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Null-plane quantization of fermions

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Massive Dirac fermions are canonically quantized on the null plane using the Dirac-Bergmann algorithm. The procedure is carried out in the framework of quantum electrodynamics as an illustration of a rigorous treatment of interacting fermion fields.

I. INTRODUCTION

With the goal of calculating the spectrum of states in relativistic field theory, one line of research consists in working in a null-plane frame. (For our notations concerning light-cone coordinates, see Appendix A.) The advantage of doing so is that the longitudinal-momentum operator P^+ is positive semidefinite; hence, the physical vacuum is identical to the bare Fock vacuum (P^+ is a Fock operator because its expression in terms of the fields does not contain any derivative with respect to "time" x^+). The hope is then to be able to do nonperturbative physics in very much the same way one works in ordinary quantum mechanics, namely, by diagonalizing the Hamiltonian P^- in large, but finite sectors of the Hilbert space. This method has been applied to various models.[1-5] In an alternative approach, one builds a path-integral formulation on the basis of the field eigenstates, and computes the energy spectrum using lattice techniques.[6]

Either way, one needs to construct the Fock space at fixed x^+. The (anti)commutation relations between particle creation and annihilation operators are derived, via the momentum-space expansions, from the (anti)commutators between the fundamental fields. As for the latter, they have been calculated in the case of scalar[7] and gauge theories[8,9] by the methods of canonical quantization.

In the case of free fermions, the functional form of $\{\psi(x),\bar{\psi}(y)\}$ is known for all x and y, so it is no problem to specialize to $x^+=y^+$ (we do this in Appendix B). As for interacting fermions, the values taken by the (anti)commutators have been proposed[10-12] in order to satisfy the following requirements: (1) identity between Heisenberg and Lagrange equations of motion, and (2) interpretation of the total four-momentum operator in terms of its particle content.

Nonetheless we feel it useful to establish these commutation relations, in theories of fermions, on the same firm footing as for the other theories, viz., by means of a canonical quantization. This calculation, to the best of our knowledge, has not been presented so far. The purpose of this paper is to fill this gap. As a by-product, we shall gain a deeper understanding of light-front fermions which can be provided only by the analysis of the structure of Hamiltonian constraints.

Given the singular nature of the system of coordinates, namely, the choice of boundaries along $x^+=\text{const}$, constraints are always present in the null-plane quantization and can be treated with the Dirac-Bergmann algorithm. In order to keep this paper as brief as possible, the reader will be assumed to possess some familiarity with the method (for a review, including the quantization of fermions in space-time, see Ref. 13).

To fix ideas about interacting Dirac fermions, we have elected to carry out the null-plane quantization in the framework of quantum electrodynamics. This should not, we hope, obscure the general features of fermionic theories. The organization of the paper is as follows: in Sec. II we introduce the Hamiltonian and derive the equations of motion; in Sec. III we classify the constraints into first and second class, and choose a suitable set of gauge conditions; in Secs. IV and V we construct the Dirac brackets. Section VI summarizes our results.

II. STRUCTURE OF CONSTRAINTS

The Lagrangian density for QED is

$$\mathcal{L}=\frac{i}{2}\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\psi-m\bar{\psi}\psi-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}-j^{\mu}A_{\mu}\ ,$$

where

$$j^{\mu}\equiv g\bar{\psi}\gamma^{\mu}\psi,\quad F_{\mu\nu}\equiv\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}\ ,$$

and

$$-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}=\tfrac{1}{2}(F_{+-})^2+F_{-k}F_{+k}-\tfrac{1}{2}(F_{12})^2\quad(k=1,2)\ .$$

In the classical theory, $\psi(x)$ and $\bar{\psi}(x)$ are independent complex Grassmann spinors. The canonical momenta are, for the photon,

$$\pi^{\nu}=F^{\nu+},\quad \text{viz.},\ \pi^{+}=0,\quad \pi^{-}=F_{+-},\quad \pi^{k}=F_{-k}\ ,$$

and for the fermion

$$\phi=\frac{i}{2}\bar{\psi}\gamma^{+},\quad \bar{\phi}=\frac{i}{2}\gamma^{+}\psi\ .$$

(We use "left derivatives," see Appendix C.) Hence the primary constraints are

$$C\equiv\pi^{+},\quad C^{k}\equiv\pi^{k}-F_{-k}\ ,$$

$$\Gamma\equiv\phi-\frac{i}{2}\bar{\psi}\gamma^{+},\quad \bar{\Gamma}\equiv\bar{\phi}-\frac{i}{2}\gamma^{+}\psi\ ,$$

while π^- is dynamical. The canonical Hamiltonian density is

$$\mathcal{H} \equiv \pi^\mu \dot{A}_\mu + \phi\dot{\psi} - \dot{\bar{\psi}}\bar{\phi} - \mathcal{L}$$
$$= \tfrac{1}{2}(\pi^-)^2 + (\pi^- \partial_- + \pi^k \partial_k) A_+ + \tfrac{1}{2}(F_{12})^2$$
$$-\frac{i}{2}\bar{\psi}(\gamma^- \overleftrightarrow{\partial}_- + \gamma^k \overleftrightarrow{\partial}_k)\psi + m\bar{\psi}\psi + j^\mu A_\mu ,$$

where the overdot denotes a derivative with respect to x^+. The primary Hamiltonian is

$$H_p = H_c + \int dx^- d^2\mathbf{x}_\perp (uC + u_k C^k + \Gamma v - \bar{v}\bar{\Gamma}) ,$$

where u, u_k are bosonic and $v, \bar{v}$ are fermionic (Grassmann) Lagrange multipliers.

In order to control the infrared divergences, it is customary to enclose the system in a longitudinal box.[14] We shall do so henceforth; in particular the integrations over x^- are to be performed from $(-L/2)$ to $(+L/2)$. The exact form of the quantum (anti)commutators depends on the boundary conditions (BC's), so to fix the ideas we choose antiperiodic BC for all the fundamental fields.

