New approach to find exact solutions for cosmological models with a scalar field

R. de Ritis, G. Marmo, G. Platania, C. Rubano, P. Scudellaro, and C. Stornaiolo
Dipartimento di Scienze Fisiche, Università di Napoli, Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Mostra d'Oltremare Pad. 19, 80125 Napoli, Italy

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We study the cosmological models whose ordinary differential equations can be given a Lagrangian description on a two-dimensional "configuration space." By requiring the existence of a Nöther symmetry for such a Lagrangian, we are able to show that the potential must have an exponential form. With the help of the constants of motion we can get from that Lagrangian, we integrate the models and analyze the behavior for all the possible varying free parameters and plot some of the solutions. We also compare our results with others available in the literature.

I. INTRODUCTION

The inflationary paradigm is currently seen as a solution to the problems arising in standard cosmology, such as flatness and horizon.[1] The correct-time behavior in the scale factor is usually obtained by introducing a scalar field whose dominant energy density drives the dynamical evolution in the primitive stages of the Universe.[2] This paradigm is not yet satisfactorily established with regard to the connection with a theory of elementary particles, so that the choice of the potential $V(\phi)$ for the scalar field ϕ is still an object of investigation. Potentials of polynomial type, such as $m^2\phi^2$ and $\lambda\phi^4$, have been studied.[3,4] Potentials of exponential type have been proposed also in connection with the four-dimensional reduction of Kaluza-Klein theories,[5] with the so-called power-law inflation,[6] and by means of arguments of phenomenological type.[7]

In the present paper, we also deal with the problem of determining the form of the potential but from a point of view that seems to us quite different. We may summarize our arguments in the following points.

(i) We try to find *exact* solutions for the field equations in the homogeneous and isotropic case. The first two are[8]

$$\ddot{a}=-\frac{1}{2}\frac{\dot{a}^2}{a^2}-\frac{1}{4}\frac{a\dot{\phi}^2}{M^2}+\frac{a}{2M^2}V(\phi)\ , \qquad (1.1a)$$

$$\ddot{\phi}=-3\frac{\dot{a}}{a}\dot{\phi}-V'(\phi)\ , \qquad (1.1b)$$

where ϕ is the scalar field, a is the scale factor, and $M^2=(8\pi G)^{-1}$.

The third equation is

$$\left[\frac{\dot{a}}{a}\right]^2=\frac{1}{3M^2}[\tfrac{1}{2}\dot{\phi}^2+V(\phi)]\ . \qquad (1.2)$$

These equations can be derived by substitution of the Friedmann metric into the Einstein action, with g_{00} left free to be varied, instead of being fixed to unity.

Our approach consists in considering the first two as a second-order dynamical system associated with the vector field

$$\Gamma=\dot{a}\frac{\partial}{\partial a}+\dot{\phi}\frac{\partial}{\partial\phi}-\left[\frac{1}{2}\frac{\dot{a}^2}{a}+\frac{1}{4}\frac{a\dot{\phi}^2}{M^2}-\frac{a}{2M^2}V(\phi)\right]\frac{\partial}{\partial\dot{a}}-\left[3\frac{\dot{a}}{a}\dot{\phi}+V'(\phi)\right]\frac{\partial}{\partial\dot{\phi}} \qquad (1.3)$$

thought of as a vector field on $T\mathbb{R}^2$, the tangent bundle of $\mathbb{R}^2$, with natural coordinates $(a,\phi,\dot{a},\dot{\phi})$. Γ may be derived by the point Lagrangian

$$\mathcal{L}=3a\dot{a}^2-\frac{1}{2M^2}a^3\dot{\phi}^2+\frac{a^3}{M^2}V(\phi) \qquad (1.4)$$

while Eq. (1.2) is considered as a constraint and is equivalent to the vanishing of the "energy function" associated with $\mathcal{L}$ (Ref. 9)

$$E_{\mathcal{L}}=3a\dot{a}^2-\frac{1}{2M^2}a^3\dot{\phi}^2-\frac{a^3}{M^2}V(\phi)\ . \qquad (1.5)$$

Thus, once we get general solutions of Eqs. (1.1a) and (1.1b) we have to specialize them with a suitable choice of the initial conditions, in order to have $E_{\mathcal{L}}=0$.

It is interesting to note that the Lagrangian (1.2) can be obtained by putting the Robertson-Walker metric into the Einstein action. This gives a second-order point Lagrangian, which can be easily reduced to (1.4).

(ii) When $V(\phi)=\Lambda$ (the cosmological constant), we are in the de Sitter case and $\mathcal{L}$ does not depend on ϕ. This means that, in the Lagrangian (1.4), ϕ is a cyclic coordinate and the vector field $\partial/\partial\phi$ is a Nöther symmetry for $\mathcal{L}$, associated with the constant of the motion[8]

$$C=a^3\dot{\phi}\ . \qquad (1.6)$$

When this is not the case, it is possible to generalize by looking for a vector field X such that

$$L_X\mathcal{L}=0\ , \qquad (1.7)$$

where L_X stands for Lie derivative with respect to X. We shall prove that this is possible only for a suitable class of potentials, and we shall find new variables which allow an explicit solution of (1.1a) and (1.1b).

