Exact Bianchi type-(I,V) solutions of the Einstein equations with scalar and spinor fields

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Exact solutions to the Einstein–Dirac–Klein-Gordon equations are found in Bianchi type-I and type-V spacetimes. When the spinor field is massive, the Bianchi type-I universe has singularities of the Zel'dovich type, and it may begin anisotropically at a singularity, leading to a dust-filled Friedmann stage at late times. For "ghost neutrinos" we get an axisymmetric Zel'dovich universe. These solutions have particle horizons. For the Bianchi type-V model there exist solutions which begin at a singularity, tending to flat spacetime at late times. In this case the neutrinos are not "ghost" ones.

I. INTRODUCTION

There is good observational evidence that at our cosmological epoch the Universe is fairly homogeneous on large scales and has been highly isotropic since the epoch in which it became definitely transparent to radiation.[1] It follows, that on a large scale, the Universe may be approximately described by one of the Robertson-Walker models. Such a model is in accord with observation of the present state of our Universe but does not explain the isotropy. Moreover, since homogeneous isotropic cosmological models are unstable[2] against perturbations near the singularity, they are not likely to describe the early Universe suitably. Therefore, it is interesting to study more appropriate cosmological models in which anisotropies, and possibly inhomogeneities, existing at the beginning of the expansion, are damped out in the course of evolution. Because of their relative simplicity, homogeneous anisotropic models are frequently investigated in current literature. These are the so-called Bianchi models.

The Bianchi types-I and -V universes are of particular interest because these cosmological models contain isotropic special cases and therefore allow arbitrarily small anisotropy levels at any instant of cosmic time. This property makes them suitable as models for our Universe.

In previous papers[3–9] exact solutions of the Einstein-Dirac equations have been given for a c-number spinor field in a spatially flat Robertson-Walker spacetime, in Bianchi type-I metrics and in cylindrical and plane-symmetric spacetimes. Ray[10] has presented exact solutions of these equations for neutrinos when the spacetime is represented by a plane-symmetric Bianchi type-V metric. Exact solutions for coupled Einstein, Dirac, Maxwell, and zero-mass scalar fields[11–13] have been found for static cylindrical and plane-symmetric metrics. Some of these solutions allow only "ghost neutrinos" (vanishing energy-momentum tensor).

In a wide class of cosmological models the initial stage is typically similar to that of the vacuum Bianchi type-I model first considered by Kasner. Investigations of Bianchi type-I models with this behavior were carried out by Heckmann and Schücking[14] and Jacobs.[15] However, later on Belinskii and Khalatnikov[16] and Barrow[17] showed that the evolutions of the Bianchi type-I models with stiff fluid ($p=\rho$) are significantly different from those of the models with dust or radiation and the initial vacuum assumption can be violated. Belinskii and Khalatnikov further showed that stiff irrotational fluid can be identified with a massless scalar field coupled minimally to the gravitational field.

Weinberg[18] and Wilczek[19] have suggested that there should exist a pseudoscalar boson, the so-called axion, of negligible mass, arising from the spontaneous breaking of the Peccei-Quinn symmetry[20] which was introduced in order to solve the strong CP problem. Because the axion is coupled to quarks, and therefore to matter, it would be copiously produced in stars and so it could contribute significantly to the energy density of the Universe as cold dark matter.

Viewed in the above perspective, the Einstein zero-mass scalar theory acquires some relevance. Therefore, it may be of some interest to find exact solutions of the Einstein–Dirac–Klein-Gordon field equations for Bianchi type-I and type-V spacetimes, and particularly those which allow nonghost neutrinos.

The aim of this work is to study the anisotropy damping of the Universe through unquantized scalar (minimally coupled to gravity) and spinor fields. We derive the general solution to the Einstein–Dirac–Klein-Gordon field equations for unquantized and homogeneous scalar and spinor fields in the Bianchi type-I case, and in the case in which the c-number spinor field represents neutrinos propagating along the symmetry axis in a plane-symmetric Bianchi type-V metric. The paper is organized as follows. In Sec. II we present the corresponding field equations. In Sec. III we find the exact Bianchi type-I solutions for the case of either massive spinor fields or neutrinos. In Sec. IV we find the exact Bianchi type-V solutions for the neutrino case. In Sec. V we discuss the results.