The Poisson brackets (PB's) between these fields are

$$\{A_\mu(x), \pi^\nu(y)\}_{x^+=y^+} = \delta_\mu{}^\nu \delta(x^- - y^-)\delta^2(\mathbf{x}_\perp - \mathbf{y}_\perp) ,$$
$$\{\psi_\alpha(x), \phi_\beta(y)\}_{x^+=y^+} = \delta_{\alpha\beta}\delta(x^- - y^-)\delta^2(\mathbf{x}_\perp - \mathbf{y}_\perp) ,$$
$$(\alpha, \beta = 1,2,3,4) ,$$

all the others being zero. Hereafter, the subscript $x^+ = y^+$ will be omitted. Using

$$\{C^k(x), C^l(y)\} = -2\delta_{kl}\partial_-^x \delta(x^- - y^-)\delta^2(\mathbf{x}_\perp - \mathbf{y}_\perp)$$
$$(k, l = 1,2) ,$$

we determine the u_k's by requiring the C^k's to be conserved in time:

$$0 = \{C^k(x), H_p\} = \partial_k \pi^- + \partial_l F_{lk} - 2\partial_- u_k - j^k .$$

The conservation of $C(x)$ yields Gauss's law:

$$0 = G(x) \equiv \{C(x), H_p\} = \partial_- \pi^- + \partial_k \pi^k - j^+ .$$

Similarly, using

$$\{\bar{\Gamma}(x), \Gamma(y)\} = -i\gamma^+ \delta(x^- - y^-)\delta^2(\mathbf{x}_\perp - \mathbf{y}_\perp) ,$$

one finds that $v, \bar{v}$ obey

$$0 = \{\Gamma(x), H_p\}$$
$$= -i\bar{\psi}(\gamma^- \overleftarrow{\partial}_- + \gamma^k \overleftarrow{\partial}_k)\psi - m\bar{\psi} - g\bar{\psi}A\!\!\!/ - i\bar{v}\gamma^+ , \qquad (2.1a)$$
$$0 = \{\bar{\Gamma}(x), H_p\}$$
$$= -i(\gamma^- \partial_- + \gamma^k \partial_k)\psi + m\psi + gA\!\!\!/\psi - i\gamma^+ v . \qquad (2.1b)$$

Finally, using

$$\{G(x), \Gamma(y)\} = -g\bar{\psi}\gamma^+ \delta(x^- - y^-)\delta^2(\mathbf{x}_\perp - \mathbf{y}_\perp) ,$$
$$\{G(x), \bar{\Gamma}(y)\} = g\gamma^+ \psi\delta(x^- - y^-)\delta^2(\mathbf{x}_\perp - \mathbf{y}_\perp) ,$$

one gets

$$\{G(x), H_p\} \equiv 0$$

with the help of Eq. (2.1). Hence Gauss's law is automatically conserved.

At this point it is nice to check that we have the correct (Euler-Lagrange) equations of motion. First one needs to calculate the time derivative of the fields:

$$\dot{A}_+ = \{A_+, H_p\} = u ,$$
$$\dot{A}_k = \{A_k, H_p\} = \partial_k A_+ + u_k ,$$
$$\dot{\pi}^- = \{\pi^-, H_p\} = \partial_k u_k - j^- ,$$
$$\dot{\pi}^k = \{\pi^k, H_p\} = -\partial_- u_k - j^k + \partial_l F_{lk} ,$$
$$\dot{\psi} = \{\psi, H_p\} = v ,$$
$$\dot{\bar{\psi}} = \{\bar{\psi}, H_p\} = \bar{v} .$$

It is then easy to obtain

$$\partial_\mu F^{\mu\nu} = j^\nu , \qquad (2.2)$$
$$(i\partial\!\!\!/ - m - gA\!\!\!/)\psi = 0, \quad 0 = \bar{\psi}(i\overleftarrow{\partial\!\!\!/} + m + gA\!\!\!/) . \qquad (2.3)$$

However, one notices immediately that the time evolution of A_+ is left undetermined since we have no constraint imposed on $u(x)$. Furthermore, Eqs. (2.1a) and (2.1b) do not determine the time evolution of ψ and $\bar{\psi}$, viz., v and $\bar{v}$ completely since γ^+ is singular; more precisely ψ and $\bar{\psi}$ each have two undetermined components (see Appendix A). As is well known,[11] a particle that is distinct from its antiparticle has in a null-plane frame half as many degrees of freedom as it has in space-time.

This means of course that one needs now to choose appropriate gauge conditions in order to fix the evolution of the system.

III. GAUGE STRUCTURE

The algorithm calls for as many gauge conditions as there are first-class constraints (the ones that have zero PB with all the others). Clearly, C is one of them, while the C^k's are second class. Closely related to $G(x)$ is the linear combination of constraints

$$\Sigma \equiv G + ig(\Gamma\psi + \bar{\psi}\bar{\Gamma}) .$$

It is first class, as one can easily verify.

Finally, what is the status of the fermionic constraints? As spinors, Γ and $\bar{\Gamma}$ are made of four constraints each. It turns out that Γ as well as $\bar{\Gamma}$ contain two first-class and two second-class constraints. This must be contrasted with the space-time quantization,[13] where the analog of Γ, $\bar{\Gamma}$ are all second class, and explains the above-mentioned difference in the number of degrees of freedom, from the point of view of the Dirac-Bergmann algorithm.

For simplicity, we switch now and for the remainder of the paper to 1+1 dimensions; the generalization to four dimensions is straightforward.

In 1+1 dimensions, the spinors have two components, so we want to show that Γ contains one first-class and one second-class component (and similarly for $\bar{\Gamma}$). This is due to the fact that the PB's of $\Gamma, \bar{\Gamma}$ imply that $(\Gamma\gamma^+)$ and $(\gamma^+\bar{\Gamma})$ contain one first-class component, while

$(\Gamma\gamma^-)$ and $(\gamma^-\bar{\Gamma})$ contain one second-class component. Let us concentrate, for example, on $(\Gamma\gamma^+)$ and $(\Gamma\gamma^-)$.

Since γ^+ is nilpotent but not zero, γ^+_{12} and γ^+_{21} cannot be simultaneously zero. So the representations of Dirac matrices fall into two categories.

$\gamma^+_{21}\neq 0$. Writing out explicitly the identities

$$[\Gamma(\gamma^+)^2]_1=0,\quad [\Gamma(\gamma^-)^2]_2=0\ ,$$

one gets

$$(\Gamma\gamma^+)_1\gamma^+_{11}+(\Gamma\gamma^+)_2\gamma^+_{21}=0\ ,$$

$$(\Gamma\gamma^-)_1\gamma^-_{12}+(\Gamma\gamma^-)_2\gamma^-_{22}=0\ .$$

As $\gamma^-_{12}=(\gamma^+)^\dagger_{12}=(\gamma^+_{21})^*$, one also has $\gamma^-_{12}\neq 0$. Hence the spinor constraint Γ is equivalent to the set

$$(\Gamma\gamma^+)_1=0\quad(\text{first class})\ ,$$

$$(\Gamma\gamma^-)_2=0\quad(\text{second class})\ .$$

In effect, assuming that $(\Gamma\gamma^-)_2$ was first class, its PB with $\bar{\Gamma}_2$ would necessitate $(\gamma^+\gamma^-)_{22}=0$, viz.,

$$0=\gamma^+_{21}\gamma^-_{12}+\gamma^+_{22}\gamma^-_{22}=|\gamma^+_{21}|^2+|\gamma^+_{22}|^2\ ,$$

which would contradict our assumption $\gamma^+_{21}\neq 0$.