The paper is organized as follows. In Sec. II we find the vector field X and the class of potentials. In Sec. III we perform the change of variables and get the solutions. In Sec. IV we discuss the solutions in connection with their physical meaning, showing that some of them give quite interesting models of the Universe, in which we find an exponential or power-law inflation. In Sec. V we deal with conclusions.

II. THE CLASS OF POTENTIALS

As sketched in the Introduction, we are looking for symmetries of the Lagrangian (1.4). Although it would be interesting to solve the problem in full generality, we shall limit ourselves to the so-called point symmetries.[9-11] These are transformations on the tangent space, derived by transformations on the base space in such a way as to preserve the second-order character of the dynamical field.[10,11] The generic infinitesimal generator of a point transformation is given by

$$X=\alpha(a,\phi)\frac{\partial}{\partial a}+\beta(a,\phi)\frac{\partial}{\partial \phi}+\frac{d\alpha}{dt}\frac{\partial}{\partial \dot a}+\frac{d\beta}{dt}\frac{\partial}{\partial \dot\phi}\ , \tag{2.1}$$

where α, β are generic functions of a, ϕ and $d\alpha/dt$, $d\beta/dt$ are their time derivatives along the dynamical vector field (1.1c). We have to determine α, β such that the Lie derivative of $\mathcal{L}$ along X is zero

$$L_X\mathcal{L}=0\ . \tag{2.2}$$

[Strictly speaking, it would be sufficient to have $L_X\mathcal{L}=df(a,\phi)/dt$, with f any function, but it is easy to see that, because $\mathcal{L}$ does not have terms linear in the "velocities," this is possible only for $f=$const.] The meaning of this equation is that $\mathcal{L}$ is constant along the flow (possibly a local flow), generated by X, i.e., Eq. (2.2) is identically verified all over TQ. Its explicit evaluation gives an expression of second degree in $\dot a$, $\dot\phi$, whose coefficients are functions of a, ϕ only. Therefore, they have to be zero separately. This gives

$$\alpha+2a\frac{\partial\alpha}{\partial a}=0\ , \tag{2.3a}$$

$$3\alpha+2a\frac{\partial\beta}{\partial\phi}=0\ , \tag{2.3b}$$

$$6\frac{\partial\alpha}{\partial\phi}-\frac{1}{M^2}a^2\frac{\partial\beta}{\partial a}=0\ , \tag{2.3c}$$

$$3\alpha V(\phi)+\beta a V'(\phi)=0\ . \tag{2.3d}$$

When $V=\Lambda$ we get $\alpha=0$, $\beta=$const, so that the $\partial/\partial\phi$ symmetry is unique; we will not consider it any longer. Let us set

$$\sigma_\pm=Ae^{\lambda\phi}\pm Be^{-\lambda\phi},\quad A,B\in\mathbb{R}\ , \tag{2.4}$$

$$\lambda=\frac{1}{2M}\left[\frac{3}{2}\right]^{1/2}\ . \tag{2.5}$$

It is easy to prove now that Eqs. (2.3a)–(2.3d) have the general solution

$$\alpha=\frac{\sigma_+(\phi)}{\sqrt{a}}\ , \tag{2.6}$$

$$\beta=-\sqrt{6}M\frac{\sigma_-(\phi)}{a\sqrt{a}}\ , \tag{2.7}$$

if and only if

$$\frac{V'}{V}=2\lambda\frac{\sigma_+}{\sigma_-}\ , \tag{2.8}$$

so that the existence of the required symmetry gives a condition on V. From (2.8) we get

$$V=V_0\sigma_-^2=V_0(A^2e^{2\lambda\phi}+B^2e^{-2\lambda\phi}-2AB)\ . \tag{2.9}$$

From now on we shall stick to this form for the potential, with the only freedom left by the arbitrary constants V_0, A, B.

From (2.3) we see that, if we perform the transformation $\alpha\to\alpha'=k\alpha$ and $\beta\to\beta'=k\beta$, the equations remain the same. This implies that only two of these constants are really necessary, i.e., V_0 and A/B or B/A. The form we have chosen allows us to consider A and B on the same footing.

At this point, the existence of the symmetry X gives us a constant of the motion, via the Nöther theorem. A possible way to find it[9] is to compute the Cartan one-form associated with $\mathcal{L}$:

$$\theta_{\mathcal{L}}=\frac{\partial\mathcal{L}}{\partial\dot a}da+\frac{\partial\mathcal{L}}{\partial\dot\phi}d\phi=6a\dot a\,da-\frac{a^3}{M^2}\dot\phi\,d\phi \tag{2.10}$$

which, when contracted with X, gives the required constant of the motion F (Refs. 9 and 10)

$$F=i_X\theta_{\mathcal{L}}=6\sigma_+\sqrt{a}\,\dot a+\frac{\sqrt{6}}{M}\sigma_-a^{3/2}\dot\phi\ . \tag{2.11}$$

In the next section we shall see that F, together with the "energy function" (1.5) associated with $\mathcal{L}$, provides the possibility of achieving complete integration of Eqs. (1.1a) and (1.1b).