II. FIELD EQUATIONS

We shall consider the Einstein field equations, with vanishing cosmological constant, coupled to a source represented by a massless free scalar field minimally coupled to gravity and a free spinor field. This system is described by the Lagrangian density[21] (units have been chosen such that $c=\hbar=1$)

$$\mathcal{L}=(\det V)\left[\frac{R(V)}{16\pi G}+\frac{1}{2}[\bar{\psi}\gamma_\alpha V^{\alpha\mu}\nabla_\mu\psi-V^{\alpha\mu}(\nabla_\mu\bar{\psi})\gamma_\alpha\psi]-m\bar{\psi}\psi+\tfrac{1}{2}(\eta^{\alpha\beta}V_\alpha{}^\mu\partial_\mu\phi V_\beta{}^\nu\partial_\nu\phi)\right], \tag{2.1}$$

where $V_\alpha{}^\mu$ is the vierbein field, satisfying[22]

$$\begin{aligned}&V^\alpha{}_\mu(x)V^\beta{}_\nu(x)\eta_{\alpha\beta}=g_{\mu\nu}(x)\,,\\&V_{\alpha\mu}(x)V_\beta{}^\mu(x)=\eta_{\alpha\beta}=\mathrm{diag}(1,-1,-1,-1)\,.\end{aligned} \tag{2.2}$$

$g_{\mu\nu}$ is the metric tensor of the spacetime (throughout this paper greek indices run as 0,1,2,3 and latin ones as 1,2,3), ψ is the spinor field, m its mass, $\bar{\psi}=-i\psi^\dagger\gamma_0$, and ϕ is the massless scalar field. $R(V)$ is the Ricci scalar curvature.

The covariant derivative of the spinor field ψ is given by[22]

$$\nabla_\mu\psi=(\partial_\mu+\sigma_\mu)\psi\,, \tag{2.3}$$

where σ_μ is the spinorial affine connection

$$\sigma_\mu=-\tfrac{1}{8}[\gamma^\alpha,\gamma^\beta]V_\alpha{}^\nu V_{\beta\nu;\mu} \tag{2.4}$$

and the semicolon denotes ordinary covariant derivative.

The representation of the Dirac matrices $\{\gamma^\alpha\}$ we have chosen is

$$\begin{aligned}\gamma_0=\gamma^0&=i\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix},\\ \gamma_1=-\gamma^1&=i\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{bmatrix},\\ \gamma_2=-\gamma^2&=\begin{bmatrix}0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0\end{bmatrix},\\ \gamma_3=-\gamma^3&=i\begin{bmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{bmatrix}\end{aligned} \tag{2.5}$$

which satisfies the usual anticommutation relations

$$\{\gamma_\alpha,\gamma_\beta\}=-2\eta_{\alpha\beta}\mathbb{1}\,. \tag{2.6}$$

Variation of the Lagrangian density (2.1) with respect to ϕ, $\bar{\psi}$, and $V_\alpha{}^\mu$ yields the Klein-Gordon, Dirac, and Einstein field equations, respectively:

$$\phi_{;\mu}{}^\mu=0\,, \tag{2.7a}$$

$$\Gamma^\mu\nabla_\mu\psi-m\psi=0\,, \tag{2.7b}$$

$$R_{\mu\nu}=-8\pi G(T_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}T^\rho{}_\rho) \tag{2.7c}$$

with $T_{\mu\nu}=T^S_{\mu\nu}+T^D_{\mu\nu}$, where

$$T^S_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi-\tfrac{1}{2}g_{\mu\nu}\partial^\rho\phi\partial_\rho\phi\,, \tag{2.8a}$$

$$T^D_{\mu\nu}=\tfrac{1}{2}[\bar{\psi}\Gamma_{(\mu}\nabla_{\nu)}\psi-(\nabla_{(\mu}\bar{\psi})\Gamma_{\nu)}\psi] \tag{2.8b}$$

are the energy-momentum tensors for scalar and Dirac fields, and

$$\Gamma^\mu=V_\alpha{}^\mu\gamma^\alpha \tag{2.9}$$

are the generalized Dirac matrices.

III. BIANCHI TYPE-I SOLUTIONS

In this section we shall deal with a Bianchi type-I homogeneous universe, and we shall seek solutions of the field equations (2.7a)–(2.7c) which are consistent with this model. The Bianchi type-I universe is characterized by the spacetime interval

$$ds^2=dt^2-a_1^2(t)(dx^1)^2-a_2^2(t)(dx^2)^2-a_3^2(t)(dx^3)^2 \tag{3.1}$$

which describes a spatially flat anisotropic geometry with three preferred directions. In (3.1) $a_i(t)$ denotes the evolution function associated with the ith principal direction.

Choosing the vierbein given by

$$V^0{}_0=1,\quad V^{(i)}{}_{(i)}=a_i(t) \tag{3.2}$$

and the other elements equal to zero, we obtain that generalized Dirac matrices (2.9), and the spinorial affine connections (2.4) take the form

$$\Gamma_0=\gamma_0,\quad \Gamma_i=a_i\gamma_i\,, \tag{3.3a}$$

$$\sigma_0=0,\quad \sigma_i=-\tfrac{1}{2}\dot{a}_{(i}\gamma^0\gamma^{i)}\,, \tag{3.3b}$$

where the dot means differentiation with respect to proper time t [summation convention is suspended for bracketed indices (ii)]. In order to maintain the Bianchi type-I form of the metric, we take homogeneous fields, that is, with no dependence on the spatial coordinates.

