Density-gradient–vorticity relation in perfect-fluid Robertson-Walker perturbations

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In a previous paper, a second-order propagation equation was derived for covariant and gauge-invariant *vector fields* characterizing density inhomogeneities in an almost-Friedmann-Lemaître-Robertson-Walker (-FLRW) perfect-fluid universe. However, an error there led to omission of a term representing an effect of vorticity on *spatial density gradients* at linear level. Here we determine this interaction (leading to an extra term in the second-order propagation equation for the spatial density gradient), and examine its geometrical and physical meaning. We define a new local decomposition of the observed density gradient and we show that the scalar variable defined in the decomposition naturally describes density clumping, and satisfies the standard Bardeen second-order equation. The physical meaning of the other variables defined in the decomposition is discussed, and their propagation equations are presented. Finally, the vorticity-induced time growth of the density gradient is derived in the long-wavelength limit.

I. INTRODUCTION

In a recent article [Ellis, Hwang, and Bruni[1] (EHB)] a new covariant and gauge-invariant approach to cosmological density inhomogeneities [Ellis and Bruni[2] (EB)] was extended to the case of a perfect fluid with barotropic equation of state. A gauge-invariant and covariant *vector field* that codes the information needed to describe density fluctuations in an almost-Friedmann-Lemaître-Robertson-Walker (-FLRW) universe model was defined in these papers (*the comoving fractional density gradient* $\mathcal{D}_a$), and a second-order homogeneous equation determined that characterizes its dynamics in the linear case. However in the course of extending previous work on the Newtonian limit of general relativity[3] to the case of almost-FLRW models,[4,5] Lottermoser (Max Planck Institute of Astrophysics, Munich) found an error in the EHB analysis. That analysis is correct for the case of vanishing vorticity, but an extra term occurs in the equations when both density and vorticity fluctuations occur (which is of course the general case). In particular, the second-order equation has a source term which can be expressed in various ways, and vanishes if the vorticity is zero. Here we derive this term and discuss possible physical implications. It should be emphasized that our analysis is based on standard general relativity with ordinary matter.

As in EB and in EHB, the basic philosophy is not to start with a FLRW universe and then perturb it, but rather to take as our starting point an anisotropic, inhomogeneous (realistic) space-time that is kinematically approximately FLRW. The approximation takes place by neglecting higher-order terms in the exact equations when the kinematic and dynamic variables take values close to those they take in a FLRW universe. Thus we do not distinguish between real and background variables, but do all calculations in the real space-time, linearizing the resulting equations; the "background solution" is just the zero-order approximation to the full solution of the exact equations.[6,7]

The key point[8-11] is that if there is vorticity, there are no 3-surfaces orthogonal to the fluid flow in the real space-time (although such surfaces do exist in the background space-time); consequently, the surfaces of constant density in an expanding universe cannot be orthogonal to the flow lines and a spatial density gradient *must* be measured by the fundamental observers (observers comoving with the fluid) if the universe is expanding and rotating (even if the Universe is exactly spatially homogeneous).[8,12,13] Equivalently, there must be a peculiar velocity of matter relative to the surfaces of constant density, and an associated dipole in the observed matter distribution.[14] If one uses the usual variable $\delta\rho$ to characterize energy density perturbations,[15-17] this effect will be reflected in variations in $\delta\rho$ of a dipole character which can only be removed by gauge choices which lead to peculiar motions of the matter relative to the chosen time surfaces. Now essentially the same argument applies to the fluid pressure. Provided the fluid has nonvanishing speed of sound, there will necessarily be a spatial gradient in the pressure in the local rest spaces of the fluid in an expanding and rotating universe; this will affect the fluid flow through the momentum-conservation equation. If the fluid is a perfect fluid with barotropic equation of state, we get simple predictions of the implications.

After the presentation of the necessary preliminary material, we derive the interaction term in Sec. II, also discussing the role it plays in the second-order propagation equation for the spatial density gradients.

In Sec. III we define a new local decomposition of our basic variable, the density gradient $\mathcal{D}_a$, and we show that it contains more information than the scalar variable[17,18] usually employed in the analysis of density inhomogeneities. The latter scalar is however recovered in the decomposition as the part that naturally describes the density clumping, which satisfies the standard Bardeen second-order equation,[17,18] with vorticity playing no role in its evolution. Thus, there is no contradiction between the vorticity–density-gradient interaction presented here and the standard results of cosmological perturbation theory:[17,18] only scalar modes contribute to describing density clumping. The physical meaning of the other variables defined in the local decomposition is also discussed in this section, and their propagation equations are presented.

Finally, in Sec. IV, we explicitly derive the vorticity-induced time growth of the density gradient $\mathcal{D}_a$ in the long-wavelength limit, for a general barotropic perfect fluid in a flat universe.

II. THE INTERACTION TERMS

The variables we use, the motivation for their use, and the equations they obey are explained in detail in EB and in EHB; only sufficient repetition of this material will be given here to establish the notation used.

A. Preliminaries

As in Refs. 9–11, 15, and 16, the 4-velocity vector tangent to the flow lines (the world lines of a fundamental observer in the Universe) is u^a ($u^a u_a = -1$). The projection tensor into the tangent 3-spaces orthogonal to u^a (the rest space of a fundamental observer) is $h_{ab} \equiv g_{ab} + u_a u_b$. The time derivative of any tensor $T^{ab}{}_{cd}$ along the fluid flow lines is $\dot{T}^{ab}{}_{cd} \equiv T^{ab}{}_{cd;e}u^e$; in particular the acceleration vector is $a^a \equiv \dot{u}^a = u^a{}_{;b}u^b$ ($\Rightarrow a_a u^a = 0$). The first covariant derivative of the 4-velocity vector is

$$\nabla_b u_a = \omega_{ab} + \sigma_{ab} + \tfrac{1}{3}\Theta h_{ab} - a_a u_b \ , \tag{1}$$

where $\Theta \equiv u^a{}_{;a}$ is the expansion, $\omega_{ab} = \omega_{[ab]}$ is the vorticity tensor ($\omega_{ab}u^b = 0$), and $\sigma_{ab} = \sigma_{(ab)}$ is the shear tensor ($\sigma_{ab}u^b = 0$, $\sigma^a{}_a = 0$). A representative length scale S along the flow lines is defined by

$$\dot{S}/S = \tfrac{1}{3}\Theta \ . \tag{2}$$

The vorticity and shear magnitudes are defined by $\omega^2 = \frac{1}{2}\omega_{ab}\omega^{ab}$, $\sigma^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab}$.

Because we are here considering the case of a barotropic perfect fluid, the conserved matter stress tensor will take the form

$$T_{ab} = \mu u_a u_b + p h_{ab} \ , \tag{3}$$

where the pressure p and energy density μ are related by a barotropic equation of state

$$p = p(\mu) \ \Rightarrow \ p_{[,a}\mu_{,c]} = 0 \ . \tag{4}$$

We will use standard notation as follows:

$$w = p/\mu \ , \ \ c_s^2 = dp/d\mu \ \Rightarrow \ \dot{w} = -(1+w)(c_s^2 - w)\Theta \tag{5}$$

(the implication following from the energy-conservation equation).

1. The projected covariant derivative

A crucial role is played by ${}^{(3)}\nabla_a$, the covariant derivative operator orthogonal to u^a, obtained by totally projecting the 4-dimensional covariant derivative operator, see, e.g, Refs. 3 and 9–11, and the Appendix; when ω=0 this is the covariant derivative in the surfaces Σ orthogonal to the fundamental flow lines, but we are concerned precisely with the case when $\omega \neq 0$ and there are no such surfaces. In terms of this derivative, which we will call "spatial derivative" for simplicity, (1) can be written

$$\begin{aligned} \nabla_b u_a &= {}^{(3)}\nabla_b u_a - a_a u_b \ , \\ {}^{(3)}\nabla_b u_a &= \omega_{ab} + \sigma_{ab} + \tfrac{1}{3}\Theta h_{ab} \ . \end{aligned} \tag{6}$$

For present purposes, the importance of ${}^{(3)}\nabla_a$ arises because it determines the fluid acceleration through the momentum equation

$$(\mu + p)a_a + {}^{(3)}\nabla_a p = 0 \ . \tag{7}$$

2. The inhomogeneity variables

In a FLRW universe model the shear, vorticity, acceleration, and Weyl tensor vanish, and the energy density μ, pressure p, and expansion Θ are functions of the cosmic time t only.[9–11] The covariantly defined and gauge-invariant quantities we use to represent the spatial variation of μ and Θ in almost-FLRW universes (EB, EHB) are the *comoving fractional density gradient* orthogonal to the fluid flow,

$$\mathcal{D}_a \equiv S\left(\frac{{}^{(3)}\nabla_a \mu}{\mu}\right) \ , \tag{8}$$

and the *comoving expansion gradient* orthogonal to the fluid flow:

$$\mathcal{Z}_a \equiv S\,{}^{(3)}\nabla_a\Theta \ . \qquad (9)$$

These quantities, which are in principle observable (EB), characterize inhomogeneity in a covariant way, and vanish in the FLRW universe models. In Sec. III we define derived quantities from these variables that further elucidate their meaning.

