Spectral representation in stochastic quantization

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A spectral representation of stationary two-point functions is investigated on the basis of the operator formalism in stochastic quantization. Assuming the existence of asymptotic noninteracting fields, we can diagonalize the total Hamiltonian in terms of asymptotic fields and show that the correlation length along the fictitious time is proportional to the physical mass expected in the usual field theory. A relation between renormalization factors in the operator formalism is derived as a by-product and its validity is checked with the perturbative results calculated in this formalism.

I. INTRODUCTION

Since its proposal by Parisi and Wu,[1] the stochastic quantization (SQ) method has been applied to various problems in field theories. It has been shown that this quantization method is indeed applicable to scalar, spinor, vector, and tensor fields consistently and their results are equivalent to those obtained by the conventional quantization methods.[2]

In this scheme, we introduce an extra degree of freedom t (fictitious time or fifth time) in addition to the ordinary four-dimensional coordinates. Though the perturbative treatments are much more involved, this extra degree of freedom has brought us many interesting and appealing features of SQ which would not be available through conventional quantization methods. In gauge theories, neither a gauge-fixing procedure nor the introduction of ghost fields is needed in principle.[1-3] Furthermore, we can consider a gauge-covariant nonconservative force as a kind of gauge-fixing term.[4] In numerical simulations, t may be regarded as a computer time and the Langevin equation which is one of the basic equations in this scheme is directly simulated to give us the desired probability distribution in the equilibrium limit.[2,5] The application to a system with a complex action has also been proposed.[2,6] New invariant regularization methods have been devised within this framework[7,8] and applied to many problems.[2]

A few years ago, another interesting possibility which is only accessible in SQ and fully utilizes its extra degree of freedom was pointed out by several authors.[9,10] They found the possibility that energy gaps or physical masses are obtainable, not only from the asymptotic behavior with respect to the usual Euclidean coordinates of equal-time[11] two-point functions, but also from the large-τ behavior of the stationary two-point functions with a time separation τ. They demonstrated its validity using a soluble potential model.[9] On the other hand, the renormalization scheme within the framework of SQ which is necessary in the field-theoretic discussion has been formulated.[10,12] The discussion in Ref. 10 is based on the operator formalism in SQ[13] which enables us to use similar techniques to those developed in canonical field theory. These frontier works have stimulated some people to calculate renormalization factors and β functions in scalar theory[14] and the O(N) nonlinear σ model[15] in their own formalisms.[16] Numerical simulation of the stationary correlation function has also been done and its asymptotic behavior from the viewpoint of the renormalization-group equation in SQ has been discussed.[15]

The aim of this paper is to investigate the structure of the stationary two-point functions on the basis of the operator formalism of Namiki and Yamanaka[13] to get a sort of spectral representation of them. This representation may give us a much firmer basis for the time correlation of two-point functions. After a brief review of their operator formalism in SQ in Sec. II, we discuss the general structure of two-point functions in Sec. III. In Sec. III, an assumption of the existence of asymptotic fields together with the boundary condition that the equal-time correlation functions should be equal to the corresponding ones in field theories in the equilibrium limit determines the structure of the physical space. And we obtain a similar spectral representation of two-point functions to that in conventional field theory.[17] From this representation we can conclude that the exponential decay rate with respect to τ of the stationary two-point functions with a time separation τ is in fact proportional to the physical mass squared. In Sec. IV, we examine the spectral condition in this formalism. First an adiabatic factor is introduced to cut the interaction off in the remote past and then the matrix element of the total Hamiltonian is calculated. The spectral condition that the total Hamiltonian can be diagonalized by asymptotic fields is proved. Section V is devoted to a summary and discussions. A relation between renormalization factors in the operator formalism is derived from the spectral representation of two-point functions and checked with the previous perturbative results for renormalization factors.[14,15] Its possible interpretation is also given. An appendix is added to present an alternative proof of the spectral condition using an operator formalism in SQ reformulated in Minkowski space.[18-21]

II. REVIEW OF THE OPERATOR FORMALISM IN SQ

In this section we briefly review the outline of the operator formalism in SQ. Details are to be found in Ref. 13.

Consider a hypothetical stochastic process governed by the Langevin equation with respect to the fictitious time t. In the equilibrium limit $t \to \infty$, equal-time correlation functions are to become correct quantum vacuum expectation values.[2] In this paper we exclusively consider a system of self-interacting scalar field ϕ for simplicity and definiteness. Let $S[\phi]$ be the corresponding classical action. We set up the Langevin equation

$$\dot{\phi}(X) = -\kappa \frac{\delta S}{\delta \phi(X)} + \eta(X) \,, \qquad \left[\frac{\delta S}{\delta \phi(X)} \equiv \left. \frac{\delta S}{\delta \phi(x)} \right|_{\phi(x)=\phi(X)} \right] , \tag{2.1}$$

where X represents the five-dimensional coordinate (x,t) with x being the Euclidean coordinates and the overdot stands for the partial derivative with respect to t. The noise $\eta(X)$ in the above equation has the Gaussian white-noise properties

$$\langle \eta(X) \rangle = 0, \quad \langle \eta(X)\eta(X') \rangle = 2\kappa \delta^5(X - X'), \text{ etc.} \tag{2.2}$$

A positive parameter κ which has no effect in the equal-time correlation functions in equilibrium has been put in the above equation to show its important role in renormalization schemes.[10,12,16]

Now it is an easy task to derive a transition probability distribution T which has the path-integral representation

$$T[\phi_2, t_2 | \phi_1, t_1] = \int_{\phi_i = \phi(t_i)} \mathcal{D}\phi \exp \left[-\frac{1}{2\kappa} \int_{t_1}^{t_2} dt \, d^4x \, \Lambda \right] \tag{2.3}$$

with

$$\Lambda = \tfrac{1}{2} \left[\dot{\phi} + \kappa \frac{\delta S}{\delta \phi} \right]^2 . \tag{2.4}$$

The resemblance between the above expression for T and the path-integral expression of the transition amplitude naturally leads us to the operator formalism in SQ. Regarding Λ as a "Lagrangian" density and replacing $i\hbar$ with 2κ, we define a momentum operator π canonically conjugate to ϕ,

$$\pi(X) = \frac{\partial \Lambda}{\partial \dot{\phi}(X)} = \dot{\phi}(X) + \kappa \frac{\delta S}{\delta \phi(X)} \,, \tag{2.5}$$

and subject to the equal-time commutation relation

$$[\phi(X), \pi(X')]_0 = 2\kappa \delta^4(x - x') \,. \tag{2.6}$$

Time development of an operator A is governed by the Heisenberg equation

$$\frac{d}{dt} A = \frac{\partial}{\partial t} A + \frac{1}{2\kappa} [A, F] \,, \tag{2.7}$$

where F is the Fokker-Planck "Hamiltonian" and is defined through the Legendre transformation of Λ:

$$F = \int d^4x [\pi(X)\dot{\phi}(X) - \Lambda] = \int d^4x \left[\tfrac{1}{2}\pi(X)^2 - \pi(X)\kappa \frac{\delta S}{\delta \phi(X)} \right] . \tag{2.8}$$

We can also define a momentum operator P_μ as in the usual manner:

$$P_\mu = \int d^4x \frac{\partial \Lambda}{\partial \dot{\phi}(X)} \partial_\mu \phi(X) = \int d^4x \, \pi(X) \partial_\mu \phi(X) \,. \tag{2.9}$$

As it commutes with F, P_μ is time independent as well as F itself. This operator causes spatial displacement for the operator

$$\partial_\mu A = \frac{1}{2\kappa} [A, P_\mu] \,. \tag{2.10}$$

For canonical operators ϕ and π, the Heisenberg equations are now written as

$$\dot{\phi}(X) = \frac{1}{2\kappa} [\phi(X), F] = \pi(X) - \kappa \frac{\delta S}{\delta \phi(X)} \,, \tag{2.11a}$$

$$\dot{\pi}(X) = \frac{1}{2\kappa} [(X), F] = \kappa \pi(X) \frac{\partial}{\partial \phi(X)} \left[\frac{\delta S}{\delta \phi(X)} \right] , \tag{2.11b}$$

or alternatively as

$$\phi(X) = e^{-Ft/2\kappa} \phi(x) e^{Ft/2\kappa} = e^{-(Ft + P\cdot x)/2\kappa} \phi(0) e^{(Ft + P\cdot x)/2\kappa} \,, \tag{2.12a}$$

$$\pi(X) = e^{-Ft/2\kappa} \pi(x) e^{Ft/2\kappa} = e^{-(Ft + P\cdot x)/2\kappa} \pi(0) e^{(Ft + P\cdot x)/2\kappa} \,, \tag{2.12b}$$

where $\phi(x) = \phi(x,0)$, $\pi(x) = \pi(x,0)$.