A. $m\neq 0$ case

Under our assumption and using (3.3a) and (3.3b), Eqs. (2.7a) and (2.7b) become

$$[(-g)^{1/2}\dot{\phi}]^{\cdot}=0\,, \tag{3.4a}$$

$$\gamma_0\dot{\chi}-m\chi=0\,, \tag{3.4b}$$

where $g=\det g_{\mu\nu}$ and the spinor field ψ was conveniently redefined by

$$\chi=(-g)^{1/4}\psi \tag{3.5}$$

so that χ satisfies the free-field Dirac equation.

Integrating Eqs. (3.4a) and (3.4b) we get

$$\dot{\phi}=\alpha(-g)^{-1/2}\ , \tag{3.6a}$$

$$\chi=\begin{bmatrix} b_1 e^{-imt} \\ b_2 e^{-imt} \\ d_1^* e^{imt} \\ d_2^* e^{imt}\end{bmatrix}\ , \tag{3.6b}$$

where α is an arbitrary real constant and b_1, b_2, d_1^*, d_2^* are complex constants.

The energy-momentum tensors (2.8a) and (2.8b) for the scalar and Dirac fields (3.6a) and (3.6b) are

$$T^S_{00}=\frac{\dot{\phi}^2}{2}=\frac{\alpha^2}{2(-g)}\ , \tag{3.7a}$$

$$T^S_{ii}=\frac{\dot{\phi}^2}{2}a_i^2=\frac{\alpha^2 a_i^2}{2(-g)}\ , \tag{3.7b}$$

$$T^D_{00}=\frac{m}{(-g)^{1/2}}(|b_1|^2+|b_2|^2-|d_1|^2-|d_2|^2)\ , \tag{3.7c}$$

$$T^D_{12}=\frac{\dot{a}_1/a_1-\dot{a}_2/a_2}{4a_3}(|b_1|^2-|b_2|^2+|d_1|^2-|d_2|^2)\ , \tag{3.7d}$$

$$T^D_{13}=i\frac{\dot{a}_1/a_1-\dot{a}_3/a_3}{4a_2}(b_1^*b_2-b_2^*b_1+d_1d_2^*-d_2d_1^*)\ , \tag{3.7e}$$

$$T^D_{23}=\frac{\dot{a}_2/a_2-\dot{a}_3/a_3}{4a_1}(b_1^*b_2+b_2^*b_1+d_1d_2^*+d_2d_1^*) \tag{3.7f}$$

and all other components vanish identically.

Since the Ricci tensor $R_{\mu\nu}$ is diagonal for the metric (3.1), the total energy-momentum tensor must be diagonal too. This imposes the following restrictions on the spinor field components:

$$b_1^*b_2+d_1d_2^*=0\ , \tag{3.8a}$$

$$|b_1|^2-|b_2|^2+|d_1|^2-|d_2|^2=0\ . \tag{3.8b}$$

Thus, substituting (3.1) and (3.7a)–(3.7c) into the Einstein field equations (2.7c) we get

$$\sum_{i=1}^{3}(\dot{H}_i+H_i^2)=\frac{8\pi G\alpha^2}{g}-\frac{K}{(-g)^{1/2}}\ , \tag{3.9a}$$

$$[(-g)^{1/2}H_i]^{\cdot}=K \tag{3.9b}$$

with $H_i=\dot{a}_i/a_i$ the directional Hubble factors, and

$$K\equiv 4\pi Gm(|b_1|^2+|b_2|^2-|d_1|^2-|d_2|^2)\ . \tag{3.10}$$

Equation (3.9b) gives

$$H_i=\frac{Kt+\lambda_i}{(-g)^{1/2}}\ , \tag{3.11}$$

where λ_i are integration constants. Then inserting (3.11) in (3.9a) we obtain, after a suitable choice of the origin of the t coordinate,

$$(-g)^{1/2}=\tfrac{3}{2}Kt(t+t_s)\ , \tag{3.12}$$

where

$$t_s=\left[\frac{3}{2}\sum_{j=1}^{3}\beta_j^2+48\pi G\alpha^2\right]^{1/2}\Big/3|K|$$

and

$$\beta_i=2\left[\lambda_i-\frac{1}{3}\sum_{j=1}^{3}\lambda_j\right]\ ,$$

so that

$$\sum_{i=1}^{3}\beta_i=0\ . \tag{3.13}$$

The solutions to Eqs. (3.11) are

$$a_i(t)=a_{0i}|t|^{1/3+\gamma_i}|t+t_s|^{1/3-\gamma_i}\ . \tag{3.14}$$

where

$$\gamma_i=\beta_i\left[\frac{3}{2}\sum_{j=1}^{3}\beta_j^2+48\pi G\alpha^2\right]^{-1/2} \tag{3.15}$$