$\gamma^+_{12}\neq 0$. Similarly the identities

$$[\Gamma(\gamma^+)^2]_2=0,\quad [\Gamma(\gamma^-)^2]_1=0$$

are written out as

$$(\Gamma\gamma^+)_1\gamma^+_{12}+(\Gamma\gamma^+)_2\gamma^+_{22}=0\ ,$$

$$(\Gamma\gamma^-)_1\gamma^-_{11}+(\Gamma\gamma^-)_2\gamma^-_{21}=0\ .$$

Here $\gamma^-_{21}\neq 0$, and Γ as a constraint is equivalent to the set

$$(\Gamma\gamma^+)_2=0\quad(\text{first class})\ ,$$

$$(\Gamma\gamma^-)_1=0\quad(\text{second class})\ .$$

We have gone through this somewhat tedious proof in order to convince the reader that, independently of the choice of representation, the set $\Gamma,\bar{\Gamma}$ is equivalent to two first-class and two second-class constraints (four of each in four dimensions). The importance of this result lies in the fact that we have to match these two first-class constraints with two gauge conditions, even in the free fermion theory. After gauge fixing, one of the components of ψ becomes an explicit function of the other (and similarly for $\bar{\psi}$).

Our next task [in two-dimensional QED (QED_2)] is to find four admissible gauge constraints (viz., compatible with the equations of motion, and turning all the constraints into second-class ones).

When the photon is coupled to fermions, one might consider the gauge $A_+=0$, but a second gauge constraint on the electromagnetic field is needed, and no one was found that would be compatible with Eq. (2.2) (this is the same problem as the temporal gauge in a space-time frame). Hence we select the gauge conditions

$$A_-=0,\quad K\equiv\pi^-+\partial_-A_+=0\ ,$$

which actually are standard in the pure gauge theory too (the so-called null-plane gauge).[8,9] To find the matching gauge conditions on the fermions, let us start from Eq. (2.3) (the Dirac equation):

$$(i\partial\!\!\!/-m-gA\!\!\!/)\psi=0\ .$$

Multiplying on the left by γ^+, one gets

$$\gamma^+(i\gamma^-\partial_--m-gA_-\gamma^-)\psi=0\ .$$

Hence we take as gauge constraints

$$\chi\equiv\gamma^+(i\gamma^-\partial_--m)\psi,\quad \bar{\chi}\equiv\bar{\psi}(i\overleftarrow{\partial}_-\gamma^-+m)\gamma^+\ .$$

The reader will immediately object that in doing so we have selected a total of six gauge conditions instead of the four to which we committed ourselves. In other words, we are providing redundant pieces of information to the system, with the result that the matrix of constraints will be inexorably singular. So how can we reduce the total number of constraints, while preserving the symmetry between ψ and $\bar{\psi}$, and without having to pick a particular representation of the γ matrices? The solution to this little puzzle will be given in the next section, but before let us just verify that with our gauge choice the multipliers are effectively determined unambiguously.

One finds that the condition $A_-=0$ is automatically conserved, while the requirement

$$0=\{\pi^-+\partial_-A_+,H_p\}=\partial_-u-j^-$$

determines $u(x)$. Using

$$\{\chi,\bar{\Gamma}\}=0,\{\bar{\chi},\Gamma\}=0\ ,$$

$$\{\chi(x),\Gamma(y)\}=\gamma^+(i\gamma^-\partial^x_--m)\delta(x^--y^-)\ ,$$

$$\{{}^T\bar{\Gamma}(x),{}^T\chi(y)\}=-{}^T[\gamma^+(i\gamma^-\partial^x_-+m)]\delta(x^--y^-)\ ,$$

$$\{\bar{\Gamma}(x),\bar{\chi}(y)\}=(-i\gamma^-\partial^x_-+m)\gamma^+\delta(x^--y^-)\ ,$$

$$\{{}^T\bar{\chi}(x),{}^T\bar{\Gamma}(y)\}={}^T[(i\gamma^-\partial^x_-+m)\gamma^+]\delta(x^--y^-)\ ,$$

where the subscript T denotes the matrix transposition, one gets

$$0=\{\chi,H_p\}=\gamma^+(i\gamma^-\partial_--m)v\ , \tag{3.1a}$$

$$0=\{\bar{\chi},H_p\}=\bar{v}(-i\gamma^-\overleftarrow{\partial}_-+m)\gamma^+\ . \tag{3.1b}$$

Manipulating Eq. (2.1b) together with Eq. (3.1a), one obtains finally

$$\begin{aligned}\partial_-v&=-\frac{m^2\psi}{2}-\frac{mg}{2}A_+\gamma^+\psi-\frac{ig}{2}\gamma^-\gamma^+\partial_-(A_+\psi)\\&=-\frac{m^2\psi}{2}-\frac{ig}{2}\gamma^-\gamma^+(\partial_-A_+)\psi-igA_+(\partial_-\psi)\ ,\end{aligned}$$

where the second line results from Dirac equation. Similarly,

$$\partial_-\bar{v}=-\frac{m^2\bar{\psi}}{2}+\frac{ig}{2}(\partial_-A_+)\bar{\psi}\gamma^+\gamma^-+igA_+(\partial_-\bar{\psi})\ .$$

Note that this expression for $\partial_-v=\partial_+\partial_-\psi$ is nothing else than the Klein-Gordon equation; in effect, using the covariant derivative $D_\mu\equiv\partial_\mu+igA_\mu$, the Dirac equation is

$$(i\not{D}-m)\psi=0 \Longrightarrow 0=(-i\not{D}-m)(i\not{D}-m)\psi=(\not{D}\not{D}+m^2)\psi=\gamma^+\gamma^-(\partial_+ +igA_+)(\partial_-\psi)+\gamma^-\gamma^+\partial_-(\partial_+ +igA_+)\psi+m^2\psi$$

$$=(2\partial_+\partial_- +m^2+ig\gamma^-\gamma^+\partial_- A_+)\psi+2igA_+\partial_-\psi \ .$$

In Sec. IV we construct a preliminary Dirac brackets by "inverting" Γ, $\bar{\Gamma}$, χ, and $\bar{\chi}$. In Sec. V we obtain the final Dirac brackets by incorporating C, G, A_-, and K.