A supplementary quantity simply related to $\mathcal{D}_a$ and $\mathcal{Z}_a$ can be defined from the 3-curvature scalar in the tangent space

$${}^{(3)}R = 2(-\tfrac{1}{3}\Theta^2 + \sigma^2 - \omega^2 + \kappa\mu + \Lambda) \qquad (10)$$

(see the Appendix), where κ is the gravitational constant and Λ the cosmological constant; when ω=0, ${}^{(3)}R$ is the Ricci scalar of the hypersurfaces orthogonal to the fluid flow lines. The covariant and gauge-invariant vector

$$C_a \equiv S^3\,{}^{(3)}\nabla_a({}^{(3)}R) = -\tfrac{4}{3}\Theta S^2\mathcal{Z}_a + 2\kappa\mu S^2\mathcal{D}_a \qquad (11)$$

is the spatial variation of this 3-curvature variable,[19] the equality being taken to linear order. This quantity plays in our formalism the same role that δk does in Lyth and Mukherjee[20] [compare their Eq. (23) and our (25) below].

3. The evolution equations

The exact nonlinear evolution equations for $\mathcal{D}_a$ and $\mathcal{Z}_a$ in a general space-time were obtained in EB by taking the spatial derivative of the energy-conservation equation and the Raychaudhuri equation. Here we use these equations (and the equations for C_a) when linearized about the FLRW models with nonvanishing pressure (see EHB). Using the subscript $\perp$ to denote projection orthogonal to u^a, i.e., writing $h_a{}^c(\mathcal{D}_c)^{\cdot\cdot} \equiv \ddot{\mathcal{D}}_{\perp a}$, $h_a{}^c(\mathcal{D}_c)^{\cdot} \equiv \dot{\mathcal{D}}_{\perp a}$, etc., they are

$$\dot{\mathcal{D}}_{\perp a} = w\Theta\mathcal{D}_a - (1+w)\mathcal{Z}_a \ , \qquad (12)$$

$$\dot{\mathcal{Z}}_{\perp a} = -\tfrac{2}{3}\Theta\mathcal{Z}_a - \tfrac{1}{2}\kappa\mu\mathcal{D}_a + S\left(\frac{3k}{S^2}a_a + A_a\right) \ , \qquad (13)$$

$$\dot{C}_{\perp a} = \frac{6k}{S^2}\,\Theta^{-1}(\tfrac{1}{2}C_a - \kappa\mu S^2\mathcal{D}_a) -\tfrac{3}{4}\Theta S^3\left(\frac{3k}{S^2}a_a + A_a\right) \ , \qquad (14)$$

where the covariant derivatives (implied by the overdot) may all be taken in the background (zero-order) model, A_a is defined by

$$A_a \equiv {}^{(3)}\nabla_a(a^c{}_{;c}) \ , \qquad (15)$$

and we have used the zero-order relation

$${}^{(3)}R = 6k/S^2, \quad \dot{k} = 0 \qquad (16)$$

(accurate to the required order of accuracy), where k corresponds to the curvature constant for the background FLRW universe model (see the Appendix). From the definition (15) of A_a and (4), (5), and (7) we see that to first order,

$$A_a = -\frac{{}^{(3)}\nabla_a\,{}^{(3)}\nabla^2 p}{(\mu+p)} = -c_s^2\frac{{}^{(3)}\nabla_a{}^{(3)}\nabla^b\mathcal{D}_b}{S(1+w)} \ , \qquad (17)$$

where we use the notation ${}^{(3)}\nabla^2 \equiv {}^{(3)}\nabla_b{}^{(3)}\nabla^b$, the second equality following from (5) and the assumption of adiabatic evolution used throughout this paper (which implies the perturbation is adiabatic).

The dynamics of our basic variable, $\mathcal{D}_a$, is given by (12) in combination with one of the two equations (13) and (14) [on using (14), one should trivially substitute for $\mathcal{Z}_a$ in (12) from (11)], or by the linear second-order equation which follows directly from (12) and (13) using (17). This equation is

$$\ddot{\mathcal{D}}_{\perp a} + \mathcal{A}(t)\dot{\mathcal{D}}_{\perp a} - \mathcal{B}(t)\mathcal{D}_a - c_s^2\,{}^{(3)}\nabla_a({}^{(3)}\nabla^b\mathcal{D}_b) = 0 \ , \qquad (18)$$

where the coefficients

$$\mathcal{A}(t) = (\tfrac{2}{3} - 2w + c_s^2)\Theta \ , \qquad (19)$$

$$\mathcal{B}(t) = (\tfrac{1}{2} + 4w - \tfrac{3}{2}w^2 - 3c_s^2)\kappa\mu + (c_s^2 - w)\frac{12k}{S^2} + (5w - 3c_s^2)\Lambda \ , \qquad (20)$$

are determined from the background model. This form of the equations allows for a variation of $w = p/\mu$ with time. However if w = const, then from (5), $c_s^2 = w$, and the coefficients simplify to

$$\mathcal{A}(t) = (\tfrac{2}{3} - w)\Theta \ , \qquad \mathcal{B}(t) = \tfrac{1}{2}(1-w)(1+3w)\kappa\mu + 2w\Lambda \ . \qquad (21)$$

B. The extra terms

The problem with (18) is that although it is a homogeneous equation, the last term is in an awkward form. We wish to commute the derivatives to bring the equation to a more standard form, with the spatial Laplacian acting on $\mathcal{D}_a$. When this was done in EHB, it was assumed that the derivatives ${}^{(3)}\nabla_a$ commute when acting on scalars. However this is true only when the vorticity is zero. When the vorticity is nonzero, there are no surfaces in space-time orthogonal to the fluid flow, and consequently these partial derivatives do not commute; rather

$${}^{(3)}\nabla_b{}^{(3)}\nabla_a p - {}^{(3)}\nabla_a{}^{(3)}\nabla_b p = 2\omega_{ab}\dot{p} = -2c_s^2(1+w)\mu\Theta\omega_{ab} \qquad (22)$$

[see the Appendix for a detailed discussion of the first equality and related results; the second equality follows from the energy-conservation equation and (4) and (5)]. It follows from (22) that

$$ {}^{(3)}\nabla_a({}^{(3)}\nabla^b \mathcal{D}_b) = \left(-\frac{2k}{S^2} + {}^{(3)}\nabla^2\right)\mathcal{D}_a + 2\Theta(1+w)S\,{}^{(3)}\nabla^b\omega_{ab} \ . \quad (23) $$

The final linear first-order equations we obtain from (13) and (14) are

$$ \dot{\mathcal{Z}}_{\perp a} = -\tfrac{2}{3}\Theta\mathcal{Z}_a - \tfrac{1}{2}\kappa\mu\mathcal{D}_a - \frac{c_s^2}{1+w}\left(\frac{k}{S^2} + {}^{(3)}\nabla^2\right)\mathcal{D}_a - 2c_s^2 S\Theta\,{}^{(3)}\nabla^b\omega_{ab} \ , \quad (24) $$

$$ \dot{C}_{\perp a} = \frac{6k}{S^2}\Theta^{-1}(\tfrac{1}{2}C_a - \kappa\mu S^2\mathcal{D}_a) + \tfrac{4}{3}\Theta S^2\frac{c_s^2}{1+w}\left(\frac{k}{S^2} + {}^{(3)}\nabla^2\right)\mathcal{D}_a + \tfrac{8}{3}c_s^2 S^3\Theta^2\,{}^{(3)}\nabla_b\omega_a{}^b \ . \quad (25) $$

The last term on the right-hand side (RHS) of (24) and (25) is the (first-order) term[21] giving the effect of vorticity on the expansion and curvature gradients, and so on the density gradient, measured by a fundamental observer. The final version of the linearized second-order equation (18) follows directly from (12) and (24) or from (18) and (23). We find

$$ \ddot{\mathcal{D}}_{\perp a} + \mathcal{A}(t)\dot{\mathcal{D}}_{\perp a} - \mathcal{B}(t)\mathcal{D}_a + \mathcal{L}(t)\mathcal{D}_a - \mathcal{C}(t)\,{}^{(3)}\nabla^b\omega_{ab} = 0 \ , \quad (26) $$

where the coefficients $\mathcal{A}(t)$, $\mathcal{B}(t)$ [given by (19) and (21)],

$$ \mathcal{C}(t) = c_s^2 2S(1+w)\Theta \ , \quad (27) $$

and the operator

$$ \mathcal{L}(t) = c_s^2\left(\frac{2k}{S^2} - {}^{(3)}\nabla^2\right) \quad (28) $$

are determined from the background model. The last term in (26) is the extra term due to (22) and (23), omitted in EHB. The key point here is that it is the operator $\mathcal{L}(t)$ that determines the effective wavelength of density inhomogeneities through its eigenvalues and eigenfunctions [unlike the last term on the right-hand side of (18)].