and the integration constants a_{0i} satisfy

$$\prod_{i=1}^{3}a_{0i}=\tfrac{3}{2}K\,\mathrm{sgn}[t(t+t_s)]\ . \tag{3.16}$$

Then, (3.5), (3.6a) and (3.6b) lead to

$$\phi(t)=\phi_0+\frac{2\alpha}{3|K|t_s}\ln\left|\frac{t}{t+t_s}\right|\ , \tag{3.17a}$$

$$\psi(t)=[\tfrac{3}{2}Kt(t+t_s)]^{-1/2}\begin{bmatrix} b_1 e^{-imt} \\ b_2 e^{-imt} \\ d_1^* e^{imt} \\ d_2^* e^{imt}\end{bmatrix}\ , \tag{3.17b}$$

where ϕ_0 is an arbitrary real constant.

B. Analysis of the solutions

The evolution functions (3.14) present some interesting features to be considered. This model has two singularities, one at $t=-t_s$ and another at $t=0$. When $K>0$ we have a contracting universe for $-\infty<t<-t_s$ which is isotropic in the remote past and collapses at $t=-t_s$ with a highly anisotropic behavior. For $0<t<+\infty$ we have an expanding universe, beginning at a singularity at $t=0$. Its expansion is highly anisotropic in the early stages ($t\lesssim t_s$); however, because of the presence of the massive spinor field, the universe tends to isotropization at late times ($t\gg t_s$) for any initial condition ($a_i\sim t^{2/3}$), and it approaches a dust-filled Robertson-Walker universe.

When $K<0$, the universe begins at a singularity at $t=-t_s$ and collapses at $t=0$. During this time interval the cosmological evolution is completely anisotropic.

For both signs of K and any initial condition, this cosmological model near the singularities behaves like the Zel'dovich universe[15,16,23] because the scalar field dominates

$$a_i \underset{t\to 0}{\sim} |t|^{q_i} \tag{3.18}$$

with $q_i = \frac{1}{3} + \gamma_i$, so that

$$\sum_{i=1}^{3} q_i = 1, \quad \sum_{i=1}^{3} q_i^2 < 1 \tag{3.19}$$

and it will be of Kasner type if there were no scalar field source. Furthermore, it can be seen that these solutions have particle horizons; that is, at a fixed epoch $t_0 > 0$, no causal interactions subsequent to the singularity $t=0$ have occurred between regions of coordinate separation:

$$|\Delta x^i| > \frac{1}{a_{0i}(1-q_i)} \frac{t_0^{1-q_i}}{t_s^{2/3-q_i}} . \tag{3.20}$$

along the x^i direction, for $t_0 \ll t_s$.

Near the singularities, where the scalar field dominates, the trace of the energy-momentum tensor $T_{\mu\nu}$ behaves as $\alpha^2 g^{-1}$, i.e., as t^{-2}, and this proves that the singularities are true physical ones. These may be classified as (i) cigar type (two evolutions tend to zero and the other to infinite), at $t=0$ when $\gamma_1 < -\frac{1}{3}$, $|\gamma_2| < \frac{1}{3}$, $\gamma_3 > -\frac{1}{3}$, and at $t=-t_s$ when $\gamma_1 < \frac{1}{3}$, $|\gamma_2| < \frac{1}{3}$, $\gamma_3 > \frac{1}{3}$, (ii) barrel type (two evolutions tend to zero and the other to a constant), at $t=0$ when $\gamma_1 > -\frac{1}{3}$, $|\gamma_2| < \frac{1}{3}$, $\gamma_3 = -\frac{1}{3}$, and at $t=-t_s$ when $\gamma_1 < \frac{1}{3}$, $|\gamma_2| < \frac{1}{3}$, $\gamma_3 = \frac{1}{3}$, and (iii) point type (all evolutions tend to zero), at $t=0$ when $\gamma_1 > -\frac{1}{3}$, $|\gamma_2| < \frac{1}{3}$, $|\gamma_3| < \frac{1}{3}$, and at $t=-t_s$ when $\gamma_1 < \frac{1}{3}$, $|\gamma_2| < \frac{1}{3}$, $|\gamma_3| < \frac{1}{3}$.

This cosmological model does not have pancake-type singularities, which exist when there is only a spinor field. For $\beta_i = 0$ $\forall i$, this model is completely isotropic ($k=0$ Robertson-Walker cosmology)[3,24] and there are no restrictions on the spinor field components [see (3.7d)–(3.7f)].