IV. PRELIMINARY DIRAC BRACKETS

To solve the problem discussed in the preceding section, we replace χ and $\bar{\chi}$ by

$$\xi\equiv\frac{\chi+m\gamma^-\bar{\Gamma}}{\sqrt{2}}, \quad \bar{\xi}\equiv\frac{\bar{\chi}+m\Gamma\gamma^-}{\sqrt{2}} \ .$$

The reason why these linear combinations work is that

$$\xi=0 \Longrightarrow 0=\gamma^+\xi=\frac{m\gamma^+\gamma^-\bar{\Gamma}}{\sqrt{2}} \Longrightarrow \gamma^-\bar{\Gamma}=0 \quad (\text{since } \gamma^-\gamma^+\gamma^-=2\gamma^-) \ ,$$

and using $\xi=0$ again we get finally

$$\xi=0 \Longrightarrow \chi=0 \quad \text{and } \gamma^-\bar{\Gamma}=0 \ .$$

Similarly

$$\bar{\xi}=0 \Longrightarrow \bar{\chi}=0 \text{ and } \Gamma\gamma^-=0 \ .$$

In this way we manage to impose not only the gauge conditions, but also one projection of each primary constraint $\Gamma,\bar{\Gamma}$. Therefore, in order to complete the set of constraints, one only needs to invert one more component of Γ (call it λ) and one component of $\bar{\Gamma}$ (call it $\bar{\lambda}$) rather than the two *spinors*. A suitable choice of λ and $\bar{\lambda}$ will be given below.

Imposing $\xi,\bar{\xi}$ on the dynamics defines a first preliminary Dirac brackets (PDB1), then requiring the full Γ, $\bar{\Gamma}$ yields PDB2.

The matrix whose elements are the PB's of ξ and $\bar{\xi}$ is

$$\Delta=\begin{bmatrix} 0 & 1-i\gamma^- -\gamma^+\gamma^- \\ {}^T(1-i\gamma^- -\gamma^+\gamma^-) & 0 \end{bmatrix} m^2\delta(x^- -y^-) \ .$$

Its inverse is

$$\Delta^{-1}=\begin{bmatrix} 0 & {}^T(1-i\gamma^- -\gamma^+\gamma^-) \\ 1-i\gamma^- -\gamma^+\gamma^- & 0 \end{bmatrix} \frac{\delta(x^- -y^-)}{m^2} \ .$$

Using

$$\{\xi(x^-),\phi(y^-)\}=\left[\gamma^+(i\gamma^-\partial_-^x-m)-i\frac{m}{2}\gamma^-\gamma^+\right]\frac{\delta(x^- -y^-)}{\sqrt{2}} \ ,$$

$$\{\bar{\phi}(x^-),\bar{\xi}(y^-)\}=\left[(i\partial_-^y\gamma^- +m)\gamma^+ -i\frac{m}{2}\gamma^+\gamma^-\right]\frac{\delta(x^- -y^-)}{\sqrt{2}} \ ,$$

one finds

$$\{\bar{\phi}(x^-),\phi(y^-)\}_{\text{PDB1}}=-\int dz_1dz_2\{\bar{\phi}(x^-),\bar{\xi}(z_1)\}\Delta^{-1}_{\bar{\xi}\xi}(z_1,z_2)\{\xi(z_2),\phi(y^-)\}$$

$$=-\tfrac{1}{2}\left[(-i\partial_-^x\gamma^- +m)\gamma^+ -i\frac{m}{2}\gamma^+\gamma^-\right]\Delta^{-1}_{\bar{\xi}\xi}(x^-,y^-)\left[\gamma^+\left[-i\overleftarrow{\partial}{}_-^y\gamma^- -m\right]-i\frac{m}{2}\gamma^-\gamma^+\right]$$

$$=[2i\gamma^-(\partial_-^x)^2-m\partial_-^x]\frac{\delta(x^- -y^-)}{m^2} \ ,$$

$$\{\psi(x^-),\phi(y^-)\}_{\text{PDB1}}=\delta(x^--y^-)-\frac{m}{2}\int dz_1dz_2\delta(x-z_1)\gamma^-\Delta^{-1}_{\bar\xi\xi}(z_1,z_2)\left[\gamma^+(i\gamma^-\partial_-^{z_2}-m)-i\frac{m}{2}\gamma^-\gamma^+\right]\delta(z_2-y^-)$$

$$=\left[\frac{\gamma^+\gamma^-}{2}+\frac{i}{m}\gamma^-\partial_-^x\right]\delta(x^--y^-)\ ,$$

$$\{\bar\phi(x^-),\bar\psi(y^-)\}_{\text{PDB1}}=\delta(x^--y^-)-\frac{m}{2}\int dz_1dz_2[(i\partial_-^{z_1}\gamma^-+m)\gamma^+-i\frac{m}{2}\gamma^+\gamma^-]\delta(x^--z_1)\Delta^{-1}_{\bar\xi\xi}(z_1,z_2)\gamma^-\delta(z_2-y^-)$$

$$=\left[\frac{\gamma^-\gamma^+}{2}+\frac{i}{m}\gamma^-\partial_-^x\right]\delta(x^--y^-)\ ,$$

$$\{\psi(x^-),\bar\psi(y^-)\}_{\text{PDB1}}=-m^2\int dz_1dz_2\{\psi(x^-),\Gamma(z_1)\}\frac{\gamma^-}{\sqrt2}\Delta^{-1}_{\bar\xi\xi}(z_1,z_2)\frac{\gamma^-}{\sqrt2}\{\bar\Gamma(z_2),\bar\psi(y^-)\}$$

$$=-\frac{m^2}{2}\gamma^-\Delta^{-1}_{\bar\xi\xi}(x^-,y^-)\gamma^-=0\ ,$$

$$\{\bar\Gamma(x^-),\Gamma(y^-)\}_{\text{PDB1}}=\{\bar\phi,\phi\}_{\text{PDB1}}-\frac{i}{2}\{\bar\phi,\bar\psi\}_{\text{PDB1}}\gamma^+-\frac{i}{2}\gamma^+\{\psi,\phi\}_{\text{PDB1}}-\tfrac14\gamma^+\{\psi,\bar\psi\}_{\text{PDB1}}\gamma^+$$

$$=\frac{2i}{m^2}\gamma^-(\partial_-^x)^2\delta(x^--y^-)\ .$$

The other brackets between the fundamental spinors stay equal to zero.

To construct PDB2, we need to distinguish again between two categories of representations, according to the value of the "antitrace," viz., $(\gamma^-_{12}+\gamma^-_{21})$.

