# Notes on solutions to the (2+1)-dimensional topologically massive Yang-Mills gauge field theories

C. H. Oh*
*Department of Theoretical Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP, England*
(Received 26 October 1989)

We demonstrate that the Bessel function solutions obtained prevoiusly are related to the (2+1)-dimensional vortex solutions when the Higgs fields are absent. Time-dependent solutions are also exhibited.

## I. INTRODUCTION

Physics in (2+1)-dimensional spacetime has some bizarre character,[1] e.g., fractional statistics is allowed and may be relevant in the understanding of high-$T_c$ superconductivity. Adding a Chern-Simons (CS) term to the Yang-Mills (YM) action will render the gauge field particle massive without violating the principle of local invariance.[2] For the SU(2) gauge group, the action is

$$S=\int d^3x(L_{\rm YM}+L_{\rm CS})\ , \tag{1a}$$

$$L_{\rm YM}=-\tfrac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a\ , \tag{1b}$$

$$L_{\rm CS}=(\xi/2)\epsilon^{\mu\nu\alpha}(\partial_\mu A_\nu^a A_\alpha^a+\tfrac{1}{3}\epsilon^{abc}A_\mu^a A_\nu^b A_\alpha^c)\ , \tag{1c}$$

where for convenience we set the gauge field coupling constant $g=1$ and the metric is $g_{\mu\nu}=(-++)$. The equations of motion can be written as

$$D_\mu F^{\mu\nu}=J^\nu\ , \tag{2a}$$

$$J^\nu\equiv-\xi\tilde{F}^{\,\nu}\equiv-(\xi/2)\epsilon^{\nu\alpha\beta}F_{\alpha\beta}\ , \tag{2b}$$

$$D_\nu J^\nu=D_\nu\tilde{F}^{\,\nu}=0\ . \tag{2c}$$

Equation (2a) is intriguing since the "external" source current of the YM field is given by the dual of the field strength itself which is covariantly conserved by virtue of the Bianchi identity (2c). In the absence of the CS term ($\xi=0$), Eq. (2c) is just a kinematic statement. The CS term couples the pure YM equation with the Bianchi identity so that Eq. (2c) is now a constraint equation for the source current $J^\nu$ of the YM field. In contrast with the (3+1)-dimensional case, the source current there is usually expressed in terms of the matter (fermionic or bosonic) fields or is simply a given fixed external source. Classical solutions play a preliminary role in our understanding of the quantized theories[3] and it will be interesting to see whether solitonlike solutions can be found for Eqs. (2). In Ref. 4 numerical solutions of Eqs. (2) were obtained and in Ref. 5 we constructed some exact analytical solutions. The first family of the solutions given in Ref. 5 is valid in Minkowski and Euclidean spacetime and is expressed in terms of the modified Bessel function of the third kind $K_\alpha$:

$$A^{ai}=[-\varphi^a\psi_2(x)+\delta_3^a/\rho]\varphi^i\ , \tag{3a}$$

$$A_0^a=\varphi^a\psi_1(x)\ , \tag{3b}$$

$$\psi_1(x)=dK_0(\rho),\quad d=\text{const}\ , \tag{3c}$$

$$\psi_2(x)=dK_1(\rho)\ , \tag{3d}$$

where $\rho=(x_1^2+x_2^2)^{1/2}$ and $\varphi^a$ denotes a unit vector:

$$\varphi^a=\epsilon^{ai}x^i/\rho\equiv\epsilon^{ai}n^i,\quad i=1,2 \tag{4}$$

with $\varphi^3=0$. The second family of solutions, valid only in Euclidean spacetime, is given by Jacobi's elliptic functions $E(u)$ with $U=\beta_\mu x^\mu$:

$$A_\mu^a=[(\alpha^a\alpha_\mu+\gamma^a\gamma_\mu)e+(\gamma^a\alpha_\mu-\alpha^a\gamma_\mu)f]E(u)-i\beta^a\beta_\mu(\xi/2)\ , \tag{5}$$

where $e$ and $f$ are constants and $\alpha,\beta,\gamma$ are mutually perpendicular unit vectors. Solutions (5) are periodic since the elliptic functions $E(u)$ are periodic in their arguments. They are however not valid as wavelike solutions in Minkowski spacetime because one of the three unit vectors $\alpha,\beta,\gamma$ must then be timelike which would lead to inconsistency.

The purpose of this paper is first to demonstrate that the first family of solutions in terms of the Bessel function given in Ref. 5 are related to the vortex solutions found in Refs. 6 and 7. Second we exhibit a new ansatz to obtain time-dependent solutions of Eqs. (2) in Minkowski spacetime. The solutions lead to zero action and vanishing energy-momentum tensor but are unfortunately complex and they can probably be understood as real solutions if one complexifies the gauge group SU(2) to SL(2,$C$) (Ref. 8). Finally we end with some remarks.