C. The interaction

The linking of vorticity to the time evolution of the density gradient is through the 3-divergence of the vorticity *tensor* (i.e., the 3-curl of the vorticity vector). The effect is nonzero provided ${}^{(3)}\nabla^c\omega_{ac} \neq 0$; if vorticity is nonzero but this divergence vanishes, there is necessarily a density gradient associated with the vorticity (see the Introduction), but the *growth* of this gradient is unaffected by the extra term. The same divergence is related to the 3-divergence of the shear and the 3-gradient of the expansion through the $(0,\nu)$ constraint equation[9,10] in its linearized form[22] (Ref. 15):

$$ {}^{(3)}\nabla^b\omega_{ab} - {}^{(3)}\nabla^b\sigma_{ba} + \tfrac{2}{3}\,{}^{(3)}\nabla_a\Theta = 0 \ . \quad (29) $$

This already shows that the vorticity and density gradients are linked in the linear approximation, because expansion and density gradients are intimately related [see (12) and (13) above]. Equation (29) restricts how initial data can be chosen, while the extra term in (26) shows how (consistent with this) there is a direct effect of vorticity, in the linear approximation, as a source of growth of density gradients. However the coefficient $\mathcal{C}(t)$ of this extra term vanishes if the speed of sound is zero, or the Universe is static.[23] The term does not occur in the case of Newtonian theory, because in that theory the hyperplanes orthogonal to the fluid flow are always tangent to hypersurfaces of absolute time, and are therefore integrable; so the equations in Ref. 5 correctly include the case of combined vorticity and density perturbations.

To linear order, the divergence of the extra term in (26) vanishes:

$$ {}^{(3)}\nabla^a({}^{(3)}\nabla^b\omega_{ab}) = 0 \quad (30) $$

(see the Appendix). This means that the contribution the extra term induces in the density gradient will have vanishing spatial divergence: ${}^{(3)}\nabla^a\mathcal{D}_a = 0$ [if this divergence and its first derivative vanish initially, the effect of the vorticity term is to leave it zero, see Eq. (48) below], so the induced growth of inhomogeneity in one direction will be compensated by a lessening in other directions. The geometric meaning of this result will be discussed below (see Sec. III B).

1. The growth of the vorticity source term

At first glance this extra term seems to imply that the equations do not close at the second order anymore, because the covariant derivative of the vorticity along the fluid flow lines involves the shear.[9,10] However this is not the case because vorticity propagation decouples in the linear approximation,[15] so we can (to this level of accuracy) determine the evolutionary behavior of the extra term.[24] In more detail, a perfect fluid with $p = w\mu$, $w = w(\mu)$ [see (4) and (5)] has an acceleration potential r (Refs. 9 and 11), where

$$ r = \exp\left(\int_{p_*}^{p}\frac{dp}{\mu(1+w)}\right) \ , \quad (31) $$

and the vorticity evolution equation (in the linear approximation) is

$$ \dot{\omega}_{ac} + \tfrac{2}{3}\Theta\omega_{ac} = {}^{(3)}\nabla_{[c}a_{a]} = -\frac{1}{\mu+p}{}^{(3)}\nabla_{[c}{}^{(3)}\nabla_{a]}p = -\frac{\dot{r}}{r}\omega_{ac} \ . \quad (32) $$

Thus

$$ (S^2 r\omega_{ac})^{\cdot} = 0 \ . \quad (33) $$

When $w = \text{const}$, $\mu = M_1/S^{3(1+w)}$, $M_1 \equiv \mu_0 S_0^{3(1+w)}$, $\dot{M}_1 = 0$, where S_0 is the present value of the scale factor, and the acceleration potential is

$$r = \left(\frac{p}{p_i}\right)^{\frac{w}{1+w}} \Rightarrow r = \left(\frac{M_1 w}{p_i}\right)^{\frac{w}{1+w}} S^{-3w} . \quad (34)$$

Hence

$$\omega_{ac} = \frac{\Omega_{ac}}{S^{2-3w}} , \quad \dot{\Omega}_{ac} = 0 , \quad \Omega_{ac} = \Omega_{[ac]} , \quad (35)$$

where all multiplying constants are now included in $\Omega_{ac} = rS^2\omega_{ac}(\frac{M_1 w}{p_i})^{-\frac{w}{1+w}}$. Finally Eqs. (93)–(95) in the Appendix show that $(S\,{}^{(3)}\nabla^a)$ is the orthogonal derivative operator which, acting on a purely spatial first-order tensor that is covariantly constant along the fluid flow lines, preserves time independence. In particular, to first order, the divergence term obeys

$$(S\,{}^{(3)}\nabla^b\omega_{ab})^{\cdot} = S\,{}^{(3)}\nabla^b(\dot{\omega}_{ab}) ,$$

thus

$$ {}^{(3)}\nabla^a\omega_{ac} = \frac{S^{(3)}\nabla^a\Omega_{ac}}{S^{3(1-w)}} , \quad (S\,{}^{(3)}\nabla^a\Omega_{ac})^{\cdot} = 0 , $$
$$ {}^{(3)}\nabla^c(S\,{}^{(3)}\nabla^a\Omega_{ac}) = 0 \quad (36)$$

[the last condition expressing the vanishing-divergence property (30)]. The nature of the interaction depends (a) on the equation of state, and (b) on the initial value ${}^{(3)}\nabla^a\Omega_{ac}$ of the spatial vorticity divergence; the interaction term always decays as the Universe expands, if $w < 1$.

As a simple example, in the case of a flat background with vanishing cosmological constant (k=0, Λ=0), the scale factor of the background model obeys

$$S = (\beta t)^{\frac{2}{3(1+w)}} , \quad \beta \equiv \tfrac{3}{2}(1+w)\sqrt{\frac{\kappa}{3}M_1} , \quad (37)$$

where t is proper time along the fluid flow lines, and so the vorticity goes as

$$\omega_{ab} = \Omega_{ab}(\beta t)^{-\frac{2(2-3w)}{3(1+w)}} . \quad (38)$$

The divergence goes as

$$ {}^{(3)}\nabla^c\omega_{ac} = \Omega_a\,(\beta t)^{-2\frac{1-w}{1+w}} , \quad (39)$$

where we defined

$$\Omega_a \equiv S\,{}^{(3)}\nabla^c\Omega_{ac} , \quad \dot{\Omega}_a = 0 , \quad {}^{(3)}\nabla^a\Omega_a = 0 . \quad (40)$$

III. INVARIANT DECOMPOSITION

The standard harmonic representation,[17,18] which combines the Arnowitt-Deser-Misner (ADM) tensor decomposition with a (logically independent) harmonic analysis, should be regarded with some reservations because it is nonlocal, whereas the physics we are concerned with is essentially local. We present an alternative local decomposition and discuss its geometrical meaning. While it is not necessary to introduce the usual harmonic decomposition to derive our equations, it is instructive to consider how they relate to harmonic representations and the ADM decomposition.