Remarks: (1) The potential (2.9), in the case when $A=0$, coincides with one of the class found by Ratra and Peebles.[7] They consider the functions

$$P=\tfrac{1}{2}\dot\phi^2-V(\phi)\ , \tag{2.12a}$$

$$\rho=\tfrac{1}{2}\dot\phi^2+V(\phi) \tag{2.12b}$$

(which are the pressure and density associated with the scalar field[7,8]) and require that the solutions should satisfy the state equation

$$P=\frac{q-3}{3}\rho\ . \tag{2.13}$$

It turns out that this is possible iff

$$V=\tilde V_0\exp\left[-\frac{\sqrt{q}\,\phi}{M}\right]\ . \tag{2.14}$$

Thus we see that our potential coincides with (2.14) when $A=0$ and $q=\frac{3}{2}$.

One may wonder why the potential happens to select just this particular value. The reason is that we are solving Eqs. (1.1) in the full tangent space. In the usual approach the tangent space is restricted by Eq. (1.2) and, in this case, by Eq. (2.13). Let us rewrite these equations as

$$\mathcal{S}=\left[\frac{\dot a}{a}\right]^2-\frac{1}{3M^2}\left[\tfrac{1}{2}\dot\phi^2+V(\phi)\right]=0\ , \tag{2.15}$$

$$\mathcal{T}=\dot\phi^2-\mu V=0\ , \tag{2.16}$$

the second being obtained by (2.12) and (2.13), with

$$\mu=\frac{2q}{6-q}\ . \tag{2.17}$$

The solutions must thus lie on the surface $\mathcal{S}=0$, $\mathcal{T}=0$. This is possible only if the dynamical vector field Γ is tangent to them. This is indeed the case if V has the form (2.14). The same argument applies to X, which can be a symmetry for the constrained system only if it is tangent to the surfaces and this is possible only for V of the form (2.14) *and* $\mu=\frac{2}{3}$, which gives $q=\frac{3}{2}$.

This comparison seems to suggest that one may try to use both procedures, namely, one can impose some "invariant relation" such as (2.13) and look for symmetries along this relation, using the same machinery that works for the Dirac-Bergmann theory of constraints.[12] We recall that $f(a,\phi,\dot a,\dot\phi)=0$ defines an invariant relation if the dynamical evolution preserves the value of f only for a selected submanifold of the initial conditions; in our case the submanifold is defined by $E_{\mathcal{L}}=0$.

(2) It may be useful to give the Hamiltonian formulation of the above results. The Legendre transformation, applied to the Lagrangian (1.4), gives the Hamiltonian

$$H=\frac{p_a^2}{12a}-\frac{M^2}{2a^3}p_\phi^2-\frac{a^3V(\phi)}{M^2} \tag{2.18}$$

with $p_a=6a\dot a$ and $p_\phi=-a^3\dot\phi/M^2$.

The constant of the motion (2.11) and the symmetry X transform into

$$F=\frac{\sigma_+}{\sqrt{a}}p_a-\frac{\sqrt{6}M\sigma_-}{\sqrt{a_3}}p_\phi\ , \tag{2.19}$$

$$X=\alpha\frac{\partial}{\partial a}+\beta\frac{\partial}{\partial\phi}+6a(L_\Gamma\alpha)\frac{\partial}{\partial p_a}-\frac{a^3}{M}(L_\Gamma\beta)\frac{\partial}{\partial p_\phi}\ , \tag{2.20}$$

with the same α and β as before, of course.

The dynamical vector field is easily derived from the Hamilton equations

$$\Gamma=6a\dot a\frac{\partial}{\partial a}-\frac{a^3\dot\phi}{M^2}\frac{\partial}{\partial\phi}+\frac{p_a}{6a}\frac{\partial}{\partial p_a}-\frac{M^2}{a^3}\frac{\partial}{\partial p_a}\ . \tag{2.21}$$

It is easy to check that the Poisson brackets $\{H,F\}$ is zero and $L_\Gamma H=0$, iff V is of the form (2.9). Unfortunately, also in this formulation, the physical meaning of F is not very clear.

(3) A paper by Kuchař[13] shows that there are no symmetries for general relativity. One may thus wonder how is it that we find one in this case. The point is that, in his paper, Kuchař considers a *general metric without scalar field*. In our case, we have a *specific metric with scalar field*. If we eliminate ϕ, it is easy to see that, with Eqs. (1.1) and (1.2), there is no metric at all. Thus we think it is not possible to compare these results.