C. $m=0$ case

The solution to the neutrino equations

$$\Gamma^\mu \nabla_\mu \psi = 0, \quad (\mathbb{1} - \gamma_5)\psi = 0 \tag{3.21}$$

is

$$\psi = (-g)^{-1/4} \begin{bmatrix} b_1 \\ b_2 \\ b_1 \\ b_2 \end{bmatrix} \tag{3.22}$$

in the representation of (2.5), where $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$.

The nonzero components of the energy-momentum tensor of the neutrino field are

$$T_{12}^D = \frac{\dot a_1/a_1 - \dot a_2/a_2}{2a_3}(|b_1|^2 - |b_2|^2) , \tag{3.23a}$$

$$T_{13}^D = i\frac{\dot a_1/a_1 - \dot a_3/a_3}{2a_2}(b_1^* b_2 - b_2^* b_1) , \tag{3.23b}$$

$$T_{23}^D = \frac{\dot a_2/a_2 - \dot a_3/a_3}{2a_1}(b_1^* b_2 + b_2^* b_1) . \tag{3.23c}$$

However, since $R_{12} = R_{13} = R_{23} = 0$, these components must vanish. For $a_1 \neq a_2 \neq a_3$ we find that this metric does not allow neutrino solutions because $b_1 = b_2 = 0$. Setting $a_1 = a_2 \neq a_3$ causes T_{12} to vanish, whereas the vanishing of T_{13} and T_{23} forces the neutrino field to have the form

$$\psi = b_1(-g)^{-1/4} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{or} \quad \psi = b_2(-g)^{-1/4} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} . \tag{3.24}$$

Thus, this metric allows only ghost neutrinos (the neutrino energy-momentum tensor vanishes, whereas the neutrino field ψ and current density S^μ do not) and since the gravitational field equations correspond to those which have only the scalar field as source, the solution is a special case of the Zel'dovich universe (axisymmetric):

$$a_1(t) = a_2(t) = a_0|t|^p, \quad a_3(t) = b_0|t|^{1-2p} , \tag{3.25}$$

where the parameter p lies in the range $0 < p < \frac{2}{3}$ and a_0, b_0 are arbitrary real constants.

The scalar field then becomes

$$\phi(t) = \phi_0 + \left[\frac{p(2-3p)}{4\pi G}\right]^{1/2} \ln|t| \tag{3.26}$$

with ϕ_0 an arbitrary real constant. This universe emerges from an initial singularity at $t=0$, which can be of the "cigar" ($\frac{1}{2} < p < \frac{2}{3}$), "point" ($0 < p < \frac{1}{2}$), or "barrel" ($p = \frac{1}{2}$) type, and the expansion rate is always highly anisotropic, even as $t \to \infty$. The time reversal of this case is another possible evolution. When $p = \frac{1}{3}$ the model is completely isotropic for all times.

IV. BIANCHI TYPE-V SOLUTIONS

We shall study now the problem in a Bianchi type-V spacetime, which has the metric

$$ds^2 = a^2(\eta)[d\eta^2 - (dx^1)^2] - b^2(\eta)e^{2x^1}[(dx^2)^2 + (dx^3)^2] . \tag{4.1}$$

This model is an anisotropic generalization of the open Robertson-Walker model. Choosing the vierbein given by

$$V^0{}_0 = V^1{}_1 = a, \quad V^2{}_2 = V^3{}_3 = be^{x^1} \tag{4.2}$$

and the other elements equal to zero, the generalized Dirac matrices (2.9) and the spinorial affine connections (2.4) take the form

$$\Gamma_0 = a\gamma_0, \quad \Gamma_1 = a\gamma_1 ,$$

$$\Gamma_2 = be^{x^1}\gamma_2, \quad \Gamma_3 = be^{x^1}\gamma_3 , \tag{4.3a}$$

$$\sigma_0=0,\quad \sigma_1=\frac{H_a}{2}\gamma_0\gamma_1\ ,$$
$$\sigma_2=\frac{e^{x^1}b}{2a}(H_b\gamma_0\gamma_2-\gamma_1\gamma_2)\ ,\qquad (4.3b)$$
$$\sigma_3=\frac{e^{x^1}b}{2a}(H_b\gamma_0\gamma_3-\gamma_1\gamma_3)\ ,$$

where $H_a=\dot{a}/a$, $H_b=\dot{b}/b$, and the dot denotes differentiation with respect to η.

Assuming a massless spinor field which depends on both η and x^1, and using (4.3a) and (4.3b), the neutrino field equations (3.21) become

$$\gamma^0\dot{\psi}+\left[\frac{H_a}{2}+H_b\right]\gamma^0\psi+\gamma^1\psi'+\gamma^1\psi=0\ , \qquad (4.4)$$

where the first and third components of ψ are equal and the same occurs between the second and fourth ones; the prime indicates partial differentiation with respect to x^1.