## II. THE BESSEL FUNCTION SOLUTIONS

For the solutions (3) the field strengths are given by

$$E_i^a=F_{0i}^a=\xi\, d\varphi^a n_i K_1(z),\quad z=\xi\rho\ , \tag{6a}$$

$$B^a=\tfrac{1}{2}\epsilon^{ij}F_{ij}^a=-\xi\, d\varphi^a K_0(z)\ . \tag{6b}$$

The magnetic field has the right asymptotic behavior for the vortex configuration[9] at large distances. This naturally leads one to suspect that solutions (3) are related to the vortex solutions found in Refs. 6 and 7 and we show below that they are in fact gauge related when the Higgs fields vanish.

In the absence of the Higgs fields the ansatz of Refs. 6 and 7 is

$$A^{ai}=-\delta_3^a\varphi^i A(\rho)/\rho, \quad A_0^a=\delta_3^a A_0(\rho) . \tag{7}$$

The YM equations with the CS term, Eq. (2), then become

$$B''-B'/\rho=-\xi\rho A_0' , \tag{8a}$$

$$A_0''+A_0'/\rho=-\xi B'/\rho , \tag{8b}$$

where $B=1+A(\rho)$ and the prime indicates differentiation with respect to $\rho$. These equations are those of Refs. 6 and 7 when the Higgs fields are set to zero and can easily be solved to give

$$B=(d\rho)K_1(z) , \tag{9a}$$

$$A_0=dK_0(z) . \tag{9b}$$

To see solution (9) is indeed related to solution (3) we apply a gauge transformation $U(X)=\exp(i\pi n_i\sigma^i/4)$ to solution (3):

$$A_\mu'=A_\mu'^a\sigma^a/2=UA_\mu U^{-1}+i\partial_\mu UU^{-1} , \tag{10}$$

where $\sigma^a$ are the Pauli matrices. Since

$$U\varphi^a\sigma^a U^{-1}=\sigma^3 , \tag{11}$$

we obtain, after some straightforward computation,

$$A'^{ai}=-\delta_3^a\varphi^i(\psi_2-1/\rho) , \tag{12a}$$

$$A_0'^a=\delta_3^a\psi_1 , \tag{12b}$$

which is just the solution (9). Thus the Bessel function solutions found in Ref. 5 are in fact the vortex solutions. They are actually Abelian solutions embedded in the SU(2), and valid everywhere except at the origin where the presence of a point source (Dirac delta function) is required.

## III. TIME-DEPENDENT SOLUTIONS

Many wavelike solutions have been found for YM equations in (3+1)-dimensional spacetime,[10] hence it is interesting to see whether the (2+1)-dimensional YM equations with the CS term also admit wavelike solutions. The Jacobi function solutions of Ref. 5 can be regarded as periodic solutions in Euclidean spacetime but we have so far been unable to construct non-Abelian progressive wave solutions for Eq. (2). However time-dependent solutions can be derived. Our starting point is to modify the ansatz (3) so that it cannot be gauge rotated to the Abelian form.

Replacing the $\varphi^a$ in the ansatz (3) by a null vector $\theta^a$ we arrive at the ansatz

$$A^{ai}=(\theta^a\bar{R}(\rho,x^0)+\delta_3^a/\rho)\varphi^i-i\varphi^a n^i H(\rho) , \tag{13a}$$

$$A_0^a=\theta^a\bar{I}(\rho,x^0) , \tag{13b}$$

$$\theta^a=n^a+i\delta_3^a, \quad \theta^a\theta^a=0 . \tag{13c}$$

Substituting the above ansatz into Eqs. (2), we get reduced coupled nonlinear equations for the unknown functions $\bar{R}$, $\bar{I}$, and $H$. Setting

$$\bar{R}(\rho,x^0)=\omega(z^0)R(\rho), \quad z^0=\xi x^0 , \tag{14a}$$

$$\bar{I}(\rho,x^0)=\omega(z^0)I(\rho) , \tag{14b}$$

the reduced equations can be simplified tremendously to yield

$$-I'+HI=\xi R , \tag{15a}$$

$$L'=HL , \tag{15b}$$

with

$$L(\rho)=R'+R/\rho-HR , \tag{16a}$$

$$\omega(z^0)=a\cos z^0+b\sin z^0, \quad a,b=\text{const} . \tag{16b}$$

We observe that ansatz (13) can reduce the YM equation (2) to effectively two coupled first-order differential equations, namely, Eq. (15a) and $L=0$, a special case of Eq. (15b). To obtain solution for Eqs. (15), we first express the function $L$ in Eq. (15b) in terms of the function $H(\rho)$, then solve for the function $R$ from Eq. (16a) and finally the function $I$ from Eq. (15a). The result is