III. THE SOLUTIONS

To find explicit solutions for Euler-Lagrange equations, we shall try to associate with X a cyclic coordinate for $\mathcal{L}$. Let us look for a nondegenerate point transformation

$$z=z(a,\phi),\quad w=w(a,\phi)\ , \tag{3.1}$$

such that

$$i_X(dz)=\text{const},\quad i_X(dw)=0\ . \tag{3.2}$$

In these new coordinates the Lagrangian will be cyclic in z. From now on, it is convenient to distinguish the case $A,B\neq0$ from the case A or $B=0$.

A. $A,B\neq0$

Let us set

$$z=a^{3/2}\sigma_+\ , \tag{3.3a}$$

$$w=a^{3/2}\sigma_-\ , \tag{3.3b}$$

which satisfies (3.2) and, provided that a, $A,B\neq0$, can be inverted to

$$\phi=\sqrt{2/3}M\ln\left[\frac{B}{A}\frac{z+w}{z-w}\right]\ , \tag{3.4a}$$

$$a=\left[\frac{z^2-w^2}{4AB}\right]^{1/3}\ . \tag{3.4b}$$

Under these transformations we get

$$V=4ABV_0\frac{w^2}{z^2-w^2}\ , \tag{3.5}$$

$$\mathcal{L}=\dot z^2-\dot w^2+\frac{3ABV_0}{M^2}w^2\ , \tag{3.6a}$$

$$F=\dot z\ , \tag{3.6b}$$

$$E_{\mathcal{L}}=\dot z^2-\dot w^2-\frac{3ABV_0}{M^2}w^2\ . \tag{3.6c}$$

(Here and below, with slight abuse of notation, we shall omit unimportant constant factors.)

Setting

$$\omega=\frac{3ABV_0}{M^2} \tag{3.7}$$

we get the equations of motion

$$\ddot z=0\ , \tag{3.8a}$$

$$\ddot w=-\omega w\ , \tag{3.8b}$$

which are trivially integrated to

$$z=v_0t+z_0\ , \tag{3.9a}$$

$$w=w_0\sin(\sqrt{\omega}t+w_1)\ \text{ if }\omega>0\ , \tag{3.9b}$$

$$w=w_0\sinh(\sqrt{-\omega}t+w_1)\ \text{ if }\omega<0\ . \tag{3.9c}$$

In the case $\omega<0$, the expression (3.9c) does not take into account the particular solution

$$w = w_0 \exp(\pm\sqrt{-\omega}t) \tag{3.10}$$

which is treated separately.

We have now to impose $E_{\mathcal{L}}=0$. Let us first consider solution (3.9). We get for both cases

$$v_0 = \pm\sqrt{|\omega|}w_0 \ . \tag{3.11}$$

The case $F=0$ is thus excluded, since this gives no dynamics at all.

If $\omega<0$, there is also the solution (3.10) for which we have

$$\begin{aligned} E_{\mathcal{L}} &= v_0^2 + w_0^2\omega\exp(\pm 2\sqrt{-\omega}t) \\ &\quad - w_0^2\omega\exp(\pm 2\sqrt{-\omega}t) \\ &= v_0 \ . \end{aligned} \tag{3.12}$$

Thus $E_{\mathcal{L}}=0 \Longrightarrow v_0=0$ and hence $F=\dot{z}=0$.

In conclusion we have the following situation: for $\omega>0$, $F\neq 0$,

$$z = \pm w_0\sqrt{\omega}t + z_0 \ , \tag{3.13a}$$

$$w = w_0\sin(\sqrt{\omega}t + w_1) \ , \tag{3.13b}$$

$$a = \left[\frac{z^2-w^2}{4AB}\right]^{1/3} = \left[\frac{(\pm w_0\sqrt{\omega}t+z_0)^2 - w_0^2\sin^2(\sqrt{\omega}t+w_1)}{4AB}\right]^{1/3} , \tag{3.13c}$$

$$\phi = \sqrt{2/3}M \ln\left[\frac{B}{A}\frac{z+w}{z-w}\right] = \sqrt{2/3}M\ln\left[\frac{B}{A}\frac{\pm\sqrt{\omega}w_0t+z_0+w_0\sin(\sqrt{\omega}t+w_1)}{\pm\sqrt{\omega}w_0t+z_0-w_0\sin(\sqrt{\omega}t+w_1)}\right] ; \tag{3.13d}$$

for $\omega<0$, $F\neq 0$,

$$z = \pm w_0\sqrt{-\omega}t + z_0 \ , \tag{3.14a}$$

$$w = w_0\sinh(\sqrt{-\omega}t + w_1) \ , \tag{3.14b}$$

$$a = \left[\frac{z^2-w^2}{4AB}\right]^{1/3} = \left[\frac{(\pm w_0\sqrt{-\omega}t+z_0)^2 - w_0^2\sinh^2(\sqrt{-\omega}t+w_1)}{4AB}\right]^{1/3} , \tag{3.14c}$$