In terms of the components of χ, defined by

$$\chi(\eta,x^1)=a^{1/2}be^{x^1}\psi(\eta,x^1)\ , \qquad (4.5)$$

the Dirac equation (4.4) yields

$$\dot{\chi}_1-\chi_2'=0,\quad \dot{\chi}_2-\chi_1'=0 \qquad (4.6)$$

whose solutions are

$$\chi_1(\eta,x^1)=\tfrac{1}{2}[F(x^1+\eta)+G(x^1-\eta)]\ ,$$
$$\chi_2(\eta,x^1)=\tfrac{1}{2}[F(x^1+\eta)-G(x^1-\eta)]\ , \qquad (4.7)$$

where $F=F_r+iF_i$ and $G=G_r+iG_i$ are arbitrary complex functions.

On the other hand, Eq. (2.7a) in the metric (4.1) for homogeneous scalar fields has the solution

$$\dot{\phi}=\alpha b^{-2}\ , \qquad (4.8)$$

where α is an arbitrary real constant. Therefore, the nonzero components of the energy-momentum tensors (2.8a) and (2.8b) are

$$T^S_{00}=T^S_{11}=\frac{\dot{\phi}^2}{2}=\frac{\alpha^2b^{-4}}{2}\ ,$$
$$T^S_{22}=T^S_{33}=\frac{a^{-2}b^2}{2}e^{2x^1}\dot{\phi}^2=\frac{\alpha^2}{2}(ab)^{-2}e^{2x^1}\ ,$$
$$T^D_{00}=b^{-2}e^{-2x^1}(F_iF_r'-F_rF_i'+G_rG_i'-G_iG_r')\ ,$$
$$T^D_{01}=b^{-2}e^{-2x^1}(F_iF_r'-F_rF_i'+G_iG_r'-G_rG_i')\ ,$$
$$T^D_{02}=\frac{(ab)^{-1}}{2}e^{-x^1}(F_rG_r+F_iG_i+G_rF_r'+G_iF_i'+F_rG_r'+F_iG_i')\ , \qquad (4.9)$$
$$T^D_{03}=\frac{(ab)^{-1}}{2}e^{-x^1}(F_rG_i-F_iG_r+G_iF_r'-G_rF_i'+F_rG_i'-F_iG_r')\ ,$$
$$T^D_{11}=b^{-2}e^{-2x^1}(F_iF_r'-F_rF_i'+G_rG_i'-G_iG_r')\ ,$$
$$T^D_{12}=-\frac{(ab)^{-1}}{2}e^{-x^1}[(H_a-H_b)(F_rG_r+F_iG_i)+F_rG_r'+F_iG_i'-G_rF_r'-G_iF_i']\ ,$$
$$T^D_{13}=\frac{(ab)^{-1}}{2}e^{-x^1}[(H_a-H_b)(F_iG_r-F_rG_i)+G_iF_r'+F_iG_r'-G_rF_i'-F_rG_i']\ ,$$

where the prime now denotes differentiation with respect to the argument of the function, and the neutrino current density is

$$S^0=\frac{e^{-2x^1}}{(ab)^2}(F_r^2+F_i^2+G_r^2+G_i^2)\ ,$$
$$S^1=-\frac{e^{-2x^1}}{(ab)^2}(F_r^2+F_i^2-G_r^2-G_i^2)\ ,$$
$$S^2=\frac{2e^{-3x^1}}{ab^3}(F_rG_i-F_iG_r)\ , \qquad (4.10)$$
$$S^3=-\frac{2e^{-3x^1}}{ab^3}(F_rG_r+F_iG_i)\ .$$

Since R_{12}, R_{13}, R_{02}, and R_{03} vanish, the Einstein field equations yield

$$T^D_{12}=T^D_{13}=T^D_{02}=T^D_{03}=0\ . \qquad (4.11)$$

Furthermore, demanding that the neutrino current propagates along the symmetry axis leads to two mutually exclusive cases:[4,10] (a) $F_r=F_i=0$, G_r, G_i arbitrary; (b) $G_r=G_i=0$, F_r, F_i arbitrary. Cases (a) and (b) correspond to neutrinos traveling in the $+x^1$ and $-x^1$ directions, respectively. We shall discuss first case (a).

For the metric (4.1), Eqs. (2.7c) become

$$\dot{H}_a+2\dot{H}_b+2H_b(H_b-H_a)=8\pi G\left[e^{-2x^1}\left[\frac{G_r}{G_i}\right]'G_i^2b^{-2}-\alpha^2b^{-4}\right]\ ,$$
$$2(H_b-H_a)=-8\pi Ge^{-2x^1}\left[\frac{G_r}{G_i}\right]'G_i^2b^{-2}\ ,$$
$$-\dot{H}_a+2-2H_aH_b=8\pi Ge^{-2x^1}\left[\frac{G_r}{G_i}\right]'G_i^2b^{-2}\ , \qquad (4.12)$$
$$e^{2x^1}a^{-2}b^2(-\dot{H}_b-2H_b^2+2)=0$$

which, together with (4.4) and (4.8), have the following solutions.