Next we introduce an abstract vector $|\psi_0\rangle$ in the Heisenberg picture to represent the probability distribution $\psi_0[\phi]$:

$$\psi_0[\phi] = \langle \phi | \psi_0 \rangle \,. \tag{2.13}$$

Here the ϕ-diagonal complete orthogonal set $|\phi\rangle$ has the following properties:

$$\phi(x)|\phi'\rangle = \phi'(x)|\phi'\rangle \,, \qquad \langle \phi | \phi' \rangle = \delta[\phi - \phi'], \quad \int d\phi |\phi\rangle\langle\phi| = 1 \,. \tag{2.14}$$

In this ϕ representation, the momentum operator π is expressed as

$$\langle \phi | \pi(x) | \phi' \rangle = -2\kappa \frac{\delta}{\delta \phi(x)} \delta[\phi - \phi'] \,, \tag{2.15}$$

which leads to the important properties of π, F, and P_μ:

$$\langle \pi(X) = 0, \quad \langle F = 0, \langle P_\mu = 0 \quad \left[\langle \equiv \int d\phi \langle \phi | \right] . \tag{2.16}$$

Note that due to the normalization condition for the probability distribution $\psi_0[\phi]$, the equation $\langle | \psi_0 \rangle = \int d\phi \, \psi_0[\phi] = 1$ holds.

Time development of the probability distribution can be realized as the Fokker-Planck equation in the Schrödinger picture, while the Langevin equation is realized by the Heisenberg equations (2.11). The probability distribution $\psi[\phi, t]$ is represented by an abstract vector

$|\psi\rangle_t$ in the Schrödinger picture,

$$\psi[\phi,t]=\langle\phi|\psi\rangle_t\ ,$$

and is subject to the Fokker-Planck equation

$$2\kappa\frac{\partial}{\partial t}\psi[\phi,t]=F[\phi,-2\kappa\delta/\delta\phi]\psi[\phi,t]$$
$$=2\kappa\int d^4x\frac{\delta}{\delta\phi(x)}\left[\frac{\delta}{\delta\phi(x)}+\frac{\delta S}{\delta\phi(x)}\right]\times\psi[\phi,t]\ , \qquad (2.17)$$

or, equivalently,

$$2\kappa\frac{d}{dt}|\psi\rangle_t=F[\phi,\pi]|\psi\rangle_t\ . \qquad (2.18)$$

If we make both pictures coincide with each other at $t=0$, $|\psi\rangle_t$ is expressed as

$$|\psi\rangle_t=e^{Ft/2\kappa}|\psi_0\rangle\ . \qquad (2.19)$$

In the operator formalism correlation functions are given by expectation values of the chronological (time-ordered) product of the corresponding field operators. For example, the two-point correlation function $\langle\phi(X)\phi(X')\rangle$ can be written as

$$\begin{aligned}\langle\phi(X)\phi(X')\rangle&=\theta(t-t')\int d\phi\, d\phi'\phi(x)T[\phi,t|\phi',t']\phi'(x')\psi[\phi',t']+\theta(t'-t)\int d\phi\, d\phi'\phi'(x')T[\phi',t'|\phi,t]\phi(x)\psi[\phi,t]\\&=\theta(t-t')\langle\phi(x)T(t,t')\phi(x')|\psi\rangle_{t'}+\theta(t'-t)\langle\phi(x')T(t',t)\phi(x)|\psi\rangle_t\\&=\langle T\phi(X)\phi(X')|\psi_0\rangle\ .\end{aligned} \qquad (2.20)$$

Here the representation of the transition probability distribution

$$T[\phi,t|\phi',t']=\langle\phi|T(t,t')|\phi'\rangle \qquad (2.21)$$

with

$$T(t,t')=e^{F(t-t')/2\kappa} \qquad (2.22)$$

has been used. In a similar manner, n-point correlation functions are obtained:

$$\langle\phi(X_1)\cdots\phi(X_n)\rangle=\langle T\phi(X_1)\cdots\phi(X_n)|\psi_0\rangle\ . \qquad (2.23)$$

The equivalence of the stochastic quantization method to the ordinary ones can be proved also in this operator formalism. The equal-time correlation functions calculated from the above equation (2.23) converge to the corresponding vacuum expectation values in Euclidean field theory in the equilibrium limit. To show this, it is necessary to investigate the spectrum of the Fokker-Planck Hamiltonian F to get an eigenvector belonging to the lowest zero eigenvalue. Note that we must consider both right- and left-eigenvalue equations of F, or in other words, eigenvalue problems for both F and $F^\dagger$,

$$F|u_i\rangle=-2\kappa\lambda_i|u_i\rangle,\quad F^\dagger|v_i\rangle=-2\kappa\lambda_i|v_i\rangle\ , \qquad (2.24)$$

because of the non-Hermite character of F. The equality of eigenvalues for $|u_i\rangle$ and $|v_i\rangle$ in the above equations follows from the fact that F can be made Hermitian by the use of a similarity transformation

$$H\equiv-\frac{1}{2\kappa}e^{S/2}Fe^{S/2}=-\frac{1}{4\kappa}\int d^4x\, Q^\dagger(x)Q(x) \qquad (2.25)$$

with

$$Q(x)=-\pi(x)+\kappa\frac{\delta S}{\delta\phi(x)}\ . \qquad (2.26)$$

It is manifest that the eigenvalues λ_i are positive semidefinite and include the lowest zero eigenvalue. Eigenvectors $|u_i\rangle$ and $|v_i\rangle$ are given by

$$|u_i\rangle=e^{-S/2}|i\rangle,\quad |v_i\rangle=e^{S/2}|i\rangle\ , \qquad (2.27)$$

where $|i\rangle$ stands for an eigenvector of H belonging to λ_i. Then the completeness and orthogonality relations between $|u_i\rangle$ and $|v_i\rangle$ read as

$$\sum_i|u_i\rangle\langle v_i|=1,\quad \langle v_i|u_j\rangle=\delta_{ij}\ . \qquad (2.28)$$

Notice that the lowest eigenstate $|0\rangle$ has its ϕ representation

$$\Psi_0[\phi]\equiv\langle\phi|0\rangle=\frac{1}{\sqrt{c}}e^{-S[\phi]/2} \qquad (2.29)$$

with $c=\int d\phi\exp(-S[\phi])$. The equal-time n-point correlation function is rewritten as

$$\begin{aligned}\langle\phi(x_1,t)\cdots\phi(x_n,t)\rangle&=\langle\phi(x_1,t)\cdots\phi(x_n,t)|\psi_0\rangle\\&=\sum_n\langle\phi(x_1)\cdots\phi(x_n)|u_n\rangle e^{-\lambda_n t}\langle v_n|\psi_0\rangle\\&=\sum_n e^{-\lambda_n t}\int d\phi\,\phi(x_1)\cdots\phi(x_n)e^{-S[\phi]/2}\langle\phi|n\rangle\int d\phi' e^{S[\phi']/2}\langle n|\phi'\rangle\psi_0[\phi']\ .\end{aligned} \qquad (2.30)$$

In the $t \to \infty$ limit, only the contribution coming from the lowest eigenstate survives to give us the desired result

$$\lim_{t\to\infty} \langle \phi(x_1,t)\cdots\phi(x_n,t)\rangle = c^{-1}\int d\phi\,\phi(x_1)\cdots\phi(x_n)e^{-S[\phi]} . \tag{2.31}$$

It is possible to construct a formal perturbation theory based on the operator formalism. We can follow the same procedure as in the usual field theory. Dividing the Hamiltonain F into free and interaction parts, F_0 and F_I, and going into the interaction picture, we get the general perturbative expression ("Dyson formula") for correlation functions. For details, see Ref. 13.

III. SPECTRAL REPRESENTATION OF TWO-POINT CORRELATION FUNCTIONS

One of the great advantages of the operator formalism in SQ lies in the fact that we can utilize various techniques developed in canonical field theory owing to the introduction of the π field conjugate to ϕ. In the stochastic diagrams generated by the original Langevin equation which have only ϕ fields as external legs, it is impossible to define "substochastic diagrams" by cutting the internal lines,[10,22] because of their specific character[8,23] which may be attributed to the existence of two different kinds of propagators in this scheme. This problem has been resolved by the introduction of the π field. We can define substochastic diagrams in the generalized stochastic diagrams with π legs as well as ϕ legs. In this formalism, we can also discuss the asymptotic behavior of stationary correlation functions $\langle T\phi(X_1)\cdots\phi(X_n)|\psi_0\rangle$ with $t_i - t_j$ being fixed.

