Geodetic precession or dragging of inertial frames?

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In metric theories of gravity the principle of general covariance allows one to describe phenomena by means of any convenient choice of coordinate system. In this paper it is shown that in an appropriately chosen coordinate system, geodetic precession of a gyroscope orbiting a spherically symmetric, spinning mass can be recast as a Lense-Thirring frame-dragging effect without invoking spatial curvature. The origin of this reference frame moves around the source but the frame axes point in fixed directions. The drag can be interpreted to arise from the orbital angular momentum of the source around the origin of the reference frame. In this reference frame the effects of geodetic precession and Lense-Thirring drag due to intrinsic angular momentum of the source have the same origin, namely, gravitomagnetism.

The theory of general relativity[1] has had remarkable success in the last few decades in describing gravitational interactions in cosmological and intergalactic as well as solar-system domains. The early predictions of the theory confirmed by the "crucial" tests[2] placed the theory on firm grounds. Observations of galactic recession strengthened the cosmological basis of the theory. Recent observations[3] of neutrino arrival times from supernova 1987A have substantiated the underlying assumption of general relativity—the principle of equivalence. Even more recently, evidence for the observation of geodetic precession has been presented.[4] It is thus of great interest to consider yet another effect which awaits confirmation—the so-called "Lense-Thirring" drag. In this paper we show that geodesic precession can, in an appropriately chosen coordinate frame, be considered as due to a Lense-Thirring drag.

According to general relativity, observers fixed with respect to distant stars will note that a spinning gyroscope, falling freely in the gravitational field of a rotating source, will undergo two kinds of effects known, respectively, as geodetic precession, and the often-called motional or "Lense-Thirring" precession.[5] Geodetic precession is usually associated with motion of the gyroscope through the static gravitational field of the source. Conventionally one derives this precession by parallel transport of the spin vector in curved spacetime near the mass;[6] the effect is present even if the mass is not rotating. The motional precession or "hyperfine precession"[7] is due to the interaction of the spin angular momentum **J** of the source, and the spin **S** of the gyroscope. This effect resembles the interaction of the spin of the electron with the magnetic field of the nucleus of an atom. Schiff[8] suggested that motional precession could be seen as a "dragging" of inertial frames, in the same way that inertial frames inside a rotating hollow shell undergo precession with respect to observers whose orientation is fixed with respect to distant stars.[9] Outside a rotating spherical source, the precession of a nearby gyroscope could be conveniently pictured by considering a spinning sphere submerged in a viscous fluid. Small toothpicks placed in the fluid near the poles rotate in the same direction as the sphere rotates, while those placed at the equator rotate in the opposite direction. The electromagnetic analogy was further pursued by Wilkins[10] and Schwinger.[11] By direct transformation of the spin vector to the local rest frame of the spin, Wilkins was able to distinguish a "gravitational" contribution to Thomas precession[12] and describe both geodetic and motional precessions in terms of an analogy with Larmor precession of a magnetic moment in a magnetic field. This work showed that geodetic and motional precessions may be considered different manifestations of the same phenomenon, much in the same spirit as the present work. Thorne[13] described the two effects in terms of interaction of the gyroscope with the "gravitoelectric" and "gravitomagnetic" fields, respectively, derived from the various components of the metric including spatial curvature contributions. It was shown that one-third of the geodetic precession effect, due to the gravitoelectric field, could be recast as a gravitomagnetic effect by a simple Lorentz boost. This, at least in part, unified the two effects into a single gravitomagnetic field phenomenon.