Zero antitrace. In this case

$$\gamma^-_{11}=\pm\gamma^-_{12}\neq0,\quad \gamma^-\bar\Gamma=0,\quad \Gamma\gamma^-=0\ \Longrightarrow\ \Gamma_1=\pm\Gamma_2,\quad \bar\Gamma=\mp\bar\Gamma_2\ ,$$

so

$$\lambda\equiv\Gamma_1=0,\quad \bar\lambda\equiv\bar\Gamma_1=0\ \Longrightarrow\ \Gamma=0,\quad \bar\Gamma=0\ .$$

The PDB1's between λ and $\bar\lambda$ form the matrix

$$M=\frac{2i}{m^2}\gamma^-_{11}(\partial_-^x)^2\delta(x^--y^-)\sigma_1\Longrightarrow M^{-1}=-\frac{im^2}{2\gamma^-_{11}}\frac{|x^--y^-|-L/2}{2}\sigma_1\ ,$$

where

$$\sigma_1\equiv\begin{bmatrix}0&1\\1&0\end{bmatrix}$$

is the usual Pauli matrix. With the help of

$$\{\psi_\alpha(x^-),\lambda(y^-)\}_{\text{PDB1}}=\left[\frac{\gamma^+\gamma^-}{2}+\frac{i}{m}\gamma^-\partial_-^x\right]_{\alpha1}\delta(x^--y^-)\ ,$$

$$\{\bar\lambda(x^-),\bar\psi_\alpha(y^-)\}_{\text{PDB1}}=\left[\frac{\gamma^-\gamma^+}{2}+\frac{i}{m}\gamma^-\partial_-^x\right]_{1\alpha}\delta(x^--y^-)\ ,$$

one finds

$$\{\psi_\alpha(x^-),\bar\psi_\beta(y^-)\}_{\text{PDB2}}=-\int dz_1dz_2\{\psi_\alpha(x^-),\lambda(z_1)\}_{\text{PDB1}}M^{-1}_{\lambda\bar\lambda}(z_1,z_2)\{\bar\lambda(z_2),\bar\psi_\beta(y^-)\}_{\text{PDB1}}$$

$$=\frac{im^2}{4\gamma^-_{11}}\left[\frac{\gamma^-\gamma^+}{2}+\frac{i}{m}\gamma^-\partial_-^x\right]_{\alpha1}\left[|x^--y^-|-\frac{L}{2}\right]\left[\frac{\gamma^-\gamma^+}{2}-\frac{i}{m}\gamma^-\overleftarrow{\partial}_-^y\right]_{1\beta}$$

$$=\frac{im^2}{4\gamma^-_{11}}\left[\frac{(\gamma^+\gamma^-)_{\alpha1}(\gamma^-\gamma^+)_{1\beta}}{4}\left[|x^--y^-|-\frac{L}{2}\right]+\frac{i}{2m}(\gamma^-)_{\alpha_1}(\gamma^-\gamma^+)_{1\beta}\,\epsilon(x^--y^-)\right.$$

$$\left.+\frac{i}{2m}(\gamma^+\gamma^-)_{\alpha1}(\gamma^-)_{1\beta}\,\epsilon(x^--y^-)-\frac{2}{m^2}(\gamma^-)_{\alpha1}(\gamma^-)_{1\beta}\,\delta\,(x^--y^-)\right]\ .$$

This is simply

$$\{\psi(x^-),\bar{\psi}(y^-)\}_{\mathrm{PDB2}}=\frac{im^2}{8}\gamma^+\left[|x^--y^-|-\frac{L}{2}\right]-\frac{m}{4}\epsilon(x^--y^-)-\frac{i}{2}\gamma^-\delta(x^--y^-)\ . \tag{4.1}$$

Nonzero antitrace. Then, with

$$\lambda\equiv\Gamma_1+\Gamma_2,\quad \bar{\lambda}\equiv\Gamma_1+\Gamma_2\ ,$$

we have

$$\xi=0,\quad \bar{\xi}=0,\quad \lambda=0,\quad \bar{\lambda}=0\ \Longrightarrow\ \Gamma=0,\quad \bar{\Gamma}=0\ .$$

The PDB1's between λ and $\bar{\lambda}$ form the matrix

$$M=\frac{2i}{m^2}(\gamma^-_{12}+\gamma^-_{21})(\partial^x_-)^2\delta(x^--y^-)\sigma_1\Longrightarrow M^{-1}=-\frac{im^2}{2(\gamma^-_{12}+\gamma^-_{21})}\frac{|x^--y^-|-L/2}{2}\sigma_1\ .$$

Using

$$\{\psi_\alpha(x^-),\lambda(y^-)\}_{\mathrm{PDB1}}=\sum_{k=1}^{2}\left[\frac{\gamma^+\gamma^-}{2}+\frac{i}{m}\gamma^-\partial^x_-\right]_{\alpha k}\delta(x^--y^-)\ ,$$

$$\{\bar{\lambda}(x^-),\bar{\psi}_\alpha(y^-)\}_{\mathrm{PDB1}}=\sum_{k=1}^{2}\left[\frac{\gamma^-\gamma^+}{2}+\frac{i}{m}\gamma^-\partial^x_-\right]_{k\alpha}\delta(x^--y^-)\ ,$$

one finds

$$\{\psi_\alpha(x^-),\bar{\psi}_\beta(y^-)\}_{\mathrm{PDB2}}=-\int dz_1dz_2\{\psi_\alpha(x^-),\lambda(z_1)\}_{\mathrm{PDB1}}M^{-1}_{\lambda\bar{\lambda}}(z_1,z_2)\{\bar{\lambda}(z_2),\bar{\psi}_\beta(y^-)\}_{\mathrm{PDB1}}$$

$$=\frac{im^2}{4(\gamma^-_{12}+\gamma^-_{21})}\sum_{k,l=1}^{2}\left[\frac{\gamma^-\gamma^+}{2}+\frac{i}{m}\gamma^-\partial^x_-\right]_{\alpha k}\left[|x^--y^-|-\frac{L}{2}\right]\left[\frac{\gamma^-\gamma^+}{2}-\frac{i}{m}\gamma^-\overleftarrow{\partial}^y_-\right]_{l\beta}$$

$$=\frac{im^2}{4(\gamma^-_{12}+\gamma^-_{21})}\sum_{k,l=1}^{2}\left[\frac{(\gamma^+\gamma^-)_{\alpha k}(\gamma^-\gamma^+)_{l\beta}}{4}\left[|x^--y^-|-\frac{L}{2}\right]+\frac{i}{2m}(\gamma^-)_{\alpha k}(\gamma^-\gamma^+)_{l\beta}\epsilon(x^--y^-)\right.$$

$$\left.+\frac{i}{2m}(\gamma^+\gamma^-)_{\alpha k}(\gamma^-)_{l\beta}\epsilon(x^--y^-)-\frac{2}{m^2}(\gamma^-)_{\alpha k}(\gamma^-)_{l\beta}\delta(x^--y^-)\right]\ .$$

Again, one can check that for all values of the indices this expression reduces to Eq. (4.1).