$$L=c_1\exp\left[\int d\rho\, H(\rho)\right] , \tag{17a}$$

$$R=(c_1\rho/2+c_2/\rho)\exp\left[\int d\rho\, H(\rho)\right] , \tag{17b}$$

$$I=-\xi(c_1\rho^2/4+c_2\ln\rho+c_3)\exp\left[\int d\rho\, H(\rho)\right] , \tag{17c}$$

where $c_1$, $c_2$, and $c_3$ are arbitrary constants. Expressions (17a)–(17c) and (16b) give the gauge field potential $A_\mu^a$ via the ansatz (13) which is then a solution of Eq. (2) in terms of the function $H(\rho)$.

The field strengths for the ansatz (13) can be written as

$$E^{ai}=\theta^a(\varphi^i d\omega/dx^0+\xi\omega n^i)R , \tag{18a}$$

$$B^a=-\theta^a\omega L . \tag{18b}$$

As $\theta^a$ is a null vector, we necessarily have

$$E_i^aE_j^a=F_{ij}^aF_{lm}^a=0 \tag{19}$$

although individually $F_{\mu\nu}^aF_{\alpha\beta}^a$ (no sum over $a$) does not vanish. Note that if the arbitrary constant $c_1$ is set to zero, the magnetic field strength $B^a$ vanishes and if furthermore $c_2$ is also set to zero, then the electric field strength $E^{ai}$ also vanishes and hence solution (17) becomes trivial. Because of the result (19), the *YM* action, the energy-momentum tensor, and the angular momentum all vanish. We proceed to compute the total charge of the system from the solution (17). The total charge density is given by $\partial_i E^{ai}$ and can be simplified to

$$\partial_i E^{ai}=[\theta^a\omega(z^0)(R'+R/z)+(\varphi^a R/z)d\omega(z^0)/dz^0]\xi^2 , \tag{20}$$

where $R'=dR/dz$. If we project along the direction $\theta^a$ in the internal group space, the charge is zero. Projecting along the direction $\varphi^i$ $(i=1,2)$, the total charge is

$$Q=\int d^2x\, \varphi^a \partial_i E^{ai}$$
$$=2\pi[d\omega(z^0)/dz^0]\int_0^\infty dz\, R(z)\ . \quad (21a)$$

By a suitable choice of the function $H(\rho)$, $Q$ is finite. Note that $Q$ here is time dependent and real. However if we project along a direction perpendicular to $\varphi^i$, say $\delta_3^a$, then $Q$ becomes imaginary and can be made finite:

$$Q=2\pi i\omega(z^0)\int_0^\infty d(zR)\ . \quad (21b)$$

## IV. REMARKS

We end with some comments.

(a) The Bessel function solution (3) can be modified to be valid in Euclidean spacetime. This Euclidean solution yields a zero action. As we discussed earlier, (3) is in fact a vortexlike solution. The physical interpretation of the time-dependent solution (13) is however not clear at the moment and it may not have any relation to solution (3) although it also leads to a zero action. In passing we note that time-dependent solutions of (3+1)-dimensional YM equations related to the Wu-Yang monopole[11] have been given in Ref. 12.

(b) Consider the Bessel function solution in the Abelian gauge frame (7). Then Eqs. (2) become

$$\partial_\mu F^{\mu\nu}=J^\nu\ , \quad (22a)$$

$$\partial_\nu J^\nu=0\ . \quad (22b)$$

For the solution (9), the continuity equation (22b) is

$$\epsilon_{ij}\partial^i E^{aj}=0\ , \quad (23a)$$

$$E^{aj}=\delta_3^a n^j \xi\, dK_1(z)\ . \quad (23b)$$

At $z\simeq 0$ we have

$$E^{aj}\simeq \delta_3^a n^j(\xi d/z) \quad (24)$$

so that Eq. (23a) is not satisfied near the origin. In other words, near the origin there is a violation of conservation of "charge."

(c) In Refs. 6 and 7 it was claimed that finite-energy vortices can have electric charge. We wish to stress here that the electric charge referred to is in fact that of the matter fields (Higgs fields), it is not the total charge of the whole system consisting of the YM fields and the Higgs fields (together with the CS term) which must be zero because of the finite-energy requirement. When Higgs fields are present, the equations of motion (2) become

$$D_\mu F^{\mu\nu}=-\xi\tilde{F}^{\,\nu}+J_m^\nu\ , \quad (25)$$

where $J_m^\nu$ is the current contributed by the Higgs fields. For the ansatz of Refs. 6 and 7 the gauge field potential $A_\mu^a$ is along the direction $\delta_3^a$; hence, the time component of Eq. (25) is

$$(\partial_i E^{ai})\delta_3^a=(J_m^{a0}-\xi F^{a0})\delta_3^a\ . \quad (26)$$

Because of the finite-energy requirement, $E^{ai}$ has to vanish faster than $\rho^{-2}$ at large distances. This leads to the result that the total charge of the whole system given by

$$\int d^2x\, \partial_i E^{ai}\delta_3^a$$

must vanish. Thus on integrating Eq. (26) over all space, one has

$$Q_m\equiv\int d^2x\, J_m^{a0}\delta_3^a=\xi\int d^2x\, B^a\delta_3^a\ . \quad (27)$$