It is hoped that in the next decade Gravity Probe B,[14] a drag-free satellite carrying a gyroscope around Earth, will be launched. For an orbit of altitude 480 km, the gyroscope's geodetic precession should be 6.9 arcsec/yr, and the Lense-Thirring precession should be 0.044 arcsec/yr. These precessions are to be measured when gyroscope orientation is checked against distant fixed stars. The Lense-Thirring drag due to Earth's rotation may also be observed using the orbit of a satellite such as the recently proposed LAGEOS III.[15]

While efforts are being put into detecting these two effects, we argue that perhaps analogies between electromagnetism and gravitomagnetism have not been taken to their full extent and may in fact have important consequences. We suggest that the two effects can be considered to be based on the same fundamental physical phenomenon—gravitomagnetism. Our purpose here is to recast the entire geodetic precession as a Lense-Thirring drag, in the framework of metric theories of gravity. This would further extend the framework introduced by Thorne to show that geodetic precession in its entirety can be described as a gravitomagnetic effect. This is via a boost from the rest frame of the source to a reference frame with origin comoving with the gyroscope, having axes pointing in fixed directions, with spatial and time units rescaled slightly due to Lorentz contraction and other small relativistic effects which we shall discuss. In this reference frame the massive source is revolving around the gyroscope, giving rise to a gravitomagnetic drag which is precisely of the magnitude necessary to explain that precession of the gyroscope which was interpreted in the original frame as geodetic precession. By the principle of general covariance, a phenomenon may be described in any convenient coordinate system provided that the experimental observations are interpreted properly, in terms of invariant quantities.

This suggests that experimental observation of geodetic precession would indirectly imply existence of gravitomagnetic phenomena within the framework of metric theories of gravity. Furthermore since it is plausible to have a theory of gravity in which the two phenomenon are not directly related, our results imply that direct verification of the effect would serve to strengthen the case for the principle of general covariance and the metric formulation.

In this paper we shall use the simplest parametrized post-Newtonian (PPN) formulation[16] and shall neglect preferred-frame and energy-momentum-nonconservation effects. The only relevant PPN parameters would be γ and β; however, we shall not find it necessary to include nonlinear effects due to β as these are of higher order.

We first consider a model of a rotating, spherically symmetric mass M placed at the origin of the PPN frame. Given the metric to post-Newtonian order, we construct an orthonormal tetrad of basis vectors moving along a curve around the source, but with orientation fixed with respect to points at infinity. The origin, at position $\mathbf{R}_0$, is thus in motion through the PPN grid. The coordinate reference frame is erected using this tetrad as a basis. We call this a quasi-inertial frame. Unless there are nongravitational forces applied, causing nongravitational accelerations of this tetrad, observers in the quasi-inertial frame would not feel radial accelerations towards the source as they are falling along geodesics; however, they "see" a mass revolving around their origin. The goal is to calculate the metric tensor in this frame by coordinate transformation. Upon expanding the metric tensor to linear order in local coordinates near the origin, the equations of motion of a spinning gyroscope at the origin may be obtained, and it will be seen that in this coordinate system the geodetic precession is entirely of gravitomagnetic origin. In other words, all contributions to precession, even including Thomas precession, arise from the space-time components g_{0i} of the metric. There is no need to invoke space curvature in the manner of Thorne.[13]

We use the formalism derived previously by us[17] to calculate the local metric $g_{\mu\nu}$ in the quasi-inertial frame. We will show that there exist terms in the components g_{0i} linear in coordinates leading to the expected geodetic precession of the gyroscope. The magnitude of the precession, and its dependence on γ is as expected, and the direction of the precessional angular velocity is the same as the direction of the orbital angular momentum of the source as seen by observers in the quasi-inertial frame. Thus the local inertial frame is truly dragged by the source as it traverses its orbit.

CALCULATION OF THE METRIC IN THE QUASI-INERTIAL FRAME

The metric in the PPN frame with the center of mass of the rotating source at the origin is given up to the desired order by

$$G_{00}=-1+2U \ , \tag{1a}$$

$$G_{0i}=-2(\gamma+1)H_i \ , \tag{1b}$$

$$G_{ij}=\delta_{ij}(1+2\gamma U) \ , \tag{1c}$$

where $-c^2U=-GM/R$ is the gravitational potential due to the central mass and H_i represents contributions arising from the intrinsic spin of the source. We emphasize that here we are only interested in physical effects associated with orbital motion, namely, geodetic precession. Inclusion of H_i is for completeness only and does not affect our results. One can construct an orthonormal tetrad moving with the gyroscope, but with spatial axes directionally fixed with respect to the PPN frame. One chooses the zeroth member of the tetrad to be tangent to the four-velocity of the gyroscope $\Lambda^{\mu}_{(0)}=dX^{\mu}/ds$, then the rest of the construction is straightforward.[18] The components of the spatial members of the tetrad are