From the formal resemblance of the operator formalism in SQ to the conventional canonical formalism, one may expect a similar form of the spectral representation of the stationary two-point correlation function in SQ. We can intuitively understand that the large time-difference behavior of the two-point functions could reflect the properties of the one-particle asymptotic states. We should, however, notice that there are essential differences: We do not have a symmetry that could provide us with a "five-dimensional dispersion relation," while the Lorentz invariance of the theory definitely prescribes the structure of the spectral function in the usual case. Furthermore, the meaning of the "time" is quite different in both cases. We need, therefore, a detailed analysis of the two-point functions within the framework of SQ.

In this section the general structure of the stationary two-point correlation functions is investigated on the basis of the operator formalism described in Sec. II. It is shown that the same physical mass as in the conventional field theory can be extracted from their time correlation length. This conclusion is closely related to the asymptotic condition in our formalism which is the subject of Sec. IV.

Let us consider the two-point function $\langle T\phi(X_1)\phi(X_2)|\psi_0\rangle$ in the Heisenberg picture. Substituting $\phi(X_i)$ by the expression (2.12a) we have

$$\langle T\phi(X_1)\phi(X_2)|\psi_0\rangle = \langle \phi(0)e^{[F\tau+P\cdot(x_1-x_2)]/2\kappa} \times\phi(0)e^{(Ft_2+P\cdot x_2)/2\kappa}|\psi_0\rangle . \tag{3.1}$$

Here we have assumed $t_1 = t_2+\tau \ge t_2$ for simplicity and used Eqs. (2.16). Inserting a unit operator $1=\sum_i |u_i\rangle\langle v_i|$ into the above equation, we easily find that the stationary correlation function is represented by the "vacuum-to-vacuum" expectation value in the large-time limit:

$$\lim_{t_2\to\infty}\langle T\phi(X_1)\phi(X_2)|\psi_0\rangle = \langle v_0|\phi(0)e^{[F\tau+P\cdot(x_1-x_2)]/2\kappa}\phi(0)|u_0\rangle . \tag{3.2}$$

The assumption of nondegeneracy of the vacuum state of F implies the equality $\langle\,=\sqrt{c}\,\langle v_0|$ which, together with the normalization condition of $|\psi_0\rangle$ and (2.16), has been used in deriving (3.2). Note that the stationary correlation function only depends on $\tau=t_1-t_2$ and x_1-x_2, which is the manifestation of the translational invariance in the large-time limit. Of course, the assumption of the existence of a discrete and nondegenerate zero eigenstate is essential at this point and care should be taken in dealing with, for example, gauge fields.

Further, by the use of simultaneous eigenstates of F and P_μ, $|u_i,s;p\rangle$ and $|v_i,s;p\rangle$, satisfying

$$\begin{aligned} F|u_i,s;p\rangle &= -2\kappa\lambda_i|u_i,s;p\rangle ,\\ F^\dagger|v_i,s;p\rangle &= -2\kappa\lambda_i|v_i,s;p\rangle ,\\ P_\mu|u_i,s;p\rangle &= 2\kappa i p_\mu|u_i,s;p\rangle ,\\ P_\mu^\dagger|v_i,s;p\rangle &= -2\kappa i p_\mu|v_i,s;p\rangle , \end{aligned} \tag{3.3}$$

the right-hand side (RHS) of (3.2) is expressed as

$$\langle v_0|\phi(0)e^{[F\tau+P\cdot(x_1-x_2)]/2\kappa}\phi(0)|v_0\rangle = \int\frac{d^4p}{(2\pi)^4}\sum_{i,s}\langle v_0|\phi(0)|u_i,s;p\rangle\langle v_i,s;p|\phi(0)|u_0\rangle \times e^{-\lambda_i\tau}e^{ip\cdot(x_1-x_2)} . \tag{3.4}$$

Here the index s refers to quantum numbers other than i and p. Setting $\tau=0$ after differentiating with respect to τ, we get

$$\langle v_0|\phi(x_1)\frac{1}{2\kappa}F\phi(x_2)|u_0\rangle = -\int\frac{d^4p}{(2\pi)^4}\left[\sum_{i,s}\langle v_0|\phi(0)|u_i,s;p\rangle\langle v_i,s;p|\phi(0)|u_0\rangle\lambda_i\right]e^{ip\cdot(x_1-x_2)} . \tag{3.5}$$

The LHS of this equation can further be reduced to

$$\left\langle v_0 \left| \phi(x_1)\frac{1}{2\kappa}[F,\phi(x_2)] \right| u_0 \right\rangle$$
$$= -\left\langle v_0 \left| \phi(x_1)\left[\pi(x_2)-\kappa\frac{\delta S}{\delta\phi(x_2)}\right] \right| u_0 \right\rangle$$
$$= -\tfrac{1}{2}\langle v_0|\phi(x_1)\pi(x_2)|u_0\rangle$$
$$= -\kappa\delta^4(x_1-x_2)\ , \tag{3.6}$$

where the stationary condition for $|u_0\rangle$,

$$\left[\pi(x)-2\kappa\frac{\delta S}{\delta\phi(x)}\right]|u_0\rangle=0\ , \tag{3.7}$$

and the relation $\langle v_0|\pi(x)=\sqrt{c}^{\,-1}\langle\pi(x)=0$ have been used. Thus we can set

$$\sum_s \langle v_0|\phi(0)|u_i,s;p\rangle\langle v_i,s;p|\phi(0)|u_0\rangle=\frac{\kappa\rho_i}{\lambda_i}\quad (i\neq 0) \tag{3.8}$$

with the normalization condition

$$\sum_{i\neq 0}\rho_i=1\ , \tag{3.9}$$

and the following expression of the stationary two-point function is obtained:

$$D(X_1,X_2)\equiv \lim_{\min(t_1,t_2)\to\infty}\langle T\phi(X_1)\phi(X_2)|\psi_0\rangle$$
$$=\int\frac{d^4p}{(2\pi)^4}\sum_{i\neq 0}\frac{\kappa\rho_i}{\lambda_i}e^{-\lambda_i|\tau|}e^{ip\cdot(x_1-x_2)}\ . \tag{3.10}$$

In this expression, the contribution coming from the intermediate vacuum state has been omitted as in the usual scalar field theory. The above expression shows the general structure of the stationary two-point function, and physical quantities are extracted from eigenvalues λ_i and the function ρ_i. The positivity of ρ_i can be shown using the relation between $|u_i\rangle$ and $|v_i\rangle$ [see (2.27)] as follows:

$$\kappa\rho_i=\lambda_i\sum_s\langle v_0|\phi(0)|u_i,s;p\rangle\langle v_i,s;p|\phi(0)|u_0\rangle$$
$$\propto\lambda_i\sum_s\langle v_0|\phi(0)|u_i,s;p\rangle\langle u_i,s;p|e^S\phi(0)e^{-S}|v_0\rangle$$
$$=\lambda_i\sum_s|\langle v_0|\phi(0)|u_i,s;p\rangle|^2\geq 0\ . \tag{3.11}$$

The function ρ_i can be considered as a sort of generalization of the spectral function in conventional field theory. Its connection with the ordinary spectral function will be discussed below.

It may be worthwhile to note that the eigenvalues λ_i and the function ρ_i characterize not only the stationary ϕ-ϕ correlation function as in (3.10) but also the stationary ϕ-π correlation function in the form

$$G(X_1,X_2)\equiv \lim_{\min(t_1,t_2)\to\infty}\langle T\phi(X_1)\pi(X_2)|\psi_0\rangle$$
$$=2\kappa\theta(\tau)\int\frac{d^4p}{(2\pi)^4}\sum_{i\neq 0}\rho_i e^{-\lambda_i\tau}e^{ip\cdot(x_1-x_2)}\ . \tag{3.12}$$

A. Free case

To get physical quantities from the above two-point functions, it is necessary to clarify the structure of the function ρ_i and the spectrum of the Fokker-Planck Hamiltonian F. As the simplest case, let us consider the free case. Though the exact expression for two-point functions is known in this case, the procedure developed below will be of help when we take interactions into account.