$$\phi = \sqrt{2/3}M \ln\left[\frac{B}{A}\frac{z+w}{z-w}\right] = \sqrt{2/3}M\ln\left[\frac{B}{A}\frac{\pm\sqrt{-\omega}w_0t+z_0+w_0\sinh(\sqrt{-\omega}t+w_1)}{\pm\sqrt{-\omega}w_0t+z_0-w_0\sinh(\sqrt{-\omega}t+w_1)}\right] ; \tag{3.14d}$$

for $\omega<0, F=0$,

$$z = z_0 \ , \tag{3.15}$$

$$w = w_0\exp(\pm\sqrt{-\omega}t) \ , \tag{3.16a}$$

$$a = \left[\frac{z_0^2 - w_0^2\exp(\pm 2\sqrt{-\omega}t)}{4AB}\right]^{1/3} , \tag{3.16b}$$

$$\phi = \sqrt{2/3}M\ln\left[\frac{B}{A}\frac{z_0+w_0\exp(\pm\sqrt{-\omega}t)}{z_0+w_0\exp(\pm\sqrt{-\omega}t)}\right] . \tag{3.16c}$$

Once V_0, A, B are fixed, the integration constants w_0, z_0, w_1 must be chosen in such a way as to get $a>0$ and ϕ well defined. This will be done is Sec. IV.

B. $A=0$ (or $B=0$)

Let us now set $A=0$, so that $\sigma_+ = -\sigma_- = Be^{-\lambda\phi}$, $V=V_0B^2e^{-2\lambda\phi}$. In this case the transformation (3.3) is degenerate, thus we shall use, instead,

$$z = \frac{a^{3/2}}{\sigma_+} \ , \tag{3.17a}$$

$$w = \frac{a^{-3/2}}{\sigma_+} \ , \tag{3.17b}$$

which again satisfies (3.2) and can be inverted to

$$a = \left[\frac{z}{w}\right]^{1/3} , \tag{3.18a}$$

$$\phi = \sqrt{2/3}M\ln(B^2zw) \ . \tag{3.18b}$$

We have then

$$\mathcal{L} = \frac{\dot{z}\dot{w}}{w^2} - \frac{3V_0}{4M^2}\frac{1}{w^2} \ , \tag{3.19a}$$

$$F = \frac{\dot{w}}{w^2} \ , \tag{3.19b}$$

$$E_{\mathcal{L}} = F\dot{z} + \frac{3V_0}{4M^2}\frac{1}{w^2} \ , \tag{3.19c}$$

with equations of motion

$$\ddot{w} = \frac{2\dot{w}^2}{w} \ , \tag{3.20a}$$

$$\ddot{z} = \frac{3V_0}{2M^2}\frac{1}{w} \ . \tag{3.20b}$$

The case $F=E_{\mathcal{L}}=0$ entails $V_0=0$, which has been excluded from the beginning. For $F\neq 0$, $E_{\mathcal{L}}=0$, we may integrate directly Eqs. (3.19b) and (3.19c)

$$w = -\frac{1}{Ft+c_1} \ , \tag{3.21a}$$

$$z=-\left[\frac{V_0F}{4M^2}t^3+\frac{3V_0c_1}{4M^2}t^2+\frac{3V_0c_1^2}{4FM^2}t+c_2\right] . \tag{3.21b}$$

The expressions for a,ϕ are

$$a=\left[(Ft+c_1)\left[\frac{V_0F}{4M^2}t^3+\frac{3V_0c_1}{4M^2}t^2+\frac{3V_0c_1^2}{4FM^2}t+c_2\right]\right]^{1/3} , \tag{3.22a}$$

$$\phi=\sqrt{2/3}M\ln\left[B^2\frac{\frac{V_0F}{4M^2}t^3+\frac{3V_0c_1}{4M^2}t^2+\frac{3V_0c_1^2}{4FM^2}t+c_2}{Ft+c_1}\right] . \tag{3.22b}$$

The case $B=0$, $A\neq 0$ is treated exactly in the same way and the results are the same, except for the sign of ϕ and substitution of A for B in Eq. (3.22b).

IV. DISCUSSION OF RESULTS

We are now going to study the solutions in order to analyze the values of the parameters V_0, A, B and of the integration constants in such a way as to satisfy the following conditions: (i) $a>0$, (ii) ϕ is well defined, i.e., the argument of the logarithm in (3.13d), (3.14d), (3.16c), and (3.22b) has to be positive, and (iii) a is not monotonically decreasing.

From Eqs. (3.4) and (3.18) we see that conditions (i) and (ii) are always satisfied together, so that it is sufficient to study the function $a(t)$ in various cases in order to make a selection.

Case 1: $A\neq 0$, $B\neq 0$, $F\neq 0$ [solution(3.9)]

Case 1A: $\omega<0$. Let us set $\epsilon=\pm 1$, $\sigma=\mathrm{sgn}(A\cdot B)$ and define the new time

$$\tau=\sqrt{-\omega}t+w_1 . \tag{4.1}$$

This means that we are fixing an *arbitrary* origin for time. We also see that ω determines the time scale and can be chosen arbitrarily by means of V_0.