$$\Lambda^{0}_{(j)}=V^j[1+(2+\gamma)U_0+V^2/2c^2]/c-2(\gamma+1)H_j \ , \tag{2a}$$

$$\Lambda^{k}_{(j)}=\delta^k_j(1-\gamma U_0)+V^kV^j/2c^2 \ , \tag{2b}$$

where $V^k=dX^k/dX^0$ are the components of velocity of the gyroscope as measured by observers at rest with respect to the PPN frame, and $U_0=U(\mathbf{R}_0)$. We shall suppose that the net velocity V^k is determined by both gravitational and nongravitational forces. If only gravitational forces act, then the gyroscope and the tetrad fall along a geodesic of the metric, Eqs. (1), and the spin vector is carried along by parallel transport. If additional nongravitationally forces act, then the path is not a geodesic and the spin vector is carried along by Fermi-Walker transport.[7] In either case, the tetrad is given by Eqs. (2). Following the procedure developed in Ref. 17 one can now construct a set of coordinate transformations, relating the PPN coordinates X^{μ} to the local coordinates x^{μ}

$$X^{\mu}=X^{\mu}|_{\rm orbit}+\Lambda^{\mu}_{(j)}x^j-\tfrac{1}{2}\Gamma^{\mu}_{\alpha\beta}\Lambda^{\alpha}_{(k)}\Lambda^{\beta}_{(j)}x^kx^j+\cdots \ , \tag{3}$$

where we have included terms to quadratic order in the local coordinates. This is required in order to obtain terms linear in coordinates in the local metric. Evaluating the coordinate transformations gives

$$X^0=\int^{x^0} K\,dx^0+(\mathbf{r}\cdot\mathbf{v})[1+(1+\gamma)U_0+(U_{,j}x^j)]/c$$
$$-2(\gamma+1)(\mathbf{H}\cdot\mathbf{r})$$
$$-\frac{\gamma+1}{2}x^ix^j(H_{i,j}+H_{j,i})-\tfrac{1}{2}\gamma r^2 U_{,0}\ , \tag{4a}$$

$$X^j=X^j|_{\text{orbit}}+x^j(1-\gamma U_0)+v^j(\mathbf{r}\cdot\mathbf{v})/2c^2$$
$$-\gamma x^j(U_{,k}x^k)+\tfrac{1}{2}\gamma r^2 U_{,j}\ , \tag{4b}$$

where $-\mathbf{v}$ is the velocity of the source as measured in the local frame and evaluated in terms of the local time coordinate x^0, $v_i=v^i$, K is given by

$$K=1+U_0+V^2/2c^2\ , \tag{4c}$$

and $\mathbf{r}$ is the local coordinate position vector. Also K, $\mathbf{H}$, and all the partial derivatives in Eqs. (4) are evaluated at the position of the gyroscope.

Having found the coordinate transformations one may simply regard them as exact transformations from PPN coordinates to another reference frame. The transformations (4) include resynchronization of clocks, Lorentz contraction, and rescaling of lengths due to the mass, as well as several quadratic terms needed to make the metric tensor at the position of the gyroscope reduce to the Minkowski values. The metric tensor in the new frame is obtained by tensor transformation

$$g_{\mu\nu}=G_{\alpha\beta}\frac{\partial X^\alpha}{\partial x^\mu}\frac{\partial X^\beta}{\partial x^\nu}\ . \tag{5}$$