That is, due to the presence of the CS term, the total electric charge of the Higgs fields is the same as the magnetic flux, and that it is discrete is due to the fact that the magnetic flux is quantized: $\Pi_1(SU(2)/Z_2)=Z_2$. We emphasize that there is no conserved Noether charge for the whole system (YM+Higgs boson+CS) since the (continuous) symmetry is completely broken in order to have topologically stable solutions. From Eq. (27) we observe that any solution which leads to nonzero magnetic flux will provide a nonzero electric charge for the Higgs fields but this in general will not be discrete. For comparison we note that the electric charge of the (3+1)-dimensional dyon[13] is in fact that of the whole system (YM+Higgs boson) and is given by

$$\int d^3x\, \partial_i F^{i0}\ .$$

Furthermore this is a Noether charge due to the U(1) symmetry and needs not be discrete at the classical level. One does not have a corresponding CS term in (3+1)-dimensional spacetime which could relate the electric charge of the matter fields to the magnetic flux.

(d) The total action for solution (17) is in general not zero, but can be easily made to vanish by setting, say, $H(\rho)=-\rho$, $c_2=c_3=0$ in Eqs. (17).

## ACKNOWLEDGMENTS

We are grateful to C. P. Soo for many helpful discussions and L. C. Sia for assistance. We thank Professor R. H. Dalitz and Department of Theoretical Physics, University of Oxford for the kind hospitality.

---

*Permanent address: Physics Department, National University of Singapore, Singapore 0511.

[1]R. MacKenzie and F. Wilczek, Int. J. Mod. Phys. A **3**, 2827 (1988).

[2]R. Jackiw and S. Templeton, Phys. Rev. D **23**, 2291 (1981); J. Schonfeld, Nucl. Phys. **B185**, 157 (1981); S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982); Ann. Phys. (N.Y.) **140**, 372 (1982).

[3]R. Jackiw, Rev. Mod. Phys. **49**, 681 (1977); A. Actor, *ibid.* **51**, 461 (1979).

[4]E. D'Hoker and L. Vinet, Ann. Phys. (N.Y.) **162**, 413 (1985).

[5]C. H. Oh, L. C. Sia, and R. Teh, Phys. Rev. D **40**, 601 (1989).

[6]H. J. de Vega and F. A. Schaposnik, Phys. Rev. Lett. **56**, 2654 (1986); Phys. Rev. D **34**, 3206 (1986); G. Lozano, M. V. Manias, and F. A. Schaposnik, *ibid.* **38**, 601 (1988).

[7]S. K. Paul and A. Khare, Phys. Lett. B **174**, 420 (1986); C. N.

Kumar and A. Khare, *ibid.* **178**, 395 (1986); Phys. Rev. D **36**. 3253 (1987); S. K. Paul, *ibid.* **35**, 3280 (1987).

[8]T. T. Wu and C. N. Yang, Phys. Rev. D **13**, 3233 (1976).

[9]H. B. Nielsen and P. Olesen, Nucl. Phys. **B61**, 45 (1973). Note that the last phrase after Eq. (19) of Ref. 5 should be "our solutions are vortexlike."

[10]S. Coleman, Phys. Lett. **70B**, 59 (1977); A. Actor, Lett. Math. Phys. **2**, 275 (1978); W. B. Campbell and T. A. Morgan, Phys. Lett. **84B**, 87 (1979); R. Casalbouoni, G. Domokos, and S. Kövesi-Domokos, *ibid.* **81B**, 45 (1979); E. Kovas and S. Y. Lo, Phys. Rev. D **19**, 3649 (1979); S. Y. Lo, P. Desmond, and E. Kovacs, Phys. Lett. **90B**, 419 (1980); C. H. Oh and R. Teh, *ibid.* **87B**, 83 (1979); J. Math. Phys. **26**, 844 (1985).

[11]T. T. Wu and C. N. Yang, in *Properties of Matter Under Unusual Conditions,* edited by M. Mark and S. Fernbach (Interscience, New York, 1969), pp. 349–354.

[12]H. Arodz, Phys. Rev. D **27**, 1903 (1983).

[13]B. Julia and A. Zee, Phys. Rev. D **11**, 2227 (1975).