Let us rewrite solutions (3.14c) and (3.14d) in the form

$$a=-\left[\frac{w_0^2}{4|AB|}\right]^{1/3}\sigma[(\sinh\tau)^2-(\epsilon\tau+h)^2]^{1/3} , \tag{4.2a}$$

$$\phi=\sqrt{2/3}M\ln\left[\sigma\frac{\epsilon\tau+h+\sinh\tau}{\epsilon\tau+h-\sinh\tau}\right]+\sqrt{2/3}M\ln\left|\frac{B}{A}\right| , \tag{4.2b}$$

where $h=z_0/w_0-\epsilon w_1$.

We see that w_0 determines only the scale of a and allows us to fix it arbitrarily. The module of the parameters A and B influence only the additive constant in ϕ, so that

$$\lim_{\tau\to\infty}\phi=\sqrt{2/3}M\ln\left|\frac{B}{A}\right|$$

and, by choosing $|A|=|B|=1$, this can be made zero. The only relevant parameters left are thus h, ϵ, σ, V_0.

Case 1A.1: $h=0$. It is easy to see that $a>0$ iff $\sigma<0$ (with $\omega<0$, this implies $V_0>0$). We have also $a(0)=0$, $a'(\tau)>0$ for $\tau>0$; moreover,

$$a(\tau)\propto\tau^{4/3} \quad \text{for } \tau\ll 1 ,$$

$$a(\tau)\propto e^{(2/3)\tau} \quad \text{for } \tau\gg 1 .$$

We see that this solution may represent a very early Universe, with $a(\tau)$ exponentially increasing and $\phi(\tau)$ starting with a singularity and very rapidly decreasing to a constant.

Case 1A.2: $h\neq 0$. In this case we have to compare the function $f(\tau)=(\sinh\tau)^2$ with $g(\tau)=(\epsilon\tau+h)^2$. It is not difficult to see that there are always two values τ_0,τ_1 (with $\tau_0<0<\tau_1$) such that

$$f(\tau)>g(\tau) \quad \text{for } \tau<\tau_0 \text{ or } \tau>\tau_1 ,$$

$$f(\tau)<g(\tau) \quad \text{for } \tau_0<\tau<\tau_1 .$$

The case $\tau<\tau_0$ is unphysical, as we have either $a(\tau)<0$ or $a(\tau)>0$, but decreasing. We have, therefore, two subcases.

Case 1A.2a: $\sigma<0$ (i.e., $V_0>0$). This is quite similar to Case 1A.1, but of course the physical origin is now τ_1. The asymptotical behavior of $a(\tau)$ is the same as before. The only difference occurs when $\tau\ll 1$, but this does not seem to be very relevant. Also $\phi(\tau)$ is very similar as before. The value of ϵ seems not to play a crucial role.

Case 1A.2b: $\sigma>0$ (i.e., $V_0>0$). This is a more interesting case. There is clearly a physical origin at $\tau=\tau_0$ and a physical *end* at $\tau=\tau_1$. A typical situation is shown in Fig. 1, for $\epsilon=1$ and $h=100$ (here and in the figures below, the scales of a and τ are arbitrary). There is a very rapid growth of $a(\tau)$ for $\tau\gtrsim\tau_0$, followed by a very slow growth in the central part and finally a very rapid fall to zero. Different from the cases discussed above, we now have to choose ω so that the time scale is suitably large. An estimation of the second derivative near the point $\tau=\tau_0$ gives $\ddot{a}(\tau)<0$, so that there is no inflation. The shape of ϕ is shown in Fig. 2. There are, of course, two singularities (at τ_0 and τ_1) and it is also $\phi(0)=$const.

Case 1B: $\omega>0$. Following the same scheme as in Case A, we have to study the functions

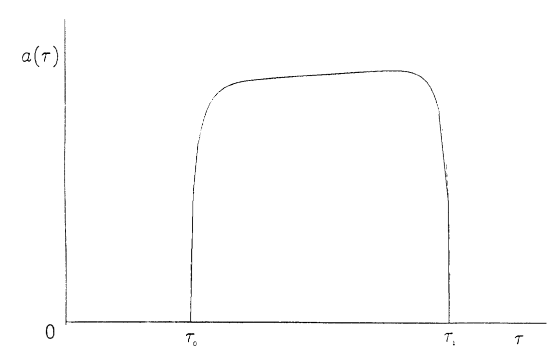

FIG. 1. $a(\tau)$ in Case 1A.2b; $\omega<0$, $h=100$, $\sigma=1$, $\epsilon=1$.

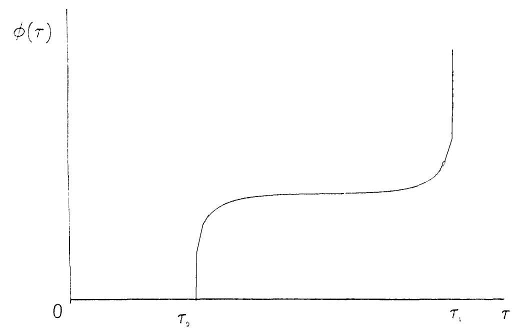

FIG. 2. $\phi(\tau)$ in Case 1A.2b; $\omega<0$, $h=100$, $\sigma=1$, $\epsilon=1$.