In a local inertial frame, the metric tensor $g_{\mu\nu}$, expanded in terms of local coordinates, consists of terms quadratic and higher in these coordinates.[17] Terms linear in coordinates cancel so that all gravitational forces vanish at the origin. In a quasi-inertial frame, terms linear in local coordinates do not cancel out and are in fact responsible for precession effects at the origin of the frame. Thus after expanding to linear order in local coordinates one finds the expressions for the metric tensor components g_{00} and g_{ij}

$$g_{00}=-1+2\mathbf{r}\cdot\mathbf{A}/c^2+O(x^ix^j)\ , \tag{6a}$$

$$g_{ij}=\delta_{ij}+O(x^ix^j)\ . \tag{6b}$$

The linear term in Eq. (6a) represents the effective gravitational potential in local quasi-inertial coordinates arising from the frames' acceleration. Before expansion of the potential $U(R)$ for small values of the local coordinates, the expression for the "gravitomagnetic" metric tensor components g_{0i}, including terms linear in x^i, is

$$g_{0i}=2(\gamma+1)\frac{GM}{c^3|\mathbf{R}_0+\mathbf{r}|}v_i-2U_0(\gamma+1)v_i/c-(\gamma+\tfrac{3}{2})U_{,j}x^jv_i/c-(\gamma+\tfrac{1}{2})U_{,i}(\mathbf{r}\cdot\mathbf{v})/c$$
$$-2(\gamma+1)x^kH_{i,k}+\gamma x^i(v^kU_{,k})/c+(\gamma+1)x^k(H_{i,k}+H_{k,i})+\tfrac{1}{2}[(\mathbf{r}\cdot\mathbf{v})A^i-(\mathbf{r}\cdot\mathbf{A})v^i]/c^3+O(x^ix^j)\ . \tag{6c}$$

The first term in the above equation is what would be expected for a mass moving with velocity $-v^i$, as is observed in the quasi-inertial frame. The second term subtracts out the constant part of the first term, leaving linear terms as the leading contribution. In Eq. (6c), the acceleration terms involving A^i arise from transformation coefficients $\partial X^\mu/\partial x^0$, when the derivative $V^i_{,0}$ is replaced by $U_{,i}+A^i/c$. The quantity A^i represents that part of the acceleration due to nongravitational forces. The physical origin of the acceleration terms in Eq. (6c) is the Lorentz contraction, and the breakdown of simultaneity, in Eqs. (2). After expansion of the above expression for g_{0i} to linear order in quasi-inertial coordinates, one finds after cancellation that

$$g_{0i}=(\gamma+\tfrac{1}{2})x^j(U_{,j}v_i-U_{,i}v_j)/c$$
$$+\gamma x^i(v^kU_{,k})/c-(\gamma+1)x^k(H_{i,k}-H_{k,i})$$
$$+[(\mathbf{r}\cdot\mathbf{V})A^i-(\mathbf{r}\cdot\mathbf{A})V^i]/2c^3+O(x^ix^j)\ . \tag{6d}$$

The equations of motion of a gyroscope with spin $\mathbf{S}$, placed at the origin of the local frame are given by Fermi-Walker transport

$$\frac{DS^\alpha}{Ds}=\left\{S_\beta\frac{Du^\beta}{Ds}\right\}u^\alpha-(S_\beta u^\beta)\frac{Du^\alpha}{Ds}\ . \tag{7}$$

However the spatial components of all terms on the right-hand side of Eq. (7) vanish identically and, remarkably, the equation of motion of the spin reduces to the equation of parallel transport. This is because the gyroscope remains at rest in this frame. The spin vector satisfies the Pirani condition $S_\beta u^\beta=0$. Then $u^k=0$ and $S_0=0$. Thomas precession, therefore, manifests itself geometrically via the presence of acceleration terms in the local metric. The only Christoffel symbols which contribute are

$$\Gamma^k_{0j}=\tfrac{1}{2}(g_{0k,j}-g_{0j,k})\ .$$

The precession of the spin of the gyroscope can now be written as

$$\frac{d\mathbf{S}}{dt}=\tfrac{1}{2}\mathbf{S}\times(\nabla\times\mathbf{g})\ , \tag{8}$$

where $\mathbf{g}=c(g_{01},g_{02},g_{03})$ is a gravitomagnetic "vector potential" given by

$$\mathbf{g}=\frac{2\gamma+1}{2}[\mathbf{r}\times(\nabla U\times\mathbf{v})]/c$$
$$-(\gamma+1)c[(\mathbf{r}\cdot\nabla)\mathbf{H}-\nabla(\mathbf{r}\cdot\mathbf{H})]$$
$$+\mathbf{r}\times(\mathbf{A}\times\mathbf{V})/2c^2+\gamma\mathbf{r}(\mathbf{v}\cdot\nabla U)/c\ , \tag{9}$$

where the gradient is taken with respect to the PPN coordinate R and evaluated at the position of the origin of the quasi-inertial frame.