The explicit forms of the Heisenberg equations for ϕ and π in (2.11) are

$$\dot\phi(X)=\frac{1}{2\kappa}[\phi(X),F_0]=\pi(X)-\kappa(-\partial^2+m_0^2)\phi(X)\ , \tag{3.13a}$$

$$\dot\pi(X)=\frac{1}{2\kappa}[\pi(X),F_0]=\kappa(-\partial^2+m_0^2)\pi(X)\ , \tag{3.13b}$$

where F_0 is the free Hamiltonian

$$F_0=\tfrac{1}{2}\int d^4x\ \pi(x)[\pi(x)-2\kappa(-\partial^2+m_0^2)\phi(x)]\ . \tag{3.14}$$

Note that F_0 can be written as a product of two operators $a(p,t)$ and $\bar a^\dagger(p,t)$:

$$F_0=-\tfrac{1}{2}\int d^4p\ \bar a^\dagger(-p)2\kappa(p^2+m_0^2)a(p) \tag{3.15}$$

with

$$a(p,t)=\int\frac{d^4x}{(2\pi)^2}\{\phi(X)-\pi(X)/[2\kappa(p^2+m_0^2)]\}\times e^{-ip\cdot x}\ , \tag{3.16a}$$

$$\bar a^\dagger(p,t)=\int\frac{d^4x}{(2\pi)^2}\pi(X)e^{-ip\cdot x}\ . \tag{3.16b}$$

These operators satisfy the equal-time commutation relation

$$[a(p,t),\bar a^\dagger(q,t)]=2\kappa\delta^4(p+q) \tag{3.17}$$

and the commutation relations with F_0:

$$[-F_0,a(p,t)]=-2\kappa^2(p^2+m_0^2)a(p,t)\ , \tag{3.18a}$$

$$[-F_0,\bar a^\dagger(p,t)]=2\kappa^2(p^2+m_0^2)\bar a^\dagger(p,t)\ . \tag{3.18b}$$

Two operators a and $\bar a^\dagger$ may be interpreted as "annihilation" and/or "creation" operators depending on which states, $|u_0\rangle$ or $\langle v_0|$, they act on. Note that the lowest zero eigenstates of F_0, $|u_0\rangle$, and $\langle v_0|$ satisfy

$$a(p)|u_0\rangle=0,\quad \langle v_0|\bar a^\dagger(p)=0\ .$$

Then we have the n-particle state with momentum p_μ:

$$|n,\lambda_n,s_n;p\rangle = \prod_{\substack{i=1,\\ \sum p_i=p}}^{n} \bar{a}^{\dagger}(p_i)|u_0\rangle \tag{3.19}$$

which satisfies

$$F_0|n,\lambda_n,s_n;p\rangle = -2\kappa^2\lambda_n|n,\lambda_n,s_n;p\rangle \ , \tag{3.20a}$$

$$P_\mu|n,\lambda_n,s_n;p\rangle = 2\kappa i p_\mu|n,\lambda_n,s_n;p\rangle \ . \tag{3.20b}$$

Here the eigenvalue λ_n of F_0 is parametrized by two real numbers α_n and m_n^2 as

$$\lambda_n = \alpha_n p^2 + m_n^2 \tag{3.21a}$$

with the conditions

$$\alpha_n \geq \frac{1}{n},\quad m_n^2 \geq n m_0^2 \quad (n>1)\ ,$$
$$\alpha_1 = 1,\quad m_1^2 = m_0^2\ , \tag{3.21b}$$
$$\lambda_0 = 0$$

and s_n represents quantum numbers other than α_n and m_n^2. In the same way, the eigenstate of $F_0^\dagger$ or left eigenstate of F_0 also belonging to the eigenvalue λ_n is constructed:

$$\langle n,\lambda_n,s_n;p| = \langle v_0| \prod_{\substack{i=1\\ \sum p_i=p}}^{n} a(p_i)\ . \tag{3.22}$$

These bras and kets, after being properly normalized, are orthogonal to each other,

$$\langle n,\lambda_n,s_n;p|n',\lambda'_{n'},s'_{n'};p'\rangle = \delta_{n,n'}\delta^4(p+p')\delta(\lambda_n-\lambda'_n)\delta(s_n-s'_n)\ , \tag{3.23}$$

and form a complete set:

$$1 = \int \frac{d^4p}{(2\pi)^4} \sum_{n=0} \int_{nm_0^2}^{\infty} d\lambda_n \int ds_n |n,\lambda_n,s_n;p\rangle\langle n,\lambda_n,s_n;p|$$
$$= |u_0\rangle\langle v_0| + \int \frac{d^4p}{(2\pi)^4}\int_0^\infty d\lambda \int ds \sum_{n=1} f_n(\lambda,s)|n,\lambda,s;p\rangle\langle n,\lambda,s;p|\ . \tag{3.24}$$

A positive function $f_n(\lambda,s)$ has been introduced to take the conditions (3.21a) and (3.21b) into account.

Now we are in a position to get the explicit expression of (3.10). Substituting the summation over i in (3.10) by the integration over λ we have

$$D(X_1,X_2) = \int \frac{d^4p}{(2\pi)^4}\int_0^\infty d\lambda \frac{\rho_0(\lambda)}{\lambda} e^{-\kappa\lambda|\tau|} e^{ip(x_1-x_2)}\ . \tag{3.25}$$

Here $\rho_0(\lambda)$ is similarly defined by (3.8),

$$\kappa\rho_0(\lambda) = \lambda \int ds \sum_{n=1} f_n(\lambda,s)\langle v_0|\phi(0)|n,\lambda,s;p\rangle \times \langle n,\lambda,s;p|\phi(0)|u_0\rangle\ , \tag{3.26}$$

and depends on p^2 through the p^2 dependence of λ (3.21a) as is shown below in the free case. Taking the p^2 dependence of λ (3.21a) into account, we can see that there exists a function $\bar{\rho}_0(\alpha,m^2)$ defined by

$$\rho_0(\lambda) = \int_0^\infty d\alpha \int_0^\infty dm^2 \delta(\alpha p^2 + m^2 - \lambda)\bar{\rho}_0(\alpha,m^2) \tag{3.27}$$

with the normalization condition (3.9):

$$\int_0^\infty d\lambda\, \rho_0(\lambda) = \int_0^\infty d\alpha \int_0^\infty dm^2 \bar{\rho}_0(\alpha,m^2) = 1\ . \tag{3.28}$$

Note that owing to the above definitions of ρ_0 and $\bar{\rho}_0$ and the normalization condition, the function $\bar{\rho}_0$ cannot depend on p^2. The above correlation function is rewritten as

$$D(X_1,X_2) = \int \frac{d^4p}{(2\pi)^4}\int_0^\infty d\alpha \int_0^\infty dm^2 \frac{\bar{\rho}_0(\alpha,m^2)}{\alpha p^2+m^2} e^{-\kappa(\alpha p^2+m^2)|\tau|} e^{ip\cdot(x_1-x_2)}\ . \tag{3.29}$$

The positivity of the integrand in (3.26) is explicitly seen from

$$\langle v_0|\phi(0)|n,\lambda,s;p\rangle\langle n,\lambda,s;p|\phi(0)|u_0\rangle \propto \left\langle v_0\left|\phi(0)\prod_{i=1}^n \bar{a}^\dagger(p_i)\right|u_0\right\rangle\left\langle v_0\left|\prod_{j=1}^n a(p_j)\phi(0)\right|u_0\right\rangle$$
$$= \left\langle u_0\left|e^{S_0}\phi(0)\prod_{i=1}^n \bar{a}^\dagger(p_i)e^{-S_0}\right|v_0\right\rangle\left\langle v_0\left|\prod_{j=1}^n a(p_j)\phi(0)\right|u_0\right\rangle$$
$$= \prod_{i=1}^n 2\kappa(p_i^2+m_0^2)\left|\left\langle v_0\left|\prod_{j=1}^n a(p_j)\phi(0)\right|u_0\right\rangle\right|^2 \geq 0\ , \tag{3.30}$$

where the relation between $\bar{a}^\dagger$ and a,

$$e^{S_0}\bar{a}^\dagger(p)e^{-S_0}=2\kappa(p^2+m_0^2)a^\dagger(p) ,$$

has been used.

We can further calculate matrix elements of ϕ in (3.26). Rewriting $\phi(0)$ as

$$\phi(0)=\int d^4p \left[a(p)+\frac{\bar{a}^\dagger(p)}{2\kappa(p^2+m_0^2)}\right]$$

we find that only the one-particle state contributes to ρ_0 which is reduced to

$$\begin{aligned}\rho_0(\lambda)&=\lambda f_1(\lambda)2\kappa(p^2+m_0^2)|\langle 1,\lambda;p|\phi(0)|u_0\rangle|^2\\&=\lambda f_1(\lambda)\frac{1}{p^2+m_0^2}\\&=\delta(p^2+m_0^2-\lambda) .\end{aligned} \tag{3.31}$$

Then the well-known results for the stationary two-point functions follow:

$$D(X_1,X_2)=\int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2+m_0^2}e^{-\kappa(p^2+m_0^2)|\tau|}e^{ip\cdot(x_1-x_2)} , \tag{3.32a}$$

$$G(X_1,X_2)=2\kappa\theta(\tau)\int \frac{d^4p}{(2\pi)^4}e^{-\kappa(p^2+m_0^2)\tau}e^{ip\cdot(x_1-x_2)} . \tag{3.32b}$$

B. Interacting case

In the usual field theory, the total Hamiltonian can be diagonalized by means of asymptotic fields. These asymptotic fields, though they are Heisenberg fields, satisfy free field equations with the physical renormalized mass term. Physical states are spanned by asymptotic states and possible bound states.