A. The ADM decomposition

This nonlocal splitting can be applied to vectors and second-rank tensors in a standard manner.[25–27] In the case of a vector field V_a, independent of any Fourier analysis, it represents V_a in terms of "scalar" and "vector" parts $\tilde{\nabla}_a\phi$, B_a relative to a chosen family of 3-surfaces:

$$V_a = \tilde{\nabla}_a\phi + B_a , \quad \tilde{\nabla}^a B_a = 0 , \quad (41)$$

where $\tilde{\nabla}_a$ is the covariant derivative in these 3-spaces. If appropriate boundary conditions are satisfied (which could be problematic if the background model has k=0), and $k \neq -1$, then B_a is unique and ϕ unique up to a constant.[26] As the first term has vanishing curl but nonvanishing 3-divergence, whereas the second has vanishing 3-divergence (it is "solenoidal"), if we take the 3-divergence of V_a we obtain an equation involving only the first term, while if we take its curl we obtain an equation involving only the second. However it must be emphasized that this splitting is nonlocal: there is no local equation determining $\tilde{\nabla}_a\phi$ and B_a uniquely from V_a.

Suppose we apply such a splitting to our first-order or second-order vector equations for $\mathcal{D}_a$ above. A vital point in the case of nonzero vorticity is that the projected derivative ${}^{(3)}\nabla_a$ in the real space-time is not the 3-derivative $\tilde{\nabla}_a$ in *any* family of 3-surfaces (although in the zero-order approximation, ${}^{(3)}\nabla_a$ is the same as $\tilde{\nabla}_a$ in the natural time surfaces in the background). Thus the importance of the distinction made above: the covariant derivatives must be correctly worked out in the real space, to the linear order, before applying such a splitting; and as no surface can be chosen orthogonal to the fluid flow, on applying the splitting (41) to (12)–(14), (24)–(26) the derivative operator $\tilde{\nabla}_a$ cannot be the operator ${}^{(3)}\nabla_a$ that affects the fluid dynamics through (7). Furthermore because this splitting is nonlocal it does not relate directly to local properties of (26). The implications of applying such a splitting to our equations are thus a bit obscure. We therefore turn to an alternative (local) decomposition whose meaning is more immediate.

B. A local decomposition

The spatial variation of the density (orthogonal to the fluid flow) is characterized by $\mathcal{D}_a$. A unique local splitting can be attained by considering the spatial derivative of this vector (multiplied by the scale factor S for convenience), and splitting this derivative into parts in analogy with (6):

$$S\,{}^{(3)}\nabla_b\mathcal{D}_a \equiv \Delta_{ab} = W_{ab} + \Sigma_{ab} + \tfrac{1}{3}\Delta h_{ab} , \quad (42)$$

where

$$W_{ab} \equiv \Delta_{[ab]} \ , \quad \Sigma_{ab} \equiv \Delta_{(ab)} - \tfrac{1}{3}\Delta h_{ab} \ , \qquad \Sigma_{ab} = \Sigma_{(ab)} \ , \quad \Sigma^a{}_a = 0 \ . \tag{43}$$

The skew-symmetric part is

$$W_{ab} = \frac{S^2}{\mu}\,{}^{(3)}\nabla_{[b}{}^{(3)}\nabla_{a]}\mu = \frac{S^2}{\mu}\omega_{ab}\dot{\mu} = -S^2(1+w)\Theta\omega_{ab} \ , \tag{44}$$

where we neglect a second-order term from ${}^{(3)}\nabla_a S$. This skew part by itself represents spatial variation of $\mathcal{D}_a$ in which its magnitude is preserved (i.e., rotations of this vector), e.g., that associated with the "tilt" of the fluid flow vector relative to the surfaces of constant density in homogeneous universes (i.e., the velocity of the matter relative to these surfaces). Thus although the associated density gradients exist and are observable[8,13,14] they are essentialy *dipolelike* in character and are not directly associated with formation of local inhomogeneities.

By contrast, the spatial divergence

$$\Delta \equiv \Delta^a{}_a = S\,{}^{(3)}\nabla^a\mathcal{D}_a = \frac{S^2}{\mu}\,{}^{(3)}\nabla^2\mu \tag{45}$$

by itself is related to spherically symmetric spatial variation of μ where density is accumulated, i.e., to *spatial aggregation of matter* that we might expect to reflect existence of high-density structures in the Universe. Finally, the trace-free symmetric part

$$\Sigma_{ab} = S\,{}^{(3)}\nabla_{(b}\mathcal{D}_{a)} - \tfrac{1}{3}\Delta h_{ab} \tag{46}$$

by itself is associated with spatial variations of $\mathcal{D}_a$ which do not represent spatial clumping of matter (as the associated divergence of $\mathcal{D}_a$ is zero) but rather represent change in the spatial anisotropy pattern of this gradient field. This seems to be what one might associate with the existence of *pancakelike* or *cigarlike* structures.

A general pattern of inhomogenity will have all the components (44)–(46) nonzero, for example, implying aggregation ($\Delta > 0$) in a pancakelike structure ($\Sigma_{ab} \neq 0$) and with turbulence present ($W_{ab} \neq 0$).

1. Evolution equations

Now the evolution equations for these quantities follow from (26) and their definitions.

(i) Antisymmetric part

Taking $S\,{}^{(3)}\nabla_b$ of (26) and antisymmetrizing over indices $[b, a]$, gives

$$\dot{W}_{\perp ab} - \left(w\Theta + \frac{\dot{\Theta}}{\Theta}\right) W_{ab} = 0 \ . \tag{47}$$

Since $W_{ab} \propto \omega_{ab}$, this is equivalent to the vorticity conservation equation (32) [it is clear this must be so from (44)]. Thus, we check the consistency of our equations, and so confirm the form (26) of the effect of the vorticity on the density anisotropy. This equation is the law controlling the dipole part of the observed density gradient, e.g., through governing the way the tilt angle of the surfaces of constant density in a Bianchi universe changes with time [see Eqs. (1.17), (1.31), and (1.32) in Ref. 13].

(ii) Symmetric part

(a) Trace: Take the divergence of (26), keeping only the linear terms that arise. While the divergence of the vorticity term is nonzero, it is second order [see the Appendix and (30)], so to linear order we obtain

$$\ddot{\Delta} + \mathcal{A}(t)\dot{\Delta} - \mathcal{B}(t)\Delta - c_s^2\,{}^{(3)}\nabla^2\Delta = 0 \tag{48}$$

for the scalar $\Delta \equiv S\,{}^{(3)}\nabla^b\mathcal{D}_b$. This is like (26) except that (a) it is a scalar equation (for Δ), (b) the linear operator $\mathcal{L}(t)$ is replaced by a simpler Laplacian term, and (c) the vorticity term does not appear. Thus we attain a "scalar mode" equation (see Woszczyna and Kulak[28] for a similar result) independent of the vorticity source term. That part of the density evolution relating to spherical aggregation of matter (and so to growth of local density inhomogeneities) is expressed in this equation (equivalent to the Bardeen[17] scalar harmonic equation, see Ref. 29): therefore, there is no contradiction between the presence of the vorticity source term in Eq. (26) and the standard results of cosmological perturbation theory in which only scalar modes contribute to describing density clumping.

(b) Trace-free symmetric part: We now take the symmetric, trace-free part of the spatial gradient of Eq. (26), finding

$$\ddot{\Sigma}_{\perp ab} + \mathcal{A}(t)\dot{\Sigma}_{\perp ab} - \mathcal{B}(t)\Sigma_{ab} + \widetilde{\mathcal{L}}(t)\Sigma_{ab} + \mathcal{C}(t)\,{}^{(3)}\nabla_{(b}{}^{(3)}\nabla^c\omega_{a)c} = 0 \ , \tag{49}$$

where

$$\widetilde{\mathcal{L}}(t) = c_s^2\left\{\frac{6k}{S^2} - \nabla^2\right\} \ . \tag{50}$$

This equation [similar to (26)] governs the growth of pancakelike or cigarlike density inhomogeneities, because it will alter Σ_{ab} in time. The effect of vorticity in this equation will be nonzero provided the initial conditions satisfy ${}^{(3)}\nabla_{(b}{}^{(3)}\nabla^c\Omega_{a)c} \neq 0$; and there seems to be no reason why this term should vanish, in general.