Hence Eq. (4.1) is always true; it is the main result of this paper. Clearly, it stands as the final Dirac brackets in the free fermion theory. Furthermore, it is unaltered by the interaction with the electromagnetic field (in the null-plane gauge), as we now show.

V. DIRAC BRACKETS

Since the constraints that have been inverted above do not involve the electromagnetic phase-space variables, the PDB2's of the latter are equal to their PB's; this is also true for the brackets between the photonic and a fermionic quantity. Specifically,

$$\{\pi^+,G\}=0,\{\pi^+,A_-\}=0\ ,$$

$$\{\pi^+(x^-),K(y^-)\}=\partial^x_-\delta(x^--y^-)\ ,$$

$$\{G(x^-),A_-(y^-)\}=-\partial^x_-\delta(x^--y^-)\ ,$$

$$\{G,K\}=0,\{A_-(x^-),K(y^-)\}=\delta(x^--y^-)\ .$$

Next, let us invert $C(x)$ and $K(x)$. The PDB2's of a fermionic variable with π^+ and K are zero, so the PDB3 of a fermionic with a photonic or fermionic variable is equal to the PDB2. Further, the brackets of C and K form the matrix

$$N=\partial^x_-\delta(x^--y^-)\sigma_1\Longrightarrow N^{-1}=\frac{\epsilon(x^--y^-)}{2}\sigma_1\ .$$

One calculates easily

$$\{A_-(x^-),\pi^-(y^-)\}_{\mathrm{PDB3}}=\delta(x^--y^-)\ ,$$

$$\{A_-(x^-),A_+(y^-)\}_{\mathrm{PDB3}}=\frac{\epsilon(x^--y^-)}{2}\ ,$$

$$\{A_+,\pi^-\}_{\mathrm{PDB3}}=0,\quad \{A_+,G\}_{\mathrm{PDB3}}=0\ .$$

Finally we invert A_- and G. Since the PDB3 of a fermionic variable with A_- is zero, the Dirac brackets between fermionic variables are equal to the PDB2's, as advertised. Then from Eq. (4.1) we deduce

$$\{j^+,j^+\}_D=0\ .$$

Further, the PDB3's of A_- and G form the matrix

$$O=-\partial^x_-\delta(x^- - y^-)\sigma_1 \Longrightarrow O^{-1}=-\frac{\epsilon(x^- - y^-)}{2}\sigma_1 .$$

With the help of the identity

$$\int dz^-\epsilon(x^- - z^-)\epsilon(z^- - y^-)=2\left[|x^- - y^-|-\frac{L}{2}\right],$$

one finds

$$\{A_+(x^-),B(y^-)\}_D=\{A_+(x^-),B(y^-)\}_{\mathrm{PDB3}}+\tfrac{1}{2}\int dz^-\left[|x^- - z^-|-\frac{L}{2}\right]\{G(z^-),B(y^-)\}_{\mathrm{PDB3}} ,$$

for any function B of the phase-space variables. In particular

$$\{A_+,A_+\}_D=0 ,$$

$$\{A_+(x^-),\psi(y^-)\}_D=-\tfrac{1}{2}\int dz^-\left[|x^- - z^-|-\frac{L}{2}\right]\{j^+(z^-),\psi(y^-)\}_D ,$$

and similarly with $\psi(y^-)$ replaced by $\bar{\psi}(y^-)$. This is in agreement with the equation of motion

$$(\partial_-)^2A_+=-j^+ \Longleftrightarrow A_+(x^-)=-\tfrac{1}{2}\int dz^-\left[|x^- - z^-|-\frac{L}{2}\right]j^+(z^-) .$$

Expressing j^+ explicitly, one finds

$$\{A_+(x^-),\psi(y^-)\}_D=-\frac{g}{2}\left[i\left[|x^- - z^-|-\frac{L}{2}\right]\psi_+(y^-)+\frac{m}{4}\int dz^-\left[|x^- - z^-|-\frac{L}{2}\right]\epsilon(y^- - z^-)\gamma^+\psi_+(z^-)\right] .$$

[See Appendix A for the definitions of ψ_+ and ψ_-.] Hence

$$\{A_+(x^-),\partial_-\psi(y^-)\}_D=-\frac{g}{2}\left[\left[|x^- - y^-|-\frac{L}{2}\right]\left[\frac{m}{2}\gamma^+ + i\partial_-\right]-i\epsilon(x^- - y^-)\right]\psi_+(y^-) . \tag{5.1}$$

Also

$$\{A_+(x^-),\psi_+(y^-)\}_D=-i\frac{g}{2}\left[|x^- - y^-|-\frac{L}{2}\right]\psi_+(y^-) . \tag{5.2}$$

This checks with our gauge condition

$$\gamma^+(i\gamma^-\partial_- - m)\psi=0 \Longleftrightarrow \partial_-\psi_-=-i\frac{m}{2}\gamma^+\psi_+ ,$$

so that

$$\{A_+(x^-),\partial_-\psi(y^-)\}_D=\{A_+(x^-),\partial_-\psi_+ + \partial_-\psi_-\}_D=\left[\partial^y_- - i\frac{m}{2}\gamma^+\right]\{A_+(x^-),\psi_+(y^-)\}_D$$

yields Eq. (5.1). Similarly, one gets

$$\{A_+(x^-),\bar{\psi}(y^-)\}_D=\frac{g}{2}\left[i\left[|x^- - y^- - \frac{L}{2}\right]\bar{\psi}(y^-)\Lambda_- - \frac{m}{4}\int dz^-\left[|x^- - z^-|-\frac{L}{2}\right]\epsilon(y^- - z^-)\bar{\psi}(z^-)\gamma^+\right]$$

and

$$\{A_+(x^-),\bar{\psi}(y^-)\gamma^+\}_D =i\frac{g}{2}\left[|x^- - y^-|-\frac{L}{2}\right]\bar{\psi}(y^-)\gamma^+ . \tag{5.3}$$

Finally, putting together Eqs. (5.2) and (5.3), one obtains

$$\{A_+,j^+\}_D=0 . \tag{5.4}$$

VI. CONCLUSIONS

Taking our results over to the quantum theory, Eq. (5.4) implies

$$j^\mu A_\mu = A_\mu j^\mu ,$$

and Eq. (4.1) becomes

$$\{\psi(x^-),\bar{\psi}(y^-)\} = -\frac{m^2}{8}\gamma^+\left[|x^- - y^-| - \frac{L}{2}\right] - i\frac{m}{4}\epsilon(x^- - y^-) + \frac{\delta(x^- - y^-)}{2}\gamma^- ,$$

where the brackets now have the meaning of anticommutators (we set $\hbar = 1$), in agreement with the result of Appendix B. Interpreting $\bar{\psi}$ as $(\psi^\dagger\gamma^0)$, and projecting out components yields

$$\{\psi_+(x^-),\psi_+^\dagger(y^-)\} = \frac{\delta(x^- - y^-)}{\sqrt{2}}\Lambda_+ ,$$

$$\{\psi_-(x^-),\psi_-^\dagger(y^-)\} = -\frac{m^2}{4\sqrt{2}}\left[|x^- - y^-| - \frac{L}{2}\right]\Lambda_- .$$