In this subsection, we solve the eigenvalue problems of F (3.3) to get eigenvalues λ_i and investigate the structure of the "spectral function" ρ_i in (3.10) in the interacting case. For this purpose we must find such a representation that makes the Hamiltonian F diagonal. As we have already got vacuum states $|u_0\rangle$ and $|v_0\rangle$ even in the interacting case [see (2.27) and (2.29)], we need to find an operator whose commutator with F is proportional to itself. Then it would give us all eigenstates of F.

Recalling that the π field itself has satisfied this condition in the free case [see (3.13b)], and in the interacting case its commutator with F is still proportional to itself but with the ϕ-dependent coefficient (2.11b), we are naturally led to look for an operator $\tilde{\pi}$ which is generated from π by a similarity transformation U:

$$\tilde{\pi}(X)=U^{-1}(t)\pi(X)U(t) . \tag{3.33}$$

The commutator between F and $\tilde{\pi}$ is easily calculated

$$\begin{aligned}[\tilde{\pi}(X), F]&=U^{-1}(t)\{[\pi(X), F]+[\pi(X),[U(t), F]U^{-1}(t)]\}U(t)\\&=2\kappa U^{-1}(t)\left[\kappa\pi(X)\frac{\partial}{\partial\phi(X)}\frac{\delta S}{\delta\phi(X)}+[\pi(X), \dot{U}(t)U^{-1}(t)]\right]U(t) .\end{aligned} \tag{3.34}$$

This expression shows that if the similarity transformation operator U is so defined as to subtract nonlinear terms in $\delta S/\delta\phi$ (the interacting part of F), then $\tilde{\pi}$ meets the requirement. The operator U should satisfy

$$\dot{U}(t)=-\frac{1}{2\kappa}F_I(t)U(t)=\frac{1}{2\kappa}\int d^4x\ \pi(X)\frac{\delta S_I}{\delta\phi(X)}U(t) \tag{3.35a}$$

or

$$\dot{U}(t)=-\frac{1}{2\kappa}U(t)\tilde{F}_I(t)=\frac{1}{2\kappa}U(t)\int d^4x\ \tilde{\pi}(X)\frac{\delta\tilde{S}_I}{\delta\tilde{\phi}(X)} , \tag{3.35b}$$

where quantities with tildes are similarly defined as in (3.33).

To determine $U(t)$ we must specify its initial value which turns out to prescribe the asymptotic behavior of fields. Here we assume, as in the usual field theory, that the interaction Hamiltonian F_I is switched off in the remote past, and as a result, field variables are governed by the free Heisenberg equations

$$\dot{\tilde{\phi}}(X)=\tilde{\pi}(X)-\kappa_R(-\partial^2+\bar{m}_R^2)\tilde{\phi}(X) , \tag{3.36a}$$

$$\dot{\tilde{\pi}}(X)=\kappa_R(-\partial^2+\bar{m}_R^2)\tilde{\pi}(X) , \tag{3.36b}$$

in this limit. Quantities with subscripts R stand for renormalized ones.[24] The operator U must satisfy

$$\lim_{t\to-\infty} U(t)\to 1 \tag{3.37}$$

and is expressed as

$$\begin{aligned}U(t)&=T\exp\left[-\frac{1}{2\kappa}\int_{-\infty}^{t}dt'F_I(t')\right]\\&=\bar{T}\exp\left[-\frac{1}{2\kappa}\int_{-\infty}^{t}dt'\tilde{F}_I(t')\right] ,\end{aligned} \tag{3.38}$$

where $\bar{T}$ is an antichronological ordering operator.

Here the limiting procedure (3.37) which implies

$$\phi_R(X)\to\tilde{\phi}(X),\quad \pi_R(X)\to\tilde{\pi}(X)\ \text{ as } t\to-\infty , \tag{3.39}$$

should be considered as a weak limit, and we must introduce an adiabatic factor to switch off the interaction in this limit. In the discussion below, however, we simply assume (3.39) and explore its consequences. More careful analysis concerning the asymptotic behavior is to be discussed in the next section.

It is worthwhile to mention that the relation

$$U(t)F=F_0(t)U(t) \tag{3.40}$$

or, in other words,

$$F[\phi,\pi]=U^{-1}(t)F_0(t)U(t)=F_0[\tilde{\phi},\tilde{\pi}] \tag{3.41}$$

is derived from (3.35). This means that the total Hamiltonian F can be diagonalized and has a simple form if it is expressed in terms of asymptotic fields $\tilde{\pi}$ and $\tilde{\phi}$.[25] The free Hamiltonian $\tilde{F}_0$ takes the same form as in (3.14) with renormalized quantities κ_R and $\overline{m}_R$ and is in fact time independent:

$$\begin{aligned}\frac{d}{dt}\tilde{F}_0=\tfrac{1}{2}\int d^4x\{\,&\dot{\tilde{\pi}}(X)[\tilde{\pi}(X)-2\kappa_R(-\partial^2+\overline{m}_R^{\,2})\tilde{\phi}(X)]\\&+\tilde{\pi}(X)[\,\dot{\tilde{\pi}}(X)-2\kappa_R(-\partial^2+\overline{m}_R^{\,2})\dot{\tilde{\phi}}(X)]\}\\=0\ .\end{aligned} \tag{3.42}$$

Now we can follow the same procedure developed in Sec. III A to construct a set of eigenstates of $\tilde{F}_0$ which also diagonalize F. It is easily shown that the stationary two-point functions are expressed as

$$D(X_1,X_2)=Z_\kappa\int\frac{d^4p}{(2\pi)^4}\int_0^\infty d\alpha\times\int_0^\infty dm^2\frac{\overline{\rho}(\alpha,m^2)}{\alpha p^2+m^2}e^{-\kappa_R(\alpha p^2+m^2)|\tau|}\times e^{ip\cdot(x_1-x_2)}\ , \tag{3.43}$$

$$G(X_1,X_2)=2\kappa\theta(\tau)\int\frac{d^4p}{(2\pi)^4}\int_0^\infty d\alpha\times\int_0^\infty dm^2\overline{\rho}(\alpha,m^2)e^{-\kappa_R(\alpha p^2+m^2)\tau}\times e^{ip\cdot(x_1-x_2)}\ , \tag{3.44}$$

where $Z_\kappa=\kappa/\kappa_R$ is a renormalization constant for κ. The spectral function $\overline{\rho}$ is similarly defined as in (3.26) and (3.27) and is a non-negative function. Here the p^2 dependence of the spectral function ρ is assumed to appear only through the eigenvalue of the asymptotic Hamiltonian $\tilde{F}_0$, which implies that the total dynamics is assumed to be completely prescribed by this asymptotic Hamiltonian. This is only an assumption, but its consistency is seen by proving the spectral condition in the next section. Under this assumption, the spectral function $\overline{\rho}$ is a momentum-independent function by construction.