Alternatively, operating by $S\,{}^{(3)}\nabla_b$ on Eq. (18), symmetrizing on the indices $[b, a]$, and taking the trace-free part, we have

$$\ddot{\Sigma}_{\perp ab} + \mathcal{A}(t)\dot{\Sigma}_{\perp ab} - \mathcal{B}(t)\Sigma_{ab} - c_s^2\left[{}^{(3)}\nabla_{(b}{}^{(3)}\nabla_{a)} - \tfrac{1}{3}h_{ab}{}^{(3)}\nabla^2\right]\Delta = 0 \tag{51}$$

giving a form of the equations for Σ_{ab} that does not explicitly contain the vorticity. Instead there is an inhomogeneous source term involving gradients of Δ. This means that if we know the evolution of Δ (and its spatial variation) from (48), we do not need to explicitly introduce the vorticity term; we have enough information to find the evolution of Σ_{ab}. However it is simpler to use the form (49) that explicitly refers to the vorticity (because of the vorticity conservation equations discussed above). The kinematic and physical effects described by these two forms of the equation are of course the same.

We do not necessarily need to specifically consider Eqs. (47)–(49), for all the information we need is in the original equation (26). However if we do wish to further reduce our equations, by contrast with applying a nonlocal decomposition (41) to them, the above procedure is (a) locally well defined and (b) independent of large-scale boundary conditions which may or may not be satisfied in the real Universe.

2. Harmonic analysis

If we apply a harmonic analysis, we can do so either to the full equation (26) or to the set of derived equations (47)–(49). We can do so without simultaneously applying the ADM splitting discussed above. The basic point is to expand every quantity in terms of eigenfunctions of the Helmholtz equations

$$^{(3)}\nabla^2 Q^{(0)} + \frac{n^2}{S^2} Q^{(0)} = 0 \ ,$$

$$^{(3)}\nabla^2 Q^{(1)}_b + \frac{n^2}{S^2} Q^{(1)}_b = 0 \ ,$$

$$^{(3)}\nabla^2 Q^{(2)}_{bc} + \frac{n^2}{S^2} Q^{(2)}_{bc} = 0 \ ,$$

obtaining effective wavelengths from the eigenvalues. The eigenfunctions however do not relate in a simple manner to the length scales associated with the extra term in the second-order equations (irrespective of which form we use to express this term). We do not pursue this matter further here, except to remark that the harmonic analysis of the scalar equation (48) is unchanged from that given in EHB and Ref. 29 (as the vorticity makes no contribution to this part), and gives the standard scalar mode equation.[17,18]

A final remark: whatever splitting or harmonic analysis is applied to the propagation equation (26) should also be applied to the constraint equation (29), where the same vorticity term occurs (indeed the ADM splitting was precisely developed to analyze the constraint equations,[25] see Ref. 26 for the cosmological case). We will again find a linking of vorticity to density perturbations, but this time in terms of initial data.

IV. IMPLICATIONS

No difference arises from the extra term in the case of pressure-free matter (EB), rotation-free matter (EHB), or the Newtonian limit;[5] our previous discussions stand in those cases. The new term takes effect when $(\mu+p)c_s^2 \neq 0$, $\Theta \neq 0$, and the fluid is rotating with $^{(3)}\nabla^a\omega_{ab} \neq 0$. This is the generic case for a fluid with nonvanishing pressure; that is, the new term will almost always have a physical effect. However, the homogeneous (source-free) solutions are unaltered, so the speed of sound is unaltered. The Jeans length criterion in EHB is unaltered, but now a new issue arises: the vorticity term can conceivably dominate the equations. Presumably this will only occur under conditions of extreme turbulence.

A. Long-wavelength solutions

Suppose we can ignore the "Laplacian part" of the second-order equation, that is, $\mathcal{L}(t)\mathcal{D}_a$ can be ignored relative to the other terms in (26) [and consequently the Laplacian term in (48) can also be ignored]. We shall call this the *long-wavelength limit.* This does not necessarily mean we can ignore the term A_a in our equations, for (23) and (28) show that now we can have

$$^{(3)}\nabla_a\Delta \simeq 2\Theta S^2(1+w)\,^{(3)}\nabla^b\omega_{ab} \ , \qquad (52)$$

a spatial gradient in Δ occurring in conjunction with the vorticity source term. Thus in general we cannot assume we can ignore the latter in the long-wavelength limit, but only $^{(3)}\nabla^2\Delta$. In this limit, (48) becomes an ordinary homogeneous differential equation; with the solution of the latter, we can then consistently integrate (18) and (51), or use the vorticity law (36) to integrate (26) and (49), neglecting (50) [and so effectively using (52)]. However in this section we prefer to solve for $\mathcal{D}_a$ through the system of first order equations introduced in Sec. II A 3.

1. A conserved quantity on large scale

While the curvature variable C_a introduced previously [see (11)] is a geometrically natural quantity which is useful in discussing the long-wavelength limit, it turns out[29] that a closely related quantity $\tilde{C}_a$ is physically significant because it is conserved in a more general set of circumstances; in particular it is suited to examining the long-wavelength limit for general k and Λ. This quantity[30] is defined by

$$\tilde{C}_a \equiv -\tfrac{4}{3}S^2\Theta\mathcal{Z}_a + 2\kappa\mu S^2\mathcal{D}_a\left(1-\frac{2k}{S^2\kappa\mu(1+w)}\right) \qquad (53)$$

and reduces to C_a when k=0. The dynamics of our basic variable $\mathcal{D}_a$ can be determined through the system of two first-order linear equations for $\mathcal{D}_a$ and $\tilde{C}_a$ that follows from (12), (24), and (53):

$$\dot{\mathcal{D}}_{\perp a} = \left[w\Theta - \left(\tfrac{3}{2}\kappa\mu(1+w) - \frac{3k}{S^2}\right)\Theta^{-1}\right]\mathcal{D}_a + \frac{3}{4}\frac{(1+w)}{S^2\Theta}\tilde{C}_a \ , \qquad (54)$$

$$\dot{\tilde{C}}_{\perp a} = \frac{4}{3}\frac{c_s^2 S^2\Theta}{(1+w)}\left({}^{(3)}\nabla^2 - \frac{2k}{S^2}\right)\mathcal{D}_a + \frac{8}{3}c_s^2 S^3\Theta^2\, {}^{(3)}\nabla^b\omega_{ab}\ . \tag{55}$$

In the assumed large-scale limit, the first term in (55) vanishes; thus if there is no vorticity term, $\tilde{C}_a$ is a conserved quantity on large length scales, for any value of k or Λ (and so C_a is conserved if k=0).

A scalar type variable $S\,{}^{(3)}\nabla^a\tilde{C}_a$ is a conserved quantity on the large scale even considering the vorticity term (in this case, an integral solution in the large-scale case can be found in Ref. 29), so the aggregation of matter to form spherically symmetric high-density concentrations (protostructures) is unaffected. Notice that this is valid for general k and Λ, thus it generalizes the conserved quantity in Refs. 20 and 31. However, the vector variable $\tilde{C}_a$ can have a contribution from the vorticity even in the large-scale case, as, for example, when there are homogeneous (Bianchi) perturbations. The effect of the vorticity is analyzed below for the case k=0=Λ.

2. Density gradient growth induced by vorticity

In the case of flat background (k = 0), the above-defined variable coincides with the previously introduced curvature gradient: i.e., $\tilde{C}_a = C_a$. In this case Eqs. (54) and (55) coincide with (12) and (25) and we can proceed to integrate them, neglecting the Laplacian term. Remembering that $\frac{1}{3}\Theta^2 = \kappa\mu$ is the zero-order equation when k=0=Λ, the equations for $\mathcal{D}_a$ and C_a become

$$\dot{\mathcal{D}}_{\perp a} + (1-w)\frac{\Theta}{2}\mathcal{D}_a = \tfrac{3}{4}\Theta^{-1}(1+w)\frac{C_a}{S^2}\ , \tag{56}$$

$$\dot{C}_{\perp a} = \tfrac{8}{3}\, c_s^2 S^3\,\Theta^2\, {}^{(3)}\nabla^b\omega_{ab}\ . \tag{57}$$

It is clear from the RHS of Eq. (57) that C_a is no longer a constant of motion, but can be determined from the source term ${}^{(3)}\nabla^b\omega_{ab}$. Then it acts as a source for $\mathcal{D}_a$. Now we can use

$$\Theta = \frac{2}{(1+w)t} \tag{58}$$

[following from (37)] to rewrite (56) as

$$\dot{\mathcal{D}}_{\perp a} + \frac{1-w}{1+w}\frac{\mathcal{D}_a}{t} = \frac{3}{8}\frac{(1+w)^2}{\beta^{\frac{4}{3(1+w)}}}t^{\frac{3w-1}{3(1+w)}}C_a\ , \tag{59}$$

while from the vorticity equation (34), assuming $c_s^2 = w$, Eq. (57) becomes

$$\dot{C}_{\perp a} = \frac{32}{3}\frac{w}{(1+w)^2}\beta^{\frac{2w}{1+w}}\Omega_a t^{-\frac{2}{1+w}}\ , \tag{60}$$

where Ω_a was defined in (40). From the above equation we have

$$w \neq 1 \ \Rightarrow\ C_a = C_a^{(\infty)}\left[1 - \left(\frac{t}{t_i}\right)^{-\frac{1-w}{1+w}}\right] + C_a^{(i)}\ , \tag{61}$$

where we have included an explicit initial time t_i, and $C_a^{(i)} = C_a(t_i)$ is the initial value of C_a (the constant of motion when $\omega = 0$) and

$$C_a^{(\infty)} \equiv \frac{32}{3}\frac{w}{1-w^2}\beta^{\frac{2w}{1+w}}\Omega_a t_i^{-\frac{1-w}{1+w}} \tag{62}$$

is the asymptotic value of C_a. Thus Eq. (61) shows how the decaying vorticity term on the RHS of (60) induces an asymptotically growing mode in C_a (for $w \neq 1$). Note that in the dust case (w=0), $C_a^{(\infty)} = 0$ by the above definition. For w=1 we obtain

$$w = 1 \ \Rightarrow\ C_a = 8\Omega_a\sqrt{\frac{\kappa}{3}M_1}\,\ln\left(\frac{t}{t_i}\right) + C_a^{(i)}\ . \tag{63}$$

Using (61) and (63) we can look at the time behavior of "curvature perturbations": then we see that for $w < 1$ the extra mode induced by the vorticity term on the RHS of (60) grows up to an asymptotic value, while for $w > 1$ (not allowed physically) there is a growing mode. Finally, C_a grows logarithmically if w=1.

With (61) we can now integrate (59) when $w \neq 1$. The general solution for $\mathcal{D}_a$ is

$$\begin{aligned}\mathcal{D}_a = {}& \mathcal{D}_a^{(i)}\left(\frac{t}{t_i}\right)^{-\frac{1-w}{1+w}} + \frac{9(1+w)^3}{8(5+3w)}\frac{C_a^{(i)}t_i^2}{(\beta t_i)^{\frac{4}{3(1+w)}}}\left(\frac{t}{t_i}\right)^{\frac{2(1+3w)}{3(1+w)}}\left[1 - \left(\frac{t}{t_i}\right)^{-\frac{5+3w}{3(1+w)}}\right]\\ &+(1+w)^3\frac{9}{8}\frac{C_a^{(\infty)}}{(\beta t_i)^{\frac{4}{3(1+w)}}}\left\{\frac{t_i^2}{5+3w}\left(\frac{t}{t_i}\right)^{\frac{2(1+3w)}{3(1+w)}}\left[1 - \left(\frac{t}{t_i}\right)^{-\frac{5+3w}{3(1+w)}}\right]\right.\\ &\left. - \frac{t_i^2}{2(1+3w)}\left(\frac{t}{t_i}\right)^{\frac{9w-1}{3(1+w)}}\left[1 - \left(\frac{t}{t_i}\right)^{-\frac{2(1+3w)}{3(1+w)}}\right]\right\}\ .\end{aligned} \tag{64}$$

A similar expression can be found for the variable

$$\Phi_a \equiv \kappa\mu S^2\mathcal{D}_a \tag{65}$$

introduced in EHB in analogy with Bardeen's variable Φ_H (Ref. 17); it is

$$\Phi_a = \Phi_a^{(i)}\left(\frac{t}{t_i}\right)^{-\frac{5+3w}{3(1+w)}} + C_a^{(i)}\frac{3(1+w)}{2(5+3w)}\left[1-\left(\frac{t}{t_i}\right)^{-\frac{5+3w}{3(1+w)}}\right]$$
$$+\frac{C_a^{(\infty)}}{2}\left\{\frac{3(1+w)}{(5+3w)}\left[1-\left(\frac{t}{t_i}\right)^{-\frac{5+3w}{3(1+w)}}\right]-\frac{3(1+w)}{2(1+3w)}\left(\frac{t}{t_i}\right)^{-\frac{1-w}{1+w}}\left[1-\left(\frac{t}{t_i}\right)^{-\frac{2(1+3w)}{3(1+w)}}\right]\right\}. \qquad (66)$$

In both of the above expressions, the first term comes from the homogeneous equation, the second arises from the constant mode in C_a (61), and the last is due to the vorticity source term in (60). The solution for Δ is immediately obtained by applying $S^{(3)}\nabla^a$ to (64); then the last term disappears, for ${}^{(3)}\nabla^a C_a^{(\infty)}=0$ by definition (62) and (40), and the second term is the term that comes from the scalar conserved quantity $C = S^{(3)}\nabla^a C_a^{(i)}$ which exists in the long-wavelength limit even if $\omega \neq 0$ (see previous section and Refs. 20, 29, and 31). Also we point out that in the very particular case in which $C_a^{(i)}$ and $C_a^{(\infty)}$ [thus Ω_a defined in (40)] have values such that $C_a^{(i)} = -C_a^{(\infty)}$, two of the growing modes in the above equations cancel. For the special case w=1, $\beta = \sqrt{3\kappa M_1}$ we obtain, from (59) and (63),

$$\mathcal{D}_a = \mathcal{D}_a^{(i)} + \tfrac{9}{8}C_a^{(i)}t_i^2\left(\frac{1}{\beta t_i}\right)^{\frac{2}{3}}\left(\frac{t}{t_i}\right)^{\frac{4}{3}}\left[1-\left(\frac{t}{t_i}\right)^{-\frac{4}{3}}\right]$$
$$+3\Omega_a\beta^{\frac{1}{3}}t_i^{\frac{4}{3}}\left(\frac{t}{t_i}\right)^{\frac{4}{3}}\left\{\ln\left(\frac{t}{t_i}\right)-\frac{3}{4}\left[1-\left(\frac{t}{t_i}\right)^{-\frac{4}{3}}\right]\right\}, \qquad (67)$$

showing that in this case the vorticity induced mode dominates. For the variable Φ_a in this case we have

$$\Phi_a = \Phi_a^{(i)}\left(\frac{t}{t_i}\right)^{-\frac{4}{3}} + \frac{3}{8}C_a^{(i)}\left[1-\left(\frac{t}{t_i}\right)^{-\frac{4}{3}}\right]$$
$$+\Omega_a\beta\left\{\ln\left(\frac{t}{t_i}\right)-\frac{3}{4}\left[1-\left(\frac{t}{t_i}\right)^{-\frac{4}{3}}\right]\right\}. \qquad (68)$$

3. Radiation

The case of pure radiation is of particular relevance to the early Universe. In this case, $\gamma = 4/3$, $w = 1/3 = c_s^2$, $\beta = 2\sqrt{\frac{\kappa}{3}M_1}$; then we find, from (64),

$$\mathcal{D}_a = \mathcal{D}_a^{(i)}\left(\frac{t}{t_i}\right)^{-\frac{1}{2}}$$
$$+\frac{4}{9}\frac{t_i}{\beta}(C_a^{(i)}+C_a^{(\infty)})\left(\frac{t}{t_i}\right)\left[1-\left(\frac{t}{t_i}\right)^{-\frac{3}{2}}\right]$$
$$-\frac{2}{3}\frac{t_i}{\beta}C_a^{(\infty)}\left(\frac{t}{t_i}\right)^{\frac{1}{2}}\left[1-\left(\frac{t}{t_i}\right)^{-1}\right], \qquad (69)$$

where we explicitly see that the growing mode induced by $C_a^{(i)}$ and one of those induced by $C_a^{(\infty)}$ (the faster growing mode) can eventually reciprocally cancel if $C_a^{(i)}$ $= -C_a^{(\infty)}$. An analogous expression for Φ_a can be found from (66).

V. CONCLUSION

The linearized dust equations of EB (c_s^2=0) and Newtonian equations of (Ref. 5) for density inhomogeneities are correct even if arbitrary (first-order) vorticity is present. The perfect fluid equations of EHB ($c_s^2 \neq 0$) are correct when the spatial divergence of the vorticity is zero (so in particular if the vorticity is zero) or the fluid is not expanding, but otherwise the extra term discussed here should appear in the density inhomogeneity evolution equations. It shows that in the linear approximation, vorticity can act as a source for the growth of density gradients with vanishing divergence.

Is this a real effect, or a mathematical creation? We have been forced to conclude it is real, arising from the fact that the acceleration of the fluid is generated by the pressure gradient orthogonal to the fluid flow lines [see (7)]; but when vorticity is nonzero, there are no integral three-spaces orthogonal to the fluid flow, so these effective pressure gradients are not integrable (indeed the gradient operators do not commute when acting on any scalars). This is the source of the extra term. It will be missed in any analysis where the commutation of these operators is calculated in the background space-time rather than the real space-time; however the local physics is determined by the latter, not the former. For example, we can consider a situation in the early Universe where a phase-transition takes place, and resulting viscosity enables local vorticity generation (the fluid is not a barotropic fluid at this time); generically this will then result in creation of density inhomogeneities (for only in rather special cases will the spatial divergence of the vorticity tensor vanish).

Given that the effect will occur, its physical importance is still open to debate. One might think that it is just a kinematic effect, because the growth of the vorticity term is governed by the vorticity conservation equations, which are essentially kinematic identities. However on the one hand, kinematic identities certainly must be obeyed; so this would not diminish the significance of the effect. On the other hand there is more than kinematics in these equations, for the evolution depends crucially on the equation of state of matter (in particular, our analysis has considered only the case of a barotropic perfect fluid; the results will be different for an imperfect fluid or for a nonbarotropic equation of state).

At a first glance, it could seem that the effect of vorticity on density gradients presented here is in contradiction with the standard results of cosmological perturbation theory in which a harmonic decomposition is carried out from the beginning:[17,18] namely, that only scalar modes contribute to density perturbation. However, introducing a new local decomposition (Sec. III), we have clarified that our basic variable, the density gradient $\mathcal{D}_a$, contains more information on the density distribution than the usual scalar variable, but we have shown that the latter is recovered as the spatial comoving divergence Δ of $\mathcal{D}_a$, and that this variable does indeed satisfy the standard Bardeen second-order equation.[17,18] Thus, there is no contradiction, and the effect presented here is related not to local aggregation of matter (clumping) but either to dipoles in the density distribution (e.g., velocity effects), or to growth of pancakelike or cigarlike structures.

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APPENDIX A: PROPERTIES OF THE SPATIAL DERIVATIVE

Given the smooth 4-velocity field u^a ($u^a u_a = -1$) at each point p of the space-time we have a subspace H_p of the tangent space T_p at p which is orthogonal to u^a, and $h_{ab} \equiv g_{ab} + u_a u_b$ is a metric in H_p. The collection of these subspaces H_p can be called a *distribution* D (see Crampin and Pirani,[32] p. 141) or a *smooth specification* (see Wald,[33] Appendix B.3). When the vorticity is nonzero we have for two vectors $X^a, Y^a \in D$,

$$[X,Y]^a - h_b{}^a [X,Y]^b = -2u^a \omega_{bc} X^b Y^c \ , \tag{70}$$

where the *defect tensor*[3]

$$D^a{}_{bc} = u^a \omega_{bc} \tag{71}$$

expresses the fact that the vector $[X,Y]^a$ does not live in D. In this case Frobenius's theorem[33] tells us that D does not possess integrable submanifolds, i.e., surfaces orthogonal to u^a.

1. The spatial derivative

By definition, acting on scalars, vectors orthogonal to u^a, and tensors orthogonal to u^a, the *orthogonal covariant derivative* ${}^{(3)}\nabla_a$ is given by[34]

$$^{(3)}\nabla_a f = h_a{}^b \nabla_b f = h_a{}^b f_{,b} \ , \tag{72}$$

$$^{(3)}\nabla_a X_b = h_a{}^c h_b{}^d \nabla_c X_d = h_a{}^c h_b{}^d X_{d;c} \ , \tag{73}$$

$$^{(3)}\nabla_a T_{bc} = h_a{}^d h_b{}^e h_c{}^f \nabla_d T_{ef} = h_a{}^d h_b{}^e h_c{}^f T_{ef;d}. \tag{74}$$

This compact notation is a convenient way of avoiding a plethora of indices:

$$^{(3)}\nabla_a S^{fg\cdots lm}{}_{bc\cdots de} = h_a{}^t h^f{}_n h^g{}_o \cdots h^l{}_u h^m{}_v h_b{}^s h_c{}^r \cdots h_d{}^p h_e{}^q \nabla_t S^{no\cdots uv}{}_{sr\cdots pq} \ . \tag{75}$$

It follows from the above definition that ${}^{(3)}\nabla_a$ preserves the orthogonal metric h_{bc}: that is, ${}^{(3)}\nabla_a h_{bc} = 0$. Consequently, we can raise and lower indices through equations acted on by ${}^{(3)}\nabla_a$ by use of h_{ab}, h^{ab}. However we cannot simply treat this operator as the standard covariant derivative of a 3-space, because the defect tensor will be nonzero when $\omega \neq 0$. Thus we cannot assume the usual commutation relations; rather we must use the expressions given in the following sections.

2. Commutators

From these definitions we can calculate the commutator of the 3-derivatives when acting on scalars, vectors, and tensors. The key point in the first case is to note that, for any function f,

$$\begin{aligned} {}^{(3)}\nabla_a({}^{(3)}\nabla_b f) &= h_a{}^c h_b{}^d \nabla_c({}^{(3)}\nabla_d f) \\ &= h_a{}^c h_b{}^d \nabla_c (h_d{}^e \nabla_e f) \end{aligned}$$

and then use the Leibniz rule for ∇_a on the last bracket, together with (6) in the form

$$^{(3)}\nabla_b u_a = \omega_{ab} + \Theta_{ab} = k_{ab} \ , \tag{76}$$

where $\Theta_{ab} = \sigma_{ab} + \frac{1}{3}\Theta h_{ab} = \Theta_{ba}$ is the expansion tensor ($\Theta_{ab} u^b = 0$). We obtain

$$^{(3)}\nabla_{[a}{}^{(3)}\nabla_{b]} f = -D^c{}_{ab}{}^{(3)}\nabla_c f = -\omega_{ab} \dot{f} \ . \tag{77}$$

Similarly, totally projecting the derivatives in the vector commutator, we find that, for all vector fields X_a orthogonal to u^a ($X_a u^a = 0$),

$$^{(3)}\nabla_{[a}{}^{(3)}\nabla_{b]} X_c + \omega_{ab} \dot{X}_{\perp c} \ = \ \tfrac{1}{2}{}^{(3)}R_{dcba} X^d \ , \tag{78}$$

where, using the above defined k_{ab}, we define[35–39]

$$\begin{aligned} {}^{(3)}R_{abcd} &= (R_{abcd})_\perp + k_{ad}k_{bc} - k_{ac}k_{bd} \\ &\Rightarrow {}^{(3)}R^{ab}{}_{cd} = (R^{ab}{}_{cd})_\perp - 2k^{[a}{}_{[c}k^{b]}{}_{d]} \end{aligned} \tag{79}$$

(which is not defined by ${}^{(3)}\nabla_a$ alone but also by its embedding). When ω=0, $k_{ab} = \Theta_{ab}$ and this is the 3-curvature of the spaces orthogonal to u^a and will have the usual curvature tensor symmetries. In the case of nonvanishing vorticity instead we have

$$ {}^{(3)}R_{abcd} = {}^{(3)}R_{[ab][cd]} \ , \quad {}^{(3)}R^{a}{}_{[bcd]} = 2k^{a}{}_{[b}\omega_{cd]} \ , \tag{80} $$

and

$$ {}^{(3)}R^{ab}{}_{cd} - {}^{(3)}R_{cd}{}^{ab} = -8\omega^{[a}{}_{[c}\Theta^{b]}{}_{d]} \ . \tag{81} $$

Further, for each tensor field T_{ab} orthogonal to u^a ($T_{ab}u^a = 0 = T_{ab}u^b = 0$),

$$ {}^{(3)}\nabla_{[a}{}^{(3)}\nabla_{b]}T_{cd} + \omega_{ab}\dot{T}_{\perp cd} = \tfrac{1}{2}({}^{(3)}R_{scba}T^{s}{}_{d} + {}^{(3)}R_{sdba}T_{c}{}^{s}) \ . \tag{82} $$

It follows from the above relations that the corresponding "3-Ricci tensor" is

$$ {}^{(3)}R_{ac} \equiv {}^{(3)}R_{a}{}^{b}{}_{cb} = {}^{(3)}R^{b}{}_{abc} = h^{bd}(R_{abcd})_{\perp} - \Theta k_{ac} + k_{ab}k^{b}{}_{c} \ , \tag{83} $$

with skew part

$$ {}^{(3)}R_{[cb]} = \tfrac{1}{3}\omega_{bc}\Theta + (\omega_{db}\sigma_{c}{}^{d} - \omega_{dc}\sigma_{b}{}^{d}) \ , \tag{84} $$

and the "Ricci scalar" is

$$ {}^{(3)}R \equiv {}^{(3)}R^{a}{}_{a} = R + 2R_{bd}u^{b}u^{d} - \tfrac{2}{3}\Theta^2 + 2\sigma^2 - 2\omega^2 \ , \tag{85} $$

giving Eq. (10) on using the Einstein field equations.

When (78) and (82) are applied to a first-order quantity, the time-derivative term can be neglected and we only need the zero-order curvature tensor term in these expressions to get the correct first-order result. From the field equations, the zero-order expression for the curvature tensor [see (28) in Ref. 40] is

$$ R_{abcd} = \tfrac{1}{2}\kappa(\mu + p)(u_a u_c g_{bd} + u_b u_d g_{ac} - u_a u_d g_{bc} - u_b u_c g_{ad}) + \tfrac{1}{3}(\kappa\mu + \Lambda)(g_{ac}g_{bd} - g_{ad}g_{bc}) \ . \tag{86} $$

Thus the zero-order versions of the 3-curvature quantities (remembering that to zero order, $u_{a;b} = \frac{1}{3}\Theta h_{ab}$) are

$$ {}^{(3)}R_{abcd} = K(h_{ac}h_{bd} - h_{ad}h_{bc}) \ , \tag{87} $$

$$ {}^{(3)}R_{ac} = 2Kh_{ac} = R_{ca} \ , \quad {}^{(3)}R = 6K \ , \tag{88} $$

where

$$ K \equiv \tfrac{1}{3}(-\tfrac{1}{3}\Theta^2 + \kappa\mu + \Lambda) = \frac{k}{S^2}, \quad \dot{k} = 0 \tag{89} $$

(the last equality following from the contracted Bianchi identities for the 3-curvature: ${}^{(3)}\nabla^{a}\,{}^{(3)}R_{ac} = \frac{1}{2}{}^{(3)}\nabla_{c}{}^{(3)}R$). These expressions can be substituted for the 3-curvatures above when performing systematic approximations of the equations.

3. 3-divergences

It follows from (82) that

$$ {}^{(3)}\nabla_{a}{}^{(3)}\nabla_{b}T^{[ab]} = -\omega_{ab}\dot{T}^{ab} + {}^{(3)}R_{[ab]}T^{ab} \ , \tag{90} $$

which shows that in particular

$$ {}^{(3)}\nabla_{a}{}^{(3)}\nabla_{b}\omega^{ab} = -\omega_{ab}\dot{\omega}^{ab} + {}^{(3)}R_{[ab]}\omega^{ab} \ , \tag{91} $$

which [see (32) and (84)] is nonzero in general, but vanishes to first order in an almost-FLRW universe model.

4. Time derivatives

Calculating

$$ {}^{(3)}\nabla_{a}(\dot{f}) - ({}^{(3)}\nabla_{a}f)^{\cdot}{}_{\perp} = (f_{;c}u^{c})_{;b}h^{b}{}_{a} - (f_{;b}h^{b}{}_{d})_{;c}u^{c}h^{d}{}_{a} $$

we find

$$ {}^{(3)}\nabla_{a}(\dot{f}) - ({}^{(3)}\nabla_{a}f)^{\cdot}{}_{\perp} = -\dot{f}a_{a} + \tfrac{1}{3}\Theta\,{}^{(3)}\nabla_{a}f + {}^{(3)}\nabla_{d}f(\sigma^{d}{}_{a} + \omega^{d}{}_{a}) \ , \tag{92} $$

where the last two terms are second order if ${}^{(3)}\nabla_{a}f$ is first order, and so can be ignored in the linear approximation. Similarly, for a first-order vector field orthogonal to u^a, we find that to first order

$$ {}^{(3)}\nabla_{a}(\dot{X}_{b}) - ({}^{(3)}\nabla_{a}X_{b})^{\cdot}{}_{\perp} = \tfrac{1}{3}\Theta\,{}^{(3)}\nabla_{a}X_{b} \Rightarrow S\,{}^{(3)}\nabla_{a}\dot{X}_{b} = (S\,{}^{(3)}\nabla_{a}X_{b})^{\cdot}{}_{\perp} \ , \tag{93} $$

where we have used (86). Similar results will then hold for a first-order tensor, e.g., if T_{bc} is orthogonal to u^a then to first order

$$ {}^{(3)}\nabla_{a}(\dot{T}_{bc}) - ({}^{(3)}\nabla_{a}T_{bc})^{\cdot}{}_{\perp} = \tfrac{1}{3}\Theta\,{}^{(3)}\nabla_{a}T_{bc} \Rightarrow S\,{}^{(3)}\nabla_{a}\dot{T}_{bc} = (S\,{}^{(3)}\nabla_{a}T_{bc})^{\cdot}{}_{\perp} \ . \tag{94} $$

We can contract this equation to obtain the result for a divergence:

$$ {}^{(3)}\nabla^{c}(\dot{T}_{bc}) - ({}^{(3)}\nabla^{c}T_{bc})^{\cdot}{}_{\perp} = \tfrac{1}{3}\Theta\,{}^{(3)}\nabla^{c}T_{bc} \Rightarrow S\,{}^{(3)}\nabla^{c}\dot{T}_{bc} = (S\,{}^{(3)}\nabla^{c}T_{bc})^{\cdot}{}_{\perp} \ . \tag{95} $$

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[4] A paper by J. Ehlers, B. Schmidt, and M. Lottermoser (in preparation), deriving simultaneously the relativistic and Newtonian versions of our results and showing that the Newtonian equations (Ref. 5) are a rigorous limit of the relativistic equations.

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[6] One may alternatively consider a one-parameter family of solutions $l(\epsilon)$, $0 \leq \epsilon \leq 1$ where $l(0)$ is the background solution and $l(1)$ the realistic solution, and take derivatives with respect to the perturbation parameter ϵ (Ref. 7); the results will be equivalent to those presented here. We shall

not explicitly use that formalism.

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[19]In EB we defined the closely related variable $\mathcal{K} = {}^{(3)}R+2\omega^2$ and its gradient $\mathcal{K}_a = {}^{(3)}\nabla_a\mathcal{K}$; it is now our view that it would be preferable to replace $\mathcal{K}$ by ${}^{(3)}R$ and $\mathcal{K}_a$ by C_a in EB and EHB. As these variables agree in the linear approximation, the linear relations in EB and EHB are unaffected, and the discussion in Appendix B of EB of the geometric meaning of the variable $\mathcal{K}$ when $\omega \neq 0$ applies equally to ${}^{(3)}R$.

[20]D. H. Lyth and M. Mukherjee, Phys. Rev. D **38**, 485 (1988).

[21]Omitted in EHB; the following equations should be taken as amending the corresponding equations in EHB, in the case that rotation occurs as well as expansion.

[22]Note that in this equation the first two derivatives ${}^{(3)}\nabla^a$ can be taken to zero order (i.e., in the background), because they act on first-order terms; but the last one must be taken to first order in the real space, because it acts on a zero-order term.

[23]Or the Universe has the exceptional (inflationary) equation of state $w = -1$, when a fluid description is not really valid.

[24]This can be extended to the nonlinear case if the propagation equation is written in terms of the Lie derivative rather than the covariant derivative along the fluid flow lines.

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[30]Which is *not* a curvature spatial gradient, as is C_a.

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[34]We can utilize the definition unchanged on vectors and tensors not orthogonal to u^a, but then must be careful to compute the extra terms that arise from the nonorthogonal parts.

[35]Note that other definitions are possible for this "3-curvature," see Ref. 3, Eq. (2.26); Refs. 36–39; and unpublished work by R. Jantzen. Here we use definition (79) because of the simple form it gives (78) and (82) (where the dot-derivative is a covariant derivative; other forms are appropriate if this is replaced by a Lie derivative, or if one wants to preserve the usual Riemann symmetries).

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