(1)

$$a(\eta)=a_0e^{\eta}|e^{-4\eta}-1|^{C/4}\ , \qquad (4.13a)$$
$$b(\eta)=b_0e^{\eta}|e^{-4\eta}-1|^{1/2}\ , \qquad (4.13b)$$
$$\psi(\eta,x^1)=\frac{a_0^{-1/2}b_0^{-1}}{2}|e^{-4\eta}-1|^{-(C+4)/8}e^{-[(3/2)\eta+x^1]}\times[G_r(x^1-\eta)+iG_i(x^1-\eta)]\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}\ , \qquad (4.13c)$$

$$G_i^2\left[\frac{G_r}{G_i}\right]'=\frac{2-C}{4\pi G}b_0^2e^{2(x^1-\eta)}\text{sgn}(e^{-4\eta}-1)\ , \tag{4.13d}$$

$$\phi(\eta)=\phi_0\pm\frac{1}{4}\left[\frac{C+1}{\pi G}\right]^{1/2}\ln\left|\frac{e^{2\eta}-1}{e^{2\eta}+1}\right|\ , \tag{4.13e}$$

where a_0, b_0, and ϕ_0 are arbitrary real constants and C is a real constant such that $C\geq -1$.

(2)

$$a(\eta)=a_0e^{\eta}(e^{-4\eta}+1)^{C/4}\ , \tag{4.14a}$$

$$b(\eta)=b_0e^{\eta}(e^{-4\eta}+1)^{1/2}\ , \tag{4.14b}$$

$$\psi(\eta,x^1)=\frac{a_0^{-1/2}b_0^{-1}}{2}(e^{-4\eta}+1)^{-(C+4)/8}e^{-[(3/2)\eta+x^1]}\times[G_r(x^1-\eta)+iG_i(x^1-\eta)]\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}\ , \tag{4.14c}$$

$$G_i^2\left[\frac{G_r}{G_i}\right]'=\frac{2-C}{4\pi G}b_0^2e^{2(x^1-\eta)}\ , \tag{4.14d}$$

$$\phi(\eta)=\phi_0\pm\frac{1}{2}\left[-\frac{C+1}{\pi G}\right]^{1/2}\arctan e^{-2\eta} \tag{4.14e}$$

with $C\leq -1$.

(3)

$$a(\eta)=a_0e^{k\eta}\ , \tag{4.15a}$$

$$b(\eta)=b_0e^{-\eta}\ , \tag{4.15b}$$

$$\psi(\eta,x^1)=\frac{a_0^{-1/2}b_0^{-1}}{2}\exp\left\{-\left[x^1+\left[\frac{k}{2}-1\right]\eta\right]\right\}\times[G_r(x^1-\eta)+iG_i(x^1-\eta)]\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}\ , \tag{4.15c}$$

$$G_i^2\left[\frac{G_r}{G_i}\right]'=\frac{1+k}{4\pi G}b_0^2e^{2(x^1-\eta)}\ , \tag{4.15d}$$

$$\phi(\eta)=\phi_0\ , \tag{4.15e}$$

where k is an arbitrary real constant.

(4)

$$a(\eta)=a_0\exp\left[\eta-\frac{C}{4}e^{-4\eta}\right]\ , \tag{4.16a}$$

$$b(\eta)=b_0e^{\eta}\ , \tag{4.16b}$$

$$\psi(\eta,x^1)=\frac{a_0^{-1/2}b_0^{-1}}{2}\exp\left[-\tfrac{3}{2}\eta+\frac{e^{-4\eta}}{8}-x^1\right]\times[G_r(x^1-\eta)+iG_i(x^1-\eta)]\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}\ , \tag{4.16c}$$

$$G_i^2\left[\frac{G_r}{G_i}\right]'=\frac{Cb_0^2}{4\pi G}e^{2(x^1-\eta)}\ , \tag{4.16d}$$

$$\phi(\eta)=\phi_0\pm\frac{1}{2}\left[\frac{C}{\pi G}\right]^{1/2}e^{-2\eta} \tag{4.16e}$$

with C a real constant such that $C\geq 0$. In order to determine the neutrino field exactly, we may choose G_i and solve (4.13d), (4.14d), (4.15d), or (4.16d) for G_r.