Unlike in the free case, we cannot get the detailed structure of $\overline{\rho}$. However, the normalization condition (3.9) or (3.28), i.e.,

$$\int_0^\infty d\alpha\int_0^\infty dm^2\overline{\rho}(\alpha,m^2)=1 \tag{3.45}$$

and the equality at $\tau=0$,

$$D(X_1,X_2)|_{\tau=0}=\langle T\phi(x_1)\phi(x_2)\rangle_{\rm FT}\ , \tag{3.46}$$

where the RHS stands for the conventional two-point function in the field theory, give us several important properties of the stationary correlation function. The usual spectral representation of the two-point function

$$\langle T\phi(x_1)\phi(x_2)\rangle_{\rm FT}=\int\frac{d^4p}{(2\pi)^4}\int_0^\infty dm^2\frac{\rho_F(m^2)}{p^2+m^2}e^{ip\cdot(x_1-x_2)} \tag{3.47}$$

and the above equation (3.46) give us

$$Z_\kappa\int_0^\infty d\alpha\int_0^\infty dm^2\frac{\overline{\rho}(\alpha,m^2)}{\alpha p^2+m^2}=\int_0^\infty dm^2\frac{\rho_F(m^2)}{p^2+m^2}\ , \tag{3.48}$$

where ρ_F is the conventional spectal function. After changing the integration variable $m^2\rightarrow m^2\alpha$ and using the uniqueness of the Laplace transformation, we obtain

$$Z_\kappa\int_0^\infty d\alpha\,\overline{\rho}(\alpha,m^2\alpha)=\rho_F(m^2)\ . \tag{3.49}$$

Assuming the nature of the physical states as[17,25]

$$\rho_F(m^2)=Z_\phi\delta(m^2-m_R^2)+\sigma_F(m^2)\ , \tag{3.50}$$

where Z_ϕ is a wave-function renormalization constant and σ_F stands for contributions from continuum states, we can determine the structure of $\overline{\rho}$:

$$\overline{\rho}(\alpha,m^2)=Z_\kappa^{-1}[Z_\phi\delta(\alpha-1)\delta(m^2-m_R^2)+\overline{\sigma}(\alpha,m^2)] \tag{3.51}$$

with

$$\int_0^\infty d\alpha\,\overline{\sigma}(\alpha,m^2\alpha)=\sigma_F(m^2)\ . \tag{3.52}$$

Note that the contribution to $\overline{\rho}$ from the one-particle state is seen in (3.51) in the form of the δ function which is a consequence of the conditions (3.21b), and that the structure of $\overline{\rho}$ derived from (3.49) implies that the one-particle state defined by the operator $\tilde{\pi}$ has exactly the same renormalized mass as that in the usual theory ($\overline{m}_R=m_R$).

Substituting $\overline{\rho}$ in (3.43) and (3.44) by (3.51), we finally obtain

$$D(X_1,X_2)=\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot(x_1-x_2)}\left[\frac{Z_\phi}{p^2+m_R^2}e^{-\kappa_R(p^2+m_R^2)|\tau|}+\int_0^\infty d\alpha\int_0^\infty dm^2\frac{\overline{\sigma}(\alpha,m^2)}{\alpha p^2+m^2}e^{-\kappa_R(\alpha p^2+m^2)|\tau|}\right]\ , \tag{3.53}$$

$$G(X_1,X_2)=2\kappa_R\theta(\tau)\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot(x_1-x_2)}\left[Z_\phi e^{-\kappa_R(p^2+m_R^2)\tau}+\int_0^\infty d\alpha\int_0^\infty dm^2\overline{\sigma}(\alpha,m^2)e^{-\kappa_R(p^2+m^2)\tau}\right] \tag{3.54}$$

with the normalization conditions (3.45) and (3.52), the former being written as

$$Z_\phi+\int_0^\infty d\alpha\int_0^\infty dm^2\bar{\sigma}(\alpha,m^2)=Z_\kappa\ . \tag{3.55}$$

It may be worthwhile to stress again that under the above assumption (3.39) concerning the asymptotic fields, the exponential decay rate with respect to the fictitious time of stationary two-point correlation functions gives us the very mass we expect in conventional field theory. It is by no means a trivial fact, though naively anticipated already in Ref. 10.

IV. SPECTRAL CONDITION

The asymptotic behavior mentioned in Sec. III clearly needs a much more careful treatment. First, we must introduce the adiabatic factor which allows us to consider noninteracting fields in the asymptotic region. Second, we must remember that the limiting procedure is taken in a weak sense, that is, the equality holds between matrix elements separately, not between operators themselves. Here we try to show that the total Hamiltonian F is indeed diagonalized by asymptotic fields (the spectral condition) taking the above points into account.

If we introduce a naive adiabatic factor $e^{-\epsilon|t|}$ to follow the usual procedure to calculate the matrix elements of F between eigenstates of $\tilde{F}_0$, we encounter a serious problem of divergent matrix elements. This problem has its origin in the fact that the basic Langevin equation is a diffusion-type one and every matrix element except for diagonal ones has an exponentially blowing up (or decaying) time dependence. So the naive adiabatic factor $e^{-\epsilon|t|}$ has nothing to do with controlling the exponentially increasing contributions at $|t|=\infty$, in contrast to the situation in conventional field theory where the oscillating time dependence can be suppressed by this factor. For our purpose, it is indispensable to find a way around this divergence.

To get the finite results, let us suppose that the interaction is confined to a finite region $[-T,t]$. We introduce the following adiabatic factor (plus an operator) $g(t)$:

$$g(t)=\theta(t+T)\left[e^{-\epsilon|t|}-e^{-\epsilon T}\frac{d_t}{\epsilon+d_t}\right], \tag{4.1}$$

where ϵ is an infinitesimal positive parameter and d_t stands for the total derivative with respect to t. Note that the second term reduces smoothly to zero in the $T\to\infty$ limit ($\epsilon>0$) and $g(t)$ satisfies

$$\int_{-\infty}^t dt'g(t')e^{\lambda t'}=\frac{1}{\epsilon+\lambda}(e^{(\epsilon+\lambda)t}-e^{\lambda t}e^{-\epsilon T})\ , \tag{4.2a}$$

$$\int_{-\infty}^t dt'\dot{g}(t')e^{\lambda t'}=\frac{\epsilon}{\epsilon+\lambda}e^{(\epsilon+\lambda)t}\ , \tag{4.2b}$$

where $t\le 0$ is assumed for simplicity. We can see from these equations that potentially divergent terms such as $e^{-\lambda T}$ are completely canceled out owing to the presence of the second term in $g(t)$. The two limiting procedures $\epsilon\to 0$ and $T\to\infty$ should be taken in such an order that the $T\to\infty$ limit is followed by the $\epsilon\to 0$ limit after integrations over t. Only in this order, the relation

$$\lim_{\epsilon\to 0}\lim_{T\to\infty}g(t)=1$$

holds and we can recover the original dynamics.

Now the matrix element of the total Hamiltonian F is defined by

$$\langle a|F|b\rangle=\lim_{\epsilon\to 0}\lim_{T\to\infty}\langle a|F(t)|b\rangle\ , \tag{4.3}$$

where $\langle a|$ and $|b\rangle$ are eigenstates of $\tilde{F}_0$ belonging to eigenvalues $-2\kappa\lambda_a$ and $-2\kappa\lambda_b$, respectively, and

$$F(t)=U(t)[\tilde{F}_0+g(t)\tilde{F}_I(t)]U^{-1}(t)\ . \tag{4.4}$$

Here the similarity transformation operator is given by

$$U(t)=\bar{T}\exp\left[-\frac{1}{2\kappa}\int_{-\infty}^t dt'g(t')\tilde{F}_I(t')\right] \tag{4.5}$$

which leads to the following Heisenberg equation for an operator $\tilde{A}$:

$$\frac{d}{dt}\tilde{A}=\dot{\tilde{A}}+\frac{1}{2\kappa}[\tilde{A},\tilde{F}_0]\ . \tag{4.6}$$

By the use of the relation

$$\begin{aligned}F(t)&=F(-\infty)+\int_{-\infty}^t dt'\frac{d}{dt'}F(t')\\&=\tilde{F}_0+\int_{-\infty}^t dt'U(t')\dot{g}(t')\tilde{F}_I(t')U^{-1}(t')\ ,\end{aligned} \tag{4.7}$$

the matrix element of $F(t)$ is expressed as[25]

$$\begin{aligned}\langle a|F(t)|b\rangle&=-2\kappa\lambda_a\delta_{ab}+\int_{-\infty}^t dt'\langle a|U(t')\dot{g}(t')\tilde{F}_I(t')U^{-1}(t')|b\rangle\\&=-2\kappa\lambda_a\delta_{ab}+\int_{-\infty}^t dt_1\dot{g}(t_1)\langle a|\tilde{F}_I(t_1)|b\rangle+\left[-\frac{1}{2\kappa}\right]\int_{-\infty}^t dt_1\int_{-\infty}^{t_1}dt_2\dot{g}(t_1)g(t_2)\langle a|[\tilde{F}_I(t_2),\tilde{F}_I(t_1)]|b\rangle\\&\quad+\left[-\frac{1}{2\kappa}\right]^2\int_{-\infty}^t dt_1\int_{-\infty}^{t_1}dt_2\int_{-\infty}^{t_2}dt_3\dot{g}(t_1)g(t_2)g(t_3)\langle a|[\tilde{F}_I(t_3),[\tilde{F}_I(t_2),\tilde{F}_I(t_1)]]|b\rangle+\cdots\ .\end{aligned} \tag{4.8}$$