Equation (8) represents interaction of spin with the gravitomagnetic components of the metric only; the first term in Eq. (9) corresponds to geodetic precession, the second to the Lense-Thirring drag, and the third to the Thomas precession. The last term is curl-free and does not contribute to the precession. It can be removed by a further coordinate transformation.

Assuming that $\mathbf{H}$ represents contributions due to intrinsic angular momentum $\mathbf{J}$ given by

$$\mathbf{H}=-\frac{G}{2c^3R^3}(\mathbf{R}\times\mathbf{J}) ,$$

we get

$$\frac{d\mathbf{S}}{dt}=\mathbf{S}\times\left[(\gamma+\tfrac{1}{2})(\nabla U\times\mathbf{v}) -\frac{\gamma+1}{2}\frac{G}{c^2}\left[\frac{2\mathbf{J}}{R^3}-\frac{3}{R^5}[\mathbf{R}\times(\mathbf{R}\times\mathbf{J})]\right]\right]\Bigg|_0 +\frac{1}{2c^2}(\mathbf{A}\times\mathbf{V})\times\mathbf{S} . \qquad (10)$$

This represents the precession of a gyroscope due to gravitomagnetic components only of the metric, in quasi-inertial coordinates. In a more illuminating form, Eq. (10) can be written as

$$\frac{d\mathbf{S}}{dt}=\mathbf{\Omega}\times\mathbf{S} , \qquad (11)$$

with

$$\mathbf{\Omega}=\frac{G}{c^2R^3}\left[\tfrac{1}{2}(\gamma+1)\left[-J+\frac{3(\mathbf{J}\cdot\mathbf{R})\mathbf{R}}{R^2}\right]+(\gamma+\tfrac{1}{2})\mathbf{L}\right]\Bigg|_0 +\tfrac{1}{2}(\mathbf{A}\times\mathbf{V})/c^2 , \qquad (12)$$

where $\mathbf{L}$ is the orbital angular momentum per unit mass of the source. The term involving $\mathbf{J}$ in Eq. (11) is the well-known Lense-Thirring contribution arising from drag of inertial frames due to the intrinsic angular momentum of the source. In the same spirit the second term represents an additional drag due to the apparent orbital angular momentum of the source. The spin axis of the gyroscope is dragging behind the revolving source with angular velocity of magnitude proportional to the angular momentum of the source measured by quasi-inertial observers. The last term in Eq. (12) is the Thomas precession, which arises in quasi-inertial coordinates entirely from gravitomagnetic contributions, in the g_{0i} components of the metric tensor.

CONCLUSIONS

The equation of motion of the spin vector has been studied by others[8,10] by transforming the spin vector to the spin's instantaneous rest frame. We have instead exhibited a coordinate transformation to quasi-inertial coordinates, and have calculated the leading contributions, linear in local coordinates, to the g_{0i} components of the metric tensor. Usually geodetic precession is derived from parallel transport of a set of basis vectors, in which significant contributions to the precession arise from spatial components of the metric. The present treatment shows that geodetic precession can be recast entirely as a gravitomagnetic (Lense-Thirring) drag effect, in which spatial curvature plays no role. The drag arises from the orbital angular momentum of a gravitational source as seen by observers in the quasi-inertial frame. The net precession of a gyroscope which is falling freely in the field of a central body is then made up of two gravitomagnetic contributions. One is due to orbital motion of the source in the quasi-orbital motion of the source in the quasi-inertial frame and a second is due to the source's intrinsic angular momentum. The seemingly different geodetic precession and the Lense-Thirring drag are then both, in the spirit of Mach's principle, aspects of a single gravitomagnetic phenomenon. If in addition the local nonrotating frame is accelerated, even the Thomas precession is a gravitomagnetic effect.

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