Case (b) is the time reversal of case (a). Therefore, making the transformation $t\to -t$, $a\to a$, $b\to b$, $\psi\to i\gamma_1\gamma_3\psi^*$ (Ref. 25), $\phi\to\phi$ on all of the solutions of case (a), we obtain the solutions corresponding to neutrinos traveling in the $-x^1$ direction.

A. Analysis of the solutions

For the solutions (4.13a) and (4.13b), the universe begins at a singularity at $\eta=0$ with an axisymmetric Zel'dovich-type local behavior, since at early times, in terms of the proper time $t\equiv\int a(\eta)d\eta$, the evolution functions are

$$a(t)\sim|t|^{1-2p},\quad b(t)\sim|t|^{p} \tag{4.17}$$

with $p\equiv 2(C+4)^{-1}\in(0,\frac{2}{3})$ [cf. (3.25)]. If there were no scalar field source ($C=-1$), then near the singularity the universe would have locally a Kasner-type behavior. At late times, the source generated by the propagating neutrino field becomes comparable to that of the scalar field, and the universe tends to flat spacetime (Milne universe). When $C\in[-1,1)$, the coordinate system has a singularity at $\eta=-\infty$. However, we still do not know whether the actual spacetime is singular. The invariant quantity

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}=\frac{4}{a^4}\{\dot H_a^2+2[(\dot H_b+H_b^2-H_aH_b)^2+(1-H_aH_b)^2-2(H_b-H_a)^2]+(1-H_b^2)^2\}$$
$$=\frac{64}{a_0^4}\frac{e^{-12\eta}}{|e^{-4\eta}-1|^{C+4}}[(C+1)(C-2)e^{-4\eta}+C^2+3C+5] \tag{4.18}$$

diverges when $\eta\to-\infty$ if $C\in(-1,0)$. Hence, for this case, the spacetime cannot be extended beyond $\eta=-\infty$, and the model is singular. Therefore, it is possible that the anisotropic universe can be singular or not in the past ($\eta\to-\infty$) where the neutrino field dominates, and then it collapses at a singularity with an axisymmetric Zel'dovich-type behavior. If there were no neutrino field source ($C=2$), then the model would be isotropic for all times.

The solutions (4.14a) and (4.14b) tend, in the remote future, to flat spacetime. When $C < -1$, this spacetime has a singularity at $\eta = -\infty$ where the neutrino field dominates. For this model an isotropic universe for all times is impossible. Moreover, if the source provided by the scalar field vanishes, then we obtain the solutions (4.15a) and (4.15b) found by Ray,[10] which describe nonisotropizing universes; the $k = -1$ case corresponds to the Milne universe.

For the solutions (4.16a) and (4.16b) the universe tends to flat spacetime at late times (the $C=0$ case corresponds to flat spacetime). As one goes back towards the singularity at $\eta = -\infty$ ($C > 0$), one would see a highly contracted universe as compared to the solutions (4.14a) and (4.14b). In this case, the sources generated by the neutrino and scalar fields are comparable during all the evolution. For solutions (4.13a), (4.13b), (4.14a), (4.14b), (4.16a), and (4.16b), there exist timelike curves which reach the spacetime singularity at $\eta = -\infty$ at finite proper time.

V. CONCLUSIONS

We have found the general solution to the Einstein–Dirac–Klein-Gordon field equations in the case of unquantized and homogeneous scalar (minimally coupled to gravity) and spinor fields in a Bianchi type-I spacetime, and in the case that the spinor field represents neutrinos propagating along the symmetry axis in a plane-symmetric Bianchi type-V metric. In the first case, when the spinor field is massive, the main result is that there exists a solution describing a universe which starts from a singularity with a Zel'dovich-type behavior (or Kasner type if the scalar source vanishes). Then, when the energy of the spinor field becomes important, the universe turns into an isotropic dust-filled Friedmann stage. The isotropization time t_s will depend on the initial energy density as well as the initial anisotropy. The remaining solutions do not have a final Friedmann stage. For massless spinor fields we get an axisymmetric Zel'dovich universe, and only ghost neutrinos with vanishing energy-momentum tensor are allowed. All these Bianchi type-I solutions have particle horizons.

In the second case we have shown that it is possible that the Bianchi type-V universe has begun at a singularity, which can be of Zel'dovich type, and at late times it tends to flat spacetime. The remaining solutions are either always anisotropic or do not tend asymptotically to flat spacetime.

For all these Bianchi type-V models the propagating neutrinos are not ghost ones. Thus, one may think that this solution may represent physical neutrinos.

Finally, when neither of the two sources of gravity vanishes, it is impossible to obtain an isotropic universe for all times as a limiting case of the Bianchi type-V metric. On the other hand, in the Bianchi type-I model, when the spinor field is massive, there is only one initial condition which gives an isotropic evolution for all times. Thus there is an *a priori* vanishing probability that the Universe has started with an isotropic initial expansion.

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