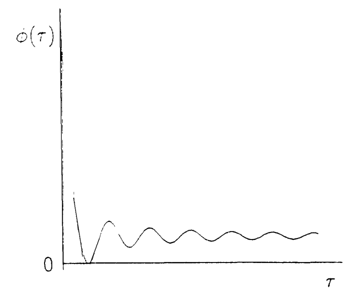

FIG. 4. $\phi(\tau)$ in Case 1B.1; $\omega>0$, $h=0$, $\sigma=1$, $\epsilon=1$.

$$a=-\left[\frac{w_0^2}{4}\right]^{1/3}\sigma[(\sin\tau)^2-(\epsilon\tau+h)^2]^{1/3}\ , \tag{4.3a}$$

$$\phi=\sqrt{2/3}M\ln\left[\sigma\frac{\epsilon\tau+h+\sin\tau}{\epsilon\tau+h-\sin\tau}\right]\ . \tag{4.3b}$$

We only have to remember that with $\omega>0$, $\sigma \gtrless 0$ implies $V_0 \gtrless 0$.

Case 1B.1: $h=n\pi$, $n\in\mathbb{Z}$. We have $a(\tau_0)=0$ for $\tau_0=-\epsilon n\pi$. Without loss of generality, we may choose $h=0$, i.e., $\tau_0=0$ so that this is the physical origin of time.

Comparing $f(\tau)=\sin^2\tau$ with $g(\tau)=\tau^2$, we see that it is always $g(\tau)>f(\tau)$ so that we have to choose $\sigma>0$ (i.e., $V_0>0$). We have thus

$$a(\tau)\propto\tau^{4/3}\quad\text{for } \tau\ll 1\ ,$$

$$a(\tau)\propto\tau^{2/3}\quad\text{for } \tau\gg 1\ ,$$

$a(\tau)$ is shown in Fig. 3. The behavior of $\phi(\tau)$ is rather peculiar, as it goes asymptotically to a constant with *oscillations*, as shown in Fig. 4. We see that this model shows a power-law inflation, with a behavior for large times like that of the matter-dominated Einstein–de Sitter universe.

Case 1B.2: $h\neq n\pi$, $n\in\mathbb{Z}$. The comparison of $f(\tau)=\sin^2\tau$ with $g(\tau)=(\epsilon\tau+h)^2$ shows that there are always two roots τ_0, τ_1 for $a(\tau)=0$ $(0<\tau_0<\tau_1)$. We have also

$$g(\tau)>f(\tau)\quad\text{for } \tau>\tau_1 \text{ or } \tau<\tau_0\ ,$$

$$g(\tau)<f(\tau)\quad\text{for } \tau_0<\tau<\tau_1\ .$$

Discarding $\tau<\tau_0$, we have two subcases.

Case 1B.2a: $\sigma>0$ (i.e., $V_0>0$). The physical origin is now τ_1 and all is quite similar to Case 1B.1, with an important difference. Examining Fig. 5 [which shows $a(\tau)$ very near to τ_1, for a particular choice of the parameters] we can see that the concavity is initially *downwards,* then there is a flexus, with concavity *upwards,* and finally the $\tau^{2/3}$ behavior.

Case 1B.2b: $\sigma<0$ (i.e., $V_0<0$). Now $a(\tau)>0$ for $\tau_0<\tau<\tau_1$ and all is very similar to Case 1A.2b.

Case 2: $\omega<0$, $A\neq 0$, $B\neq 0$, $F=0$ [solution (3.16)]

We have

$$a(t)=-\left[\frac{w_0}{4|AB|}\right]^{1/3}\sigma\exp(\pm 2\sqrt{-\omega}t-h^2)\ , \tag{4.4a}$$

$$\phi(t)=\sqrt{2/3}M\ln\left[\sigma\frac{B}{A}\frac{h+\exp(\pm\sqrt{-\omega}t)}{h-\exp(\pm\sqrt{-\omega}t)}\right]+\sqrt{2/3}M\ln\left|\frac{B}{A}\right|\ , \tag{4.4b}$$

with $h=z_0/w_0$. Thus, we must choose $\sigma<0$ (i.e., $V_0>0$) and the plus sign in (4.4a). In particular, for $h=0$, we have a model of the Universe without singularities, steadily increasing exponentially, with *constant* ϕ (which may be taken to be zero), that is a de Sitter universe. In this case the state equation is

$$P=-\rho \tag{4.5}$$

which is typical of exponential inflation.