Similarly, Eqs. (5.2) and (5.3) become

$$[A_+(x^-),\psi_+(y^-)] = \frac{g}{2}\left[|x^- - y^-| - \frac{L}{2}\right]\psi_+(y^-) ,$$

$$[A_+(x^-),\psi_+^\dagger(y^-)] = -\frac{g}{2}\left[|x^- - y^-| - \frac{L}{2}\right]\psi_+^\dagger(y^-) .$$

We have thus recovered all the fundamental (anti)commutators used in the literature. The point of this paper was to illuminate their origin and provide a rigorous derivation of their value by means of the machinery of canonical quantization. In our view, these achievements are worth the effort of carrying out this procedure, which turns out to be significantly more complicated than the spacetime version. Now that these techniques are under control, they can be applied to any other theory with coupled fermions, e.g., Yukawa models or QCD.

The only limitation of this work is that the fermion is assumed to be massive. As is well known,[15] massless fermions in a null-plane frame have peculiarities of their own, and the application of canonical quantization to this case would require a different treatment altogether.

APPENDIX A: LIGHT-CONE COORDINATES

We present here our notations regarding light-cone coordinates, along with a few simple properties of the corresponding γ matrices.

The light-cone time and "longitudinal" coordinate are defined, respectively, as

$$x^+ \equiv \frac{x^0 + x^3}{\sqrt{2}}, \quad x^- \equiv \frac{x^0 - x^3}{\sqrt{2}} ,$$

with the "transverse" coordinates $\mathbf{x}_\perp \equiv (x^1, x^2)$ kept unchanged. Hence in the space of four-vectors $x = (x^+, x^1, x^2, x^-)$, the metric is

$$g = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} .$$

Explicitly,

$$x^+ = x_-, \quad x^- = x_+, \quad x\cdot y = x^+y^- + x^-y^+ - \mathbf{x}_\perp\cdot\mathbf{y}_\perp .$$

The same orthogonal transformation is applied to the Dirac matrices, who still obey

$$\{\gamma^\mu,\gamma^\nu\} = 2g_{\mu\nu} ,$$

but with the new metric. This makes γ^+ and γ^- singular:

$$(\gamma^+)^2 = 0 = (\gamma^-)^2 .$$

Since

$$(\gamma^0)^\dagger = \gamma^0, \quad (\gamma^j)^\dagger = -\gamma^j \quad (j = 1,2,3) ,$$

one gets

$$(\gamma^+)^\dagger = \gamma^-, \quad (\gamma^-)^\dagger = \gamma^+, \quad (\gamma^k)^\dagger = -\gamma^k \quad (k = 1,2) ;$$

therefore, the matrices

$$\Lambda_\pm \equiv \tfrac{1}{2}\gamma^\mp\gamma^\pm$$

are Hermitian. The latter are also projection operators, viz.,

$$(\Lambda_\pm)^2 = \Lambda_\pm, \quad \Lambda_\pm\Lambda_\mp = 0, \quad \Lambda_+ + \Lambda_- = 1 .$$

Their action on Dirac spinors yields

$$\psi_\pm = \Lambda_\pm\psi .$$

Note that all the matrices $\Lambda_\pm, \gamma^\pm$ have the same rank since

$$\gamma^0\Lambda_\mp = \frac{\gamma^\pm}{\sqrt{2}}, \quad \gamma^-\gamma^+\gamma^0 = \gamma^- .$$

Because of the projection property of $\Lambda_\pm$, this rank must be equal to one-half the dimensionality of the spinors, viz., two in four space-time dimensions.

In $1+1$ dimensions, these matrices are rank one (this is obvious since they are singular). It is interesting to note that in this case the chiral matrix is

$$\gamma_5 \equiv \gamma^0\gamma^3 = 1 - \gamma^+\gamma^- ,$$

so that ψ_+ and ψ_- are, respectively, the right and left spinor projections. For example, in the representation

$$\gamma^0 = \sigma_1, \quad \gamma^3 = -i\sigma_2 \implies \gamma_5 = \sigma_3 ,$$

where the σ_j are the standard Pauli matrices, one has simply

$$\psi \equiv \begin{bmatrix} \psi_A \\ \psi_B \end{bmatrix} \implies \psi_+ = \begin{bmatrix} \psi_A \\ 0 \end{bmatrix}, \quad \psi_- = \begin{bmatrix} 0 \\ \psi_B \end{bmatrix} .$$

APPENDIX B: FREE-FERMION ANTICOMMUTATOR

The free-fermion anticommutator at equal x^+ has been derived from the standard general formula in Ref. 16.

Since $\{\psi(x),\bar{\psi}(y)\}$ is a function of $(x-y)$, one can simply consider

$$\{\psi(x),\bar{\psi}(0)\}=\int \frac{d^3\mathbf{p}}{(2\pi)^3 2\omega_{\mathbf{p}}}[(\not{p}+m)e^{-ipx}+(\not{p}-m)e^{ipx}] ,$$

where $\omega_{\mathbf{p}}\equiv(\mathbf{p}^2+m^2)^{1/2}$. We have expressed the integration measure in the familiar space-time coordinates, but it is straightforward to convert it to light-cone coordinates:

$$\int \frac{d^3\mathbf{p}}{2\omega_p}=\int d^4p\,\delta(p^2-m^2)\theta(p^0)=\int d^2\mathbf{p}_\perp\int_{-\infty}^{+\infty}dp^+dp^-\frac{1}{2|p^+|}\delta\left[p^- -\frac{\mathbf{p}_\perp^2+m^2}{2p^+}\right]\theta\left[\frac{p^+ + p^-}{\sqrt{2}}\right]$$

$$=\int d^2\mathbf{p}_\perp\int_0^{+\infty}\frac{dp^+}{2p^+} .$$

On the null plane,

$$\{\psi(x),\bar{\psi}(0)\}_{x^+=0}=\int\frac{d^2\mathbf{p}_\perp}{(2\pi)^3}\int_{-\infty}^{+\infty}\frac{dp^+}{4p^+}[(\not{p}+m)e^{-i(p^+x^- -\mathbf{p}_\perp\mathbf{x}_\perp)}+(\not{p}-m)e^{i(p^+x^- -\mathbf{p}_\perp\mathbf{x}_\perp)}] ,$$

where

$$\not{p}=p^+\gamma^- + \frac{\mathbf{p}_\perp^2+m^2}{2p^+}\gamma^+ - p^k\gamma^k \quad (k=1,2) .$$

