developments. The link between the associated Hamiltonians is realized via unitary Foldy-Wouthuysen transformations having the original motivation to get even matrices or diagonal transformed Hamiltonians: the FW transformation eliminates the odd part of the Dirac Hamiltonian (readily expressed in terms of the odd supercharge) and leads to an even result explicitly given in terms of the corresponding nonrelativistic supersymmetric Hamiltonian. Applied to the three-dimensional harmonic oscillator through the spin-orbit coupling procedure of supersymmetrization, the Balantekin Hamiltonian is recovered and related with a relativistic Dirac Hamiltonian which is linear in the conjugated (three) pairs of bosonic operators [see Eq. (3.12), for example]. We thus have the opportunity to define from such a study the notion of relativistic harmonic oscillators through associated relativistic Dirac wave equations such as Eqs. (3.15) in Hamiltonian form and (3.16) in covariant form. Finally we have determined the Poincaré subsymmetries of such equations: they only admit spatial-rotational and time-translational usual Lie invariances.

Polarized by oscillator problems, our developments have been here subtended by very simple assumptions. Let us, for example, mention that the form (2.2) for the relativistic Dirac Hamiltonian could be easily generalized by including other *even* terms in addition to the mass term. The contexts of magnetic and electromagnetic interactions having already been considered in the FW extensions (see the nice review article by De Vries[39] and references therein) apply also here in principle but this is not the purpose of this paper. Open further problems also appear in connection with other successful physical applications treated in supersymmetric quantum mechanics (such as the Coulomb problem, the hydrogen atom, the quantum Hall effect, other types of potentials, etc.[3,13,40]) but these are once again developments which, maybe, have the merit to be considered elsewhere. The main physical relevance of this study lies in the connection between relativistic (Dirac) developments with supersymmetric (nonrelativistic) quantum mechanics.

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