Using the relations (4.2a) and (4.2b) we can easily calculate the lowest-order contribution:

$$\int_{-\infty}^t dt_1\dot{g}(t_1)\langle a|\tilde{F}_I(t_1)|b\rangle=\frac{\epsilon}{\epsilon+\lambda_{ab}}e^{(\epsilon+\lambda_{ab})t}\langle a|\tilde{F}_I(0)|b\rangle\quad(\lambda_{ab}\equiv\lambda_a-\lambda_b)\ . \tag{4.9}$$

Thus after eliminating the adiabatic cutoff we find that only the diagonal term survives to contribute to the total Hamiltonian, i.e.,

$$\lim_{\epsilon\to 0}\lim_{T\to\infty}\int_{-\infty}^{t} dt_1 \dot{g}(t_1)\langle a|\tilde{F}_I(t_1)|b\rangle=\delta_{ab}\langle a|\tilde{F}_I(0)|a\rangle \ . \tag{4.10}$$

We subtract this contribution from the interaction Hamiltonian $\tilde{F}_I$ and define a new Hamiltonian $\tilde{F}_I^{(1)}$ which has no diagonal matrix element. This subtraction causes the shift in eigenvalues of $\tilde{F}_0$,

$$-2\kappa\lambda_a \to -2\kappa\lambda_a+\langle a|\tilde{F}_I(0)|a\rangle \equiv -2\kappa\lambda_a^{(1)} \ ,$$

and we have

$$\langle a|F(t)|b\rangle=-2\kappa\lambda_a^{(1)}\delta_{ab}+\left[-\frac{1}{2\kappa}\right]\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\dot{g}(t_1)g(t_2)\langle a|[\tilde{F}_I^{(1)}(t_2),\tilde{F}_I^{(1)}(t_1)]|b\rangle+\cdots \ . \tag{4.11}$$

The next-order contribution is similarly calculated to be

$$\begin{aligned}
&\left[-\frac{1}{2\kappa}\right]\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\dot{g}(t_1)g(t_2)\langle a|[\tilde{F}_I^{(1)}(t_2),\tilde{F}_I^{(1)}(t_1)]|b\rangle \\
&=\left[-\frac{1}{2\kappa}\right]\sum_{n\neq a,b}\left[\frac{e^{(2\epsilon+\lambda_{ab}^{(1)})t}}{(\epsilon+\lambda_{an}^{(1)})(2\epsilon+\lambda_{ab}^{(1)})}-\frac{\epsilon e^{(\epsilon+\lambda_{ab}^{(1)})t-\epsilon T}}{(\epsilon+\lambda_{an}^{(1)})(\epsilon+\lambda_{ab}^{(1)})}-\frac{\epsilon e^{(2\epsilon+\lambda_{ab}^{(1)})t}}{(\epsilon+\lambda_{nb}^{(1)})(2\epsilon+\lambda_{ab}^{(1)})}+\frac{\epsilon e^{(\epsilon+\lambda_{ab}^{(1)})t-\epsilon T}}{(\epsilon+\lambda_{nb}^{(1)})(\epsilon+\lambda_{ab}^{(1)})}\right] \\
&\qquad\times\langle a|\tilde{F}_I^{(1)}(0)|n\rangle\langle n|\tilde{F}_I^{(1)}(0)|b\rangle \\
&\underset{T\to\infty}{\longrightarrow}\left[-\frac{1}{2\kappa}\right]\sum_{n\neq a,b}\frac{\epsilon e^{(2\epsilon+\lambda_{ab}^{(1)})t}}{2\epsilon+\lambda_{ab}^{(1)}}\left[\frac{1}{\epsilon+\lambda_{an}^{(1)}}-\frac{1}{\epsilon+\lambda_{nb}^{(1)}}\right]\langle a|\tilde{F}_I^{(1)}(0)|n\rangle\langle n|\tilde{F}_I^{(1)}(0)|b\rangle \\
&\underset{\epsilon\to 0}{\longrightarrow}\left[-\frac{1}{2\kappa}\right]\delta_{ab}\sum_{n\neq a}\langle a|\tilde{F}_I^{(1)}(0)|n\rangle\frac{1}{\lambda_{an}^{(1)}}\langle n|\tilde{F}_I^{(1)}(0)|a\rangle \ .
\end{aligned} \tag{4.12}$$

It is easily seen that the T dependence only appears through such terms as $e^{-\epsilon T}$ even in higher-order terms, which enables us to take $T\to\infty$ without any serious problems. Repeating the same procedure we arrive at the conclusion that the total Hamiltonian F is diagonalized in the space spanned by the properly renormalized basis of the asymptotic Hamiltonian $\tilde{F}_0$.

Alternatively, we can reformulate the operator formalism in Minkowski space to prove the spectral condition using the naive adiabatic factor $e^{-\epsilon|t|}$. (See Appendix.)

V. SUMMARY AND DISCUSSIONS

In this paper, we have investigated the structure of the stationary two-point functions on the basis of the operator formalism in SQ. In the operator formalism, correlation functions in the stationary state are given by vacuum expectation values of T products of operators and the techniques used in canonical field theory are available. In deriving the spectral representation of the stationary two-point functions, knowledge of the eigenvalues and eigenstates of the total Hamiltonian is indispensable. We have assumed the existence of asymptotic free fields. This assumption which prescribes the structure of the physical space, together with the canonical commutation relations of ϕ and π, the stationary condition for the vacuum state, and the boundary condition in the equilibrium limit, has played an important role in the derivation of the spectral representation of two-point functions (3.43) and (3.44).

The assumption of the existence of asymptotic fields in SQ which is an essential point in our formalism clearly needs some interpretation. In conventional field theory, the existence of asymptotic fields is required from the physical background, i.e., the physical setup of the scattering process. Although there is no such physical situation in the stochastic process with respect to fictitious time, the notion of noninteracting fields seems necessary in considering the spectrum of the total Hamiltonian even in this case. In the usual terminology, the asymptotic fields considered here are *in* fields. We may imagine the "physical" situation as follows. In the remote past, there is no interaction and field variables are subject to free field equations. As t proceeds, the interaction is turned on adiabatically and in the $t\to\infty$ limit the quantum field theory is reproduced irrespective of the initial distribution $\psi_0[\phi]$. And what has been shown in this paper is that if we assume the existence of such an asymptotic field, then it should satisfy the free Heisenberg equation with the same physical mass expected in the usual field theory. The exact expression of two-point correlation functions in the O(N) nonlinear σ model in the large-N limit[26] which has essentially the same form as (3.43) seems to support our assumption about asymptotic fields in this scalar field case. Of course, whether we can expect asymptotic fields or not depends on the details of the dynamics and is an important problem to be explored.

To prove the spectral condition, we have had to choose the special adiabatic factor (4.1). Any other choices would result in divergent matrix elements of the total Hamiltonian due to their exponentially blowing up time dependences. This fact implies that we have very small leeway in which to achieve adiabatic switchings in dissipative systems. This may be an origin of the difficulty in the analytic continuation between the Euclidean space formulation and the Minkowski one. However, we have only assumed its existence in the Appendix where the naive adiabatic factor $e^{-\epsilon|t|}$ is shown to be sufficient to prove the spectral condition. There is another comment in connection with the proof given in the Appendix. The limiting procedures should be taken in such an order that the imaginary part of the mass is first set equal to zero followed by the $|t|\to\infty$ limit.[27] If the order were reversed, we would encounter the problem of divergent matrix elements.

One of the conclusions reached here is that under the assumption mentioned above the physical mass can be derived from the correlation length along the fictitious time direction of the stationary two-point functions. To extract the physical mass, we only have to integrate them over x and to consider their exponential decay rates:

$$\int d^4x\, D(X_1,X_2) \underset{|\tau|\to\infty}{\sim} \frac{Z_\phi}{m_R^2} e^{-\kappa_R m_R^2 |\tau|} , \tag{5.1}$$

$$\int d^4x\, G(X_1,X_2) \underset{\tau\to\infty}{\sim} 2\kappa_R Z_\phi e^{-\kappa_R m_R^2 \tau} . \tag{5.2}$$

This property has already been anticipated and numerically investigated in the frontier works[9,10] and in the recent analysis based on the renormalization-group equation.[15] It is also consistent with the exact solution of the O(N) nonlinear σ model in the large-N limit.[26] The spectral representation explored here justifies their anticipation and gives a firmer basis to the time correlation of stationary correlation functions in SQ. To get the physical mass m_R^2 from the exponential decay rate in the numerical simulation, however, there remain some problems to be clarified. For example, the uniqueness of κ_R which is one of the crucial points in numerical investigations must be proved. But if we approve the assumption about asymptotic fields mentioned above, the consistency of the theory seems to require the uniqueness of κ_R. To determine the value of κ_R we need much more detailed information about the physical content of the underlying asymptotic theory.