Case 3: $A=0$, $F\neq 0$ [solutions (3.22a) and (3.22b)]

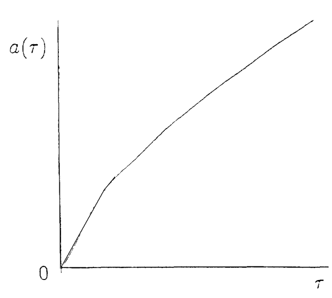

FIG. 3. $a(\tau)$ in Case 1B.1; $\omega>0$, $h=0$, $\sigma=1$, $\epsilon=1$.

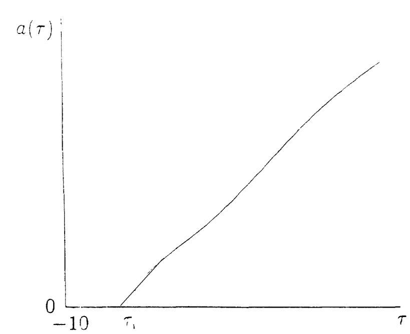

FIG. 5. $a(\tau)$ in Case 1B.2a; $\omega>0$, $h=10$, $\sigma=1$, $\epsilon=1$.

The case $B=0$, as already observed, is practically identical with this one and therefore will not be treated. In the solutions (3.22a) and (3.22b) we may set $c_1=0$, which again corresponds to an arbitrary origin of time and considerably simplifies the computations. We assume $F>0$, which only fixes the time direction. We also set

$$h=\frac{4c_2M^2F^2}{|V_0|}, \quad \tau=Ft, \quad \sigma=\mathrm{sgn}(V_0)\ , \tag{4.6}$$

so that

$$a(\tau)=\left[\frac{|V_0|}{4M^2F^2}\right]^{1/3}[\sigma\tau(\tau^3+h)]^{1/3}\ , \tag{4.7a}$$

$$\phi(\tau)=\sqrt{2/3}M\ln\left[\sigma\frac{\tau^3+h}{-\tau}\right]+\sqrt{2/3}M\ln\left[\frac{B^2|V_0|}{4M^2F^2}\right]\ . \tag{4.7b}$$

We see that this situation is rather different from Cases 1 and 2. Here the scale of $a(\tau)$ is determined by $|V_0|$, while F gives the time scale; the sign of V_0 is again crucial. The additive constant for $\phi(\tau)$ may be chosen to be zero by a suitable value for B.

Case 3A: $h=0$. We must set $\sigma>0$, $\tau>0$, so that

$$a(\tau)=\left[\frac{|V_0|}{4M^2F^2}\right]^{1/3}\tau^{4/3}\ , \tag{4.8a}$$

$$\phi(\tau)=2\sqrt{2/3}M\ln\tau+\sqrt{2/3}M\ln\left[\frac{B^2|V_0|}{4M^2F^2}\right]\ . \tag{4.8b}$$

As discussed above, this solution corresponds to the one found by Ratra and Peebles, and it is easy to reobtain the equation of state (2.13) from (4.8b).

Case 3B: $h\neq0$. Essentially we are in the same situation of Case 3A or 1B.2b.

V. CONCLUSIONS

The aim of the approach pursued in this paper consists in using the full machinery of classical Lagrangian dynamics in order to study the properties of the system of equations associated with a homogeneous isotropic universe filled with a scalar field ϕ and a generic potential $V(\phi)$. From this method it is possible to obtain a reduction, and possibly full integration, of the system, whenever a Nöther constant of the motion is found.

In this paper we have found that it is possible to obtain a Nöther constant by imposing, in addition to the spatial symmetry, a particular symmetry in the "configuration space" $(a,\phi,\dot a,\dot\phi)$. We showed that the existence of this symmetry is possible iff the potential $V(\phi)$ has the form (2.9).

Clearly, there is no immediate physical justification for this choice for $V(\phi)$, but we did not have, until now, phenomenological evidence of the law governing scalar fields. The study of universes with scalar fields has been stimulated when it has been realized that a scalar field might be responsible for inflation. But there is still no convincing way to realize this scenario.[14] Thus there is a lack of theoretical information about the form of $V(\phi)$.

The main consequence of all this is that we have selected the class of potentials (2.9) and indicated the most reasonable, specific ones directly from the physical interpretation of the explicit solutions. Contact with other approaches is assured by the fact that exponential potentials have been well studied in the literature.[5,7] We thus have classified all the possible models varying the free parameters of the theory, and we have plotted the time behavior of the scale function and the field for the more interesting ones.

It must be noted that our results include some already known models. In particular, setting the parameter $A=0$, we obtained the solution of Ratra and Peebles[7] in their case $q=\frac{3}{2}$. Solutions (4.4a), and (4.4b) give us the the typical situation of the de Sitter universe or of exponential inflation.

Of particular interest are solutions (4.3a) and (4.3b), which give us a smooth transition from a power-law inflationary universe (with expansion law $a^{4/3}$) to a matterlike universe (with expansion law $a^{2/3}$). We think that it is necessary to give a more detailed analysis of the viability of the model of a universe with this kind of potential, but this work will be done in the near future.

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