Assuming antiperiodic boundary conditions in x^+, one must take

$$\int\frac{dp^+}{p^+}e^{\pm ip^+x^-}=\pm i\pi\epsilon(x^-) , \quad \int\frac{dp^+}{(p^+)^2}e^{\pm ip^+x^-}=-\pi|x^-| .$$

Then one gets easily

$$\{\psi(x),\bar{\psi}(y)\}_{x^+=y^+}=-i\frac{m}{4}\epsilon(x^- -y^-)\delta^2(\mathbf{x}_\perp-\mathbf{y}_\perp)+\tfrac{1}{2}\gamma^-\delta(x^- -y^-)\delta^2(\mathbf{x}_\perp-\mathbf{y}_\perp)$$
$$-\tfrac{1}{8}\gamma^+|x^- -y^-|[m^2-\Delta_\perp^x]\delta^2(\mathbf{x}_\perp-\mathbf{y}_\perp)+\tfrac{1}{4}\epsilon(x^- -y^-)\gamma_\perp\cdot\partial_\perp^x\delta^2(\mathbf{x}_\perp-\mathbf{y}_\perp) ,$$

where explicit translational invariance has been restored.

APPENDIX C: GENERALIZED DIRAC-BERGMANN ALGORITHM

The generalization of the Dirac-Bergmann formalism needed to include Grassmann variables was worked out by Casalbuoni.[17] The basic definitions and properties are as follows.

A Grassmann algebra contains bosonic (self-commuting) and fermionic (self-anticommuting) variables:

$$AB=(-1)^{n_A n_B}BA ,$$

where $n=0$ for a bosonic, and $n=1$ for a fermionic variable. Note that the product of two fermionic variables is bosonic, and the product of a fermionic and a bosonic variable is fermionic.

The left-derivative of a fermionic variable is defined as

$$\frac{\partial\psi}{\partial\psi}\equiv-1 ,$$

and, for a product of those,

$$\frac{\partial}{\partial\psi_\alpha}(\psi_{\alpha_1}\cdots\psi_{\alpha_i})\equiv-\delta_{\alpha\alpha_1}\psi_{\alpha_2}\cdots\psi_{\alpha_i}+\delta_{\alpha\alpha_2}\psi_{\alpha_1}\psi_{\alpha_3}\cdots\psi_{\alpha_i}+\cdots+(-1)^i\delta_{\alpha\alpha_i}\psi_{\alpha_1}\cdots\psi_{\alpha_{i-1}} .$$

The generalized Poisson brackets (PB) is given by

$$\{B_1,B_2\}=-\{B_2,B_1\}\equiv\left[\frac{\partial B_1}{\partial q_i}\frac{\partial B_2}{\partial p^i}-\frac{\partial B_2}{\partial q_i}\frac{\partial B_1}{\partial p^i}\right]+\left[\frac{\partial B_1}{\partial\psi_\alpha}\frac{\partial B_2}{\partial\phi^\alpha}-\frac{\partial B_2}{\partial\psi_\alpha}\frac{\partial B_1}{\partial\phi^\alpha}\right] ,$$

$$\{F,B\}=-\{B,F\}\equiv\left[\frac{\partial F}{\partial q_i}\frac{\partial B}{\partial p^i}-\frac{\partial B}{\partial q_i}\frac{\partial F}{\partial p^i}\right]+\left[\frac{\partial F}{\partial \psi_\alpha}\frac{\partial B}{\partial \phi^\alpha}+\frac{\partial B}{\partial \psi_\alpha}\frac{\partial F}{\partial \phi^\alpha}\right],$$

$$\{F_1,F_2\}=+\{F_2,F_1\}\equiv\left[\frac{\partial F_1}{\partial q_i}\frac{\partial F_2}{\partial p^i}+\frac{\partial F_2}{\partial q_i}\frac{\partial F_1}{\partial p^i}\right]+\left[\frac{\partial F_1}{\partial \psi_\alpha}\frac{\partial F_2}{\partial \phi^\alpha}+\frac{\partial F_2}{\partial \psi_\alpha}\frac{\partial F_1}{\partial \phi^\alpha}\right],$$

where B denotes a bosonic and F a fermionic function of the phase-space variables (the canonical momentum ϕ^α associated to ψ_α is of course fermionic). For example, if the canonical Hamiltonian is

$$H=\phi\dot{\psi}-L(\psi)\ ,$$

one obtains

$$\{\psi,H\}=-\frac{\partial H}{\partial \phi},\quad \{\phi,H\}=-\frac{\partial H}{\partial \psi}\ .$$

In the case of a regular system, Poisson's equation of motion $\dot{A}=\{A,H\}$ yields

$$\dot{\psi}=-\frac{\partial H}{\partial \phi},\quad \dot{\phi}=-\frac{\partial H}{\partial \psi}\ ,$$

namely, the correct Hamilton's equations (the Lagrange equation of motion is $\dot{\phi}=\partial L/\partial\psi$).

It follows from its definition that the PB has the properties

$$\{A,B\}=-(-1)^{n_A n_B}\{B,A\}\ ,$$

$$\{A,B+C\}=\{A,B\}+\{A,C\}\ ,$$

$$\{A,BC\}=(-1)^{n_A n_B}B\{A,C\}+\{A,B\}C\ ,$$

$$\{AB,C\}=(-1)^{n_B n_C}\{A,C\}B+A\{B,C\}\ ,$$

$$(-1)^{n_A n_C}\{A,\{B,C\}\}+(-1)^{n_B n_A}\{B,\{C,A\}\}+(-1)^{n_C n_B}\{C,\{A,B\}\}=0\ .$$

The Dirac-Bergmann algorithm applies to a Grassmann algebra without modification from the purely bosonic case, and the Dirac brackets enjoy all the properties listed above for the PB's. The last step in the canonical quantization consists in promoting the Dirac brackets (or the PB in a regular system) between phase-space functions A and B to a quantum (anti)commutator between *operators* $\hat{A}$ and $\hat{B}$ according to the rule

$$\{A,B\}_D\mapsto-\frac{i}{\hbar}\{\hat{A},\hat{B}\}$$

if both A and B are fermionic, and

$$\{A,B\}_D\mapsto-\frac{i}{\hbar}[\hat{A},\hat{B}]$$

otherwise.

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