As a by-product, we get a relation between renormalization factors from the spectral representation. Because only asymptotic fields $\tilde{\phi}$ and $\tilde{\pi}$ have one-particle state contributions to two-point functions, we can extract them from (3.53) and (3.54) to obtain

$$\tilde{D}(X_1,X_2) \equiv \lim \langle T\tilde{\phi}(X_1)\tilde{\phi}(X_2)|\psi_0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2+m_R^2} e^{-\kappa_R(p^2+m_R^2)|\tau|} , \tag{5.3}$$

$$\tilde{G}(X_1,X_2) \equiv \lim \langle T\tilde{\phi}(X_1)\tilde{\pi}(X_2)|\psi_0\rangle = 2\kappa_R \theta(\tau)\sqrt{Z_\phi/Z_\pi} \int \frac{d^4p}{(2\pi)^4} e^{-\kappa_R(p^2+m_R^2)\tau} , \tag{5.4}$$

where field variables are renormalized as[28]

$$\phi(X)=\sqrt{Z_\phi}\phi_R(X), \quad \pi(X)=\sqrt{Z_\pi}\pi_R(X) \tag{5.5}$$

and the asymptotic conditions (3.39) are assumed. Recalling that the asymptotic fields are free fields and subject to the same equal-time commutation relations as (2.6) with κ being replaced by $\kappa_R=\kappa/Z_\kappa$, we are led to the relation

$$\sqrt{Z_\phi/Z_\pi}=1 \quad \text{(finite)} . \tag{5.6}$$

The notation used in this paper is slightly different from those used in Namiki and Yamanaka[10] and in Okano and Schülke.[15] In their notation, the above relation is written

$$Z'_\kappa \sqrt{Z_\phi/Z_\pi}=1 \quad \text{(finite)} , \tag{5.7}$$

where Z'_κ stands for Z_γ in Ref. 10 and Z_λ in Ref. 15. We can easily check the validity of this relation using the explicit expressions for Z factors calculated in the lowest-order perturbation of ϕ^4 theory[14] and of the O(N) nonlinear σ model.[15]

We may understand the above equality between Z factors as follows. In our notation, the π field is given by (2.11a). As far as the equal-time commutation relations are concerned, the π field may be considered essentially equivalent to the $\dot{\phi}$ field. The $\dot{\phi}$ field usually gets the same renormalization factor as that for ϕ itself and the relation (5.6) follows. The equality (5.7) is understood following the same line of thought if the appropriate translations are made between different notations.

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APPENDIX

Here an alternative proof of the spectral condition is given using a Minkowski-space formulation of the operator formalism in SQ.

To begin with, we briefly describe SQ formulated in Minkowski space. As a basic equation we set up a complex Langevin equation

$$\dot{\phi}(X)=i\kappa\frac{\delta S_M}{\delta\phi(X)}+\eta(X) , \tag{A1}$$

where x in X stands for the Minkowski coordinates and S_M is the Minkowski action corresponding to S in (2.1). The Gaussian white-noise $\eta(X)$ is assumed to have the same statistical properties as in (2.2). The Fokker-Planck equation has been derived from the complex Langevin equation (A1) and its explicit solution in the free case has clarified the important role of the damping factor in S_M in the large-t limit.[19] Though the Fokker-Planck equation directly derived from (A1) takes a somewhat complicated form,[29] the effective Fokker-Planck equation which governs the time development of a complex distribution defined for functionals of a real ϕ field takes a simple form with an effective Hamiltonian:[21]

$$F=2\kappa\int d^4x\frac{\delta}{\delta\phi(x)}\left[\frac{\delta}{\delta\phi(x)}-i\frac{\delta S_M}{\delta\phi(x)}\right] . \quad (A2)$$

It is manifest that the Feynman measure e^{iS_M} is nothing but the stationary solution of the effective Fokker-Planck equation.

Now we can formulate an operator formalism on the basis of the above Hamiltonian F which is rewritten as

$$F=\tfrac{1}{2}\int d^4x\,\pi(x)\left[\pi(x)+2\kappa i\frac{\delta S_M}{\delta\phi(x)}\right] \quad (A3)$$

with a momentum operator π. We assume here the same equal-time commutation relations between ϕ and π as in (2.6). Although the above Hamiltonian resembles that in the Euclidean case (2.8), the presence of i in it results in an essential change in our previous argument. It is very difficult to make F Hermitian by similarity transformations. Its eigenvalues are complex in general.

First, consider the free case. In this case the free Hamiltonian F_0 is expressed as

$$F_0=\frac{i}{2}\int d^4p\,\bar{a}^{\dagger}(-p)2\kappa(p^2-m_0^2)a(p) . \quad (A4)$$

Here operators a and $\bar{a}^{\dagger}$ similarly defined as in (3.16) satisfy the equal-time commutation relation (3.17) and an infinitesimal imaginary mass is understood to be in m_0^2. From the commutation relations of these operators with F_0,

$$\begin{aligned}[F_0, a(p,t)]&=-2\kappa^2 i(p^2-m_0^2)a(p,t) ,\\ [F_0, \bar{a}^{\dagger}(p,t)]&=2\kappa^2 i(p^2-m_0^2)\bar{a}^{\dagger}(p,t) ,\end{aligned} \quad (A5)$$

eigenstates of F_0 belonging to eigenvalues

$$-2\kappa i\lambda_n=-2\kappa^2 i(\alpha_n p^2-m_n^2) \quad (\alpha_n>0) \quad (A6)$$

are similarly constructed as in (3.19) and (3.22). Note that $\lambda_1=\kappa(p^2-m_0^2)$.

Let us consider the interacting case. We assume the existence of asymptotic fields satisfying the free Heisenberg equations

$$\dot{\tilde{\phi}}(X)=\tilde{\pi}(X)+i\kappa_R(-\partial^2-m_R^2)\tilde{\phi}(X) , \quad (A7a)$$

$$\dot{\tilde{\pi}}(X)=-i\kappa_R(-\partial^2-m_R^2)\tilde{\pi}(X) . \quad (A7b)$$

These fields are connected to the Heisenberg fields through a similarity transformation

$$U(t)=\bar{T}\exp\left[-\frac{1}{2\kappa}\int_{-\infty}^{t}dt'e^{-\epsilon|t'|}\tilde{F}_I(t')\right] \quad t<\infty , \quad (A8)$$

where the naive adiabatic factor $e^{-\epsilon|t|}$ has been introduced. Then the matrix element of the total Hamiltonian F is given in the limit

$$\langle a|F|b\rangle=\lim_{\epsilon\to 0}\langle a|F(t)|b\rangle , \quad (A9)$$

where $\langle a|$ and $|b\rangle$ are eigenstates of $\tilde{F}_0$ belonging to eigenvalues $-2\kappa i\lambda_a$ and $-2\kappa i\lambda_b$, respectively, and

$$\begin{aligned}F(t)&=F_0(t)+e^{-\epsilon|t|}F_I(t)\\ &=U(t)[\tilde{F}_0+e^{-\epsilon|t|}\tilde{F}_I(t)]U^{-1}(t) .\end{aligned} \quad (A10)$$

Using the same techniques as in Sec. IV we can easily see that the lowest contribution to $\langle a|F|b\rangle$ turns out to be diagonal in the $\epsilon\to 0$ limit ($\lambda_{ab}\equiv\lambda_a-\lambda_b$):

$$\begin{aligned}\epsilon\int_{-\infty}^{t}dt'e^{\epsilon t'}\langle a|\tilde{F}_I(t')|b\rangle&=\frac{\epsilon e^{(\epsilon+i\lambda_{ab})t}}{\epsilon+i\lambda_{ab}}\langle a|\tilde{F}_I(0)|b\rangle\\ &\underset{\epsilon\to 0}{\longrightarrow}\delta_{ab}\langle a|\tilde{F}_I(0)|a\rangle .\end{aligned} \quad (A11)$$

After a similar subtraction of the diagonal part of $\tilde{F}_I$, we can calculate the next-order contribution and get

$$\epsilon\left[-\frac{1}{2\kappa}\right]\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2 e^{\epsilon(t_1+t_2)}\langle a|[\tilde{F}_I^{(1)}(t_2),\tilde{F}_I^{(1)}(t_1)]|b\rangle\underset{\epsilon\to 0}{\longrightarrow}\delta_{ab}\sum_{n\neq a}\frac{i}{2\kappa\lambda_{an}^{(1)}}\langle a|\tilde{F}_I^{(1)}(0)|n\rangle\langle n|\tilde{F}_I^{(1)}(0)|a\rangle . \quad (A12)$$

Repetition of the above procedure proves the spectral condition in Minkowski space.

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