R^2 inflation in anisotropic universes

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The evolution of Bianchi type-I and type-IX universes for a theory of gravity with an ϵR^2 term added to the usual Lagrangian is considered. As in the spatially flat Robertson-Walker case considered previously by others, inflation is found to occur. For any amount of initial anisotropy, the anisotropy decays quickly relative to the length of the inflationary epoch, and the amount of expansion is enhanced by the anisotropy. The exceptions are Bianchi type-IX universes near or at isotropy. In these cases a wide range of initial parameters causes the universe to recollapse, thus reducing the phase space in which inflation can occur. The diagonal metric is shown to be the most general form in the R^2 theory for both Bianchi type-I universes with a perfect fluid and vacuum Bianchi type-IX models.

I. INTRODUCTION

Several mechanisms have been introduced to generate expansion in inflationary universe models. In theories such as new inflation[1] and chaotic inflation,[2] field theory is used to create an effective cosmological constant which drives an exponential expansion of the scale factor. Another interesting mechanism, dubbed R^2 inflation,[3] adds an ϵR^2 term to the usual gravitational Lagrangian. For a certain range of initial parameters the effect of the R^2 term is to generate sufficient expansion to solve many cosmological problems.

Among the problems which inflation purports to solve are why our observable universe is so remarkably homogeneous and isotropic and why it is so close to being spatially flat. In the standard hot big bang model this smoothness and flatness is put in by hand as initial conditions. The inflationary scenario explains the observed smoothness by having our present Universe evolve from a causally connected region. However, most calculations involving inflation have been done using the isotropic Robertson-Walker metric to describe initial geometry; hence the smoothness is still put in by hand. For this explanation to be meaningful, it is crucial that arbitrary initial geometries be used, to discover if inflation still occurs and if the anisotropy is dissipated.

In this paper, I examine R^2 inflation with Bianchi type-I and type-IX universes as initial geometries. These geometries reduce to the spatially flat and positively curved Robertson-Walker universes, respectively, in the limit of zero anisotropy. Anisotropic R^2 inflation has been considered previously by other authors,[4] but their results were not applicable to Bianchi type-IX universes. In addition, the techniques of this paper yield quantitative data and are adaptable to more complicated Lagrangians or matter which violates the energy condition used in previous works.[4,5] I use both numerical results and analytical arguments to show that in both cases inflation is actually enhanced by the presence of anisotropy, and that this anisotropy is indeed dissipated. The exceptions are Bianchi type-IX universes near or at isotropy. For these cases a wide range of initial parameters leads to a recollapse of the universe before an inflationary era can occur.

II. BIANCHI TYPE-I UNIVERSES

I start by considering the Lagrangian density $L = R + \epsilon R^2$ to describe gravity, with R the Ricci scalar and ϵ an arbitrary constant. For simplicity and to facilitate comparison with previous work, terms involving the squares of the Riemann and Ricci tensors are not considered. The field equations including matter are[6]

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} + 2\epsilon[R(R_{\mu\nu} - \tfrac{1}{4} R g_{\mu\nu}) + R_{;\kappa\lambda}(g^{\kappa\lambda} g_{\mu\nu} - \delta^\kappa_\mu \delta^\lambda_\nu)] = 8\pi G T_{\mu\nu} \; . \tag{2.1}$$

Throughout this paper I will use a perfect fluid to describe matter, so that

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} \; , \tag{2.2}$$

where ρ is the energy density, p the pressure, and u_μ the four-velocity of the matter. I will furthermore use the equation of state

$$p = (\gamma - 1)\rho \; , \tag{2.3}$$

where γ is a constant.

I first consider the Bianchi type-I universes, whose form of the metric may be written without loss of generality in the case of a perfect fluid as

$$ds^2 = -dt^2 + a^2(t) \sum_{i=1}^{3} e^{2\beta_i(t)} dx^{i\,2} \; , \tag{2.4}$$

with $\sum_{i=1}^{3} \beta_i = 0$. To see that this form of the metric is most general, write the metric as an orthonormal tetrad[7]

$$ds^2 = -dt^2 + (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2 \; , \tag{2.5}$$

where

$$\sigma^i = b_{ij} dx^j, \quad b_{is} b_{sj} = g_{ij}, \quad b_{ij} = b_{ji} ,$$

and summation over repeated indices implied. The differential forms σ^i thus obey

$$d\sigma^i = k_{ij} dt \wedge \sigma^j , \tag{2.6a}$$

with

$$k_{ij} = \dot{b}_{is} b_{sj}^{-1} , \tag{2.6b}$$

and an overdot indicates a time derivative. I will also use the following matrix notation: **B**, **G**, and **K** represent b_{ij}, g_{ij}, and k_{ij}, respectively. **K** may be broken up into its symmetric and antisymmetric parts **L** and **M** as

$$\mathbf{L} = \tfrac{1}{2}(\mathbf{K} + \mathbf{K}^T), \quad \mathbf{M} = \tfrac{1}{2}(\mathbf{K} - \mathbf{K}^T) .$$

By proper choice of coordinates, I may take both **B** and $\dot{\mathbf{B}}$ to be diagonal at some time t_0. Then at this time $\mathbf{K} = \mathbf{L}$ will also be diagonal, while $\mathbf{M} = 0$. Now, the $(0i)$ equations imply that $u_i = 0$ at all times. The (ij) field equations then give, at $t = t_0$,

$$R_{ij}(1 + 2\epsilon R) = 0 , \quad i \neq j . \tag{2.7}$$

In general this equation is solved by

$$R_{ij} = \dot{l}_{ij} + l_{ij} l_{ss} + l_{is} m_{sj} + l_{js} m_{si} = 0 \tag{2.8}$$

at t_0, which implies that $\dot{\mathbf{L}}$ is diagonal at t_0. Differentiating the relationship

$$\mathbf{BL} - \mathbf{LB} = \mathbf{BM} + \mathbf{MB} \tag{2.9}$$

then gives $\dot{\mathbf{M}} = 0$ at t_0, and differentiating the relationship

$$\dot{\mathbf{B}} = \mathbf{LB} + \mathbf{MB} \tag{2.10}$$

shows that $\ddot{\mathbf{B}}(t_0)$ is diagonal. Using the same arguments as above with time derivatives of the equations shows that arbitrarily many time derivatives of **B** remain diagonal at t_0. By constructing a Taylor series about t_0, **B** is seen to remain diagonal at all times.

Straightforward calculations for the diagonal metric (2.4) then show that

$$R = 6\dot{H} + 12H^2 + 6Q , \tag{2.11}$$

where H is the Hubble parameter,

$$H = \frac{\dot{a}}{a} , \tag{2.12}$$

and Q is the anisotropy parameter

$$Q = \frac{1}{6} \sum_{i=1}^{3} \dot{\beta}_i^2 . \tag{2.13}$$

Q is proportional to the trace of the square of the shear tensor, and is therefore a measure of the shear.

The contracted field equations give

$$\ddot{R} + 3H\dot{R} + \frac{1}{6\epsilon} R = \frac{4\pi G(4 - 3\gamma)\rho}{3\epsilon} . \tag{2.14}$$

The (00) equation gives

$$\dot{R} = \frac{R^2}{12H} - RH - \frac{H}{2\epsilon} + \frac{RQ}{H} + \frac{Q}{2\epsilon H} + \frac{4\pi G\rho}{3\epsilon H} , \tag{2.15}$$

while the (ii) equations yield

$$\dot{Q} = \frac{1}{1 + 2\epsilon R} \left[-4QH - \frac{2Q^2}{H} - 8\epsilon QRH - \frac{\epsilon QR^2}{3H} - \frac{4\epsilon RQ^2}{H} - \frac{16\pi G\rho Q}{3H} \right] . \tag{2.16}$$

Equation (2.13) implies that as Q goes to zero so do the individual $\dot{\beta}_i$. In all cases the anisotropy does become zero, so there is no need to consider equations for the individual β_i. Equation (2.14) may be considered superfluous because of the contracted Bianchi identities and the covariant conservation of the stress-energy tensor, and provides a check of the accuracy of the program. In a vacuum, Eq. (2.14) was numerically never larger than 10^{-7}.

I first consider the case with no matter, and hence, Eqs. (2.11), (2.15), and (2.16) form a set of three first-order differential equations for the three variables H, R, and Q. They may easily be solved numerically using a fourth-order Runge-Kutta routine.[8] For the rest of this paper I will consider the case $\epsilon > 0$, $R_i > 0$, $H_i > 0$, where the subscript indicates the value of the quantity at the beginning of the classical regime. This case was shown in Ref. 3 to be the only one of interest.

Typical results are shown in Fig. 1. The anisotropy always increases the rate and amount of inflation and decays to zero. This behavior can be justified by simple analytic arguments. Equation (2.16) shows that $\dot{Q} \leq 0$, with the equality holding only when $Q = 0$, so long as H remains positive. Hence, anisotropy will only decay with time, and the greater the initial anisotropy, the quicker it will decay. In fact, in the limit $\epsilon \to 0$, Eqs. (2.12), (2.15), and (2.16) yield the standard general relativity result $Q \propto a^{-6}$ for a vacuum. The inclusion of the ϵ terms further decreases the anisotropy, as may be seen in Fig. 2.

To see why anisotropy helps enhance expansion, I first consider the isotropic case using an analysis in R-H phase space (see Fig. 3). For $Q = 0$, Eqs. (2.11) and (2.15) give

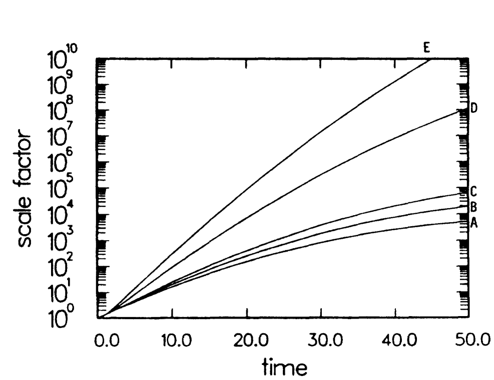

FIG. 1. Scale factor vs time for a Bianchi type-I universe with different initial values of the shear: $Q_i = 0.0$ (curve A), 0.1 (curve B), 0.2 (curve C), 0.5 (curve D), 1.0 (curve E). In this and all subsequent figures, $\epsilon = 5.0$, $H_i = 0.3$, and $R_i = 1.08$, unless otherwise noted.

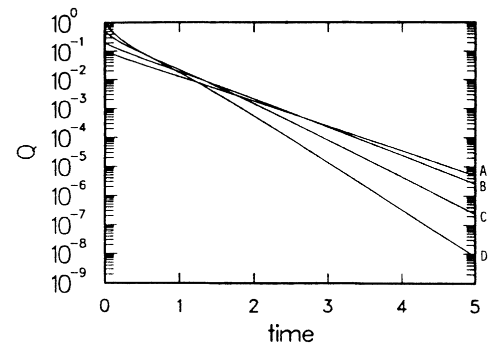

FIG. 2. Shear vs time in a Bianchi type-I universe for $Q_i = 0.1, 0.2, 0.5, 1.0$ (curves A–D, respectively). The case of $Q_i = 0$ remains shear-free for all time.

$$\dot{H} = 0 \Longrightarrow R = 12H^2 , \tag{2.17}$$

$$\dot{R} = 0 \Longrightarrow R = 6H^2 \left[1 + \left[1 + \frac{1}{6\epsilon H^2} \right]^{1/2} \right] . \tag{2.18}$$

Note that as $\epsilon \to \infty$, these two relations become identical. In this case, any universe which starts with $R_i > 12H_i^2$ will evolve to the curve $\dot{H} = \dot{R} = 0$ and then stay there. Since $\dot{H} = 0$ on this curve, H will be constant, and the Universe will exponentially expand indefinitely.

For a finite but large ϵ, curves I and II (Fig. 3) describing $\dot{H} = 0$ and $\dot{R} = 0$ will be close together. The Universe will evolve slowly while near these curves and thus undergo almost exponential expansion; hence the ϵ term provides the inflationary mechanism. In greater detail, if the Universe begins the classical era with

$$R_i > 6H_i^2 \left[1 + \left[1 + \frac{1}{6\epsilon H_i^2} \right]^{1/2} \right] , \tag{2.19}$$

then $\dot{R} < 0$ and $\dot{H} > 0$. The Universe will evolve to the $\dot{R} = 0$ curve, with H increasing. Once at this curve, if $H \geq 1/\sqrt{6\epsilon}$ then $\dot{H} \approx 1/12\epsilon$. Hence, H will grow linearly but slowly until reaching the $\dot{H} = 0$ curve while rapid expansion occurs. At the $\dot{H} = 0$ curve, $\dot{R} \approx -H/2\epsilon$, so only now will the Universe move down in the phase diagram into the $\dot{H} < 0$ region. H will decrease, and the Universe will exit the inflationary phase. Hence, in the isotropic case, any universe with initial values above or near the $\dot{H} = 0$ curve will inflate. If the universe enters the classical regime with values significantly below this curve, then not enough expansion will occur to solve the desired cosmological problems.

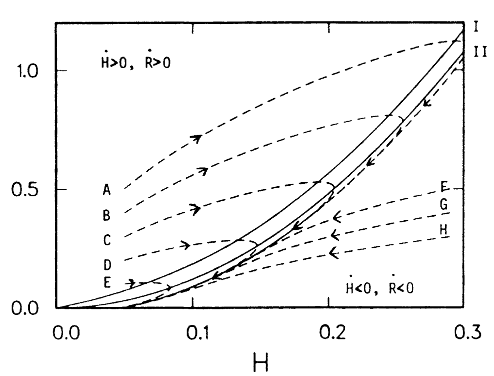

FIG. 3. R-H phase space in flat Robertson-Walker universe. The dashed curves represent various initial values, with the arrows indicating the evolution of the parameters in time. Curve I is $\dot{R} = 0$, curve II is $\dot{H} = 0$. Near II inflation occurs.

Now consider the inclusion of anisotropy. As seen by Eqs. (2.11) and (2.15), the presence of initial anisotropy increases $\dot{R}$ and decreases $\dot{H}$. This change moves the Universe up and to the left in the phase diagram, and hence boosts some additional initial values into the inflationary region of phase space, as seen in Fig. 4. One may worry that an excessive amount of anisotropy may force H to be negative. However, because $\dot{Q}$ has a quadratic dependence on Q, the anisotropy always decays away before H can become negative, and the increase in R further prevents the decrease of H.

If matter is included, these results remain the same. From Eq. (2.15), the presence of matter increases the value of R faster than in the vacuum case, and hence further enhances inflation. The matter contribution to $\dot{Q}$ is negative, and thus will cause an even quicker decrease in the shear. The fate of matter is independent of Q. The $(0i)$ field equations show that $u_i = 0$, and using this fact along with the covariant conservation of $T_{\mu\nu}$ gives

$$\rho \propto a^{-3\gamma} . \tag{2.20}$$

Therefore, once the Universe starts to inflate, the matter content will rapidly lose any influence it may have had.

III. BIANCHI TYPE-IX UNIVERSES

I first consider the diagonal Bianchi type-IX universe, whose metric is given by

$$ds^2 = -dt^2 + a^2 \sum_{i=1}^{3} e^{2\beta_i} {\omega^i}^2 , \tag{3.1}$$

where a and the β_i are functions of time only. The one-forms ω^i obey the relation

$$d\omega^i = \frac{1}{2} \epsilon_{ijk} \omega^j \wedge \omega^k , \tag{3.2}$$

where ϵ_{ijk} is the fully antisymmetric symbol with $\epsilon_{123} = 1$. The β sum to zero, and a useful parametrization is

$$\beta_1 = \beta_+ + \sqrt{3}\beta_- , \quad \beta_2 = \beta_+ - \sqrt{3}\beta_- , \quad \beta_3 = -2\beta_+ . \tag{3.3}$$

The field equations (2.1) then yield, for a Bianchi type-IX universe,

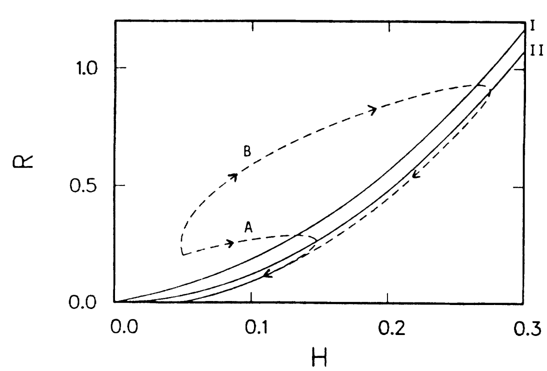

FIG. 4. R-H phase space in a Bianchi type-I universe for $Q_i = 0$ (curve A) and $Q_i = 0.05$ (curve B). Curve I is $\dot{R} = 0$, curve II is $\dot{H} = 0$.

$$\dot{H} = \tfrac{1}{6}R - 2H^2 - (B_+^2 + B_-^2) + \frac{V-1}{4a^2} , \tag{3.4a}$$

$$\dot{R} = \frac{R^2}{12H} - RH - \frac{H}{2\epsilon} + \frac{B_+^2 + B_-^2}{2\epsilon H} + \frac{R}{H}(B_+^2 + B_-^2) + \frac{V-1}{8\epsilon H a^2} + \frac{R(V-1)}{4Ha^2} + \frac{4\pi G\rho}{3\epsilon H} , \tag{3.4b}$$

$$\dot{a} = aH , \tag{3.4c}$$

$$\dot{B}_\pm = \frac{1}{1+2\epsilon R}\left[-2HB_\pm - \frac{J_\pm}{a^2} - \frac{B_\pm}{H}(B_+^2 + B_-^2) - \frac{B_\pm(V-1)}{4Ha^2} - \frac{8\pi G\rho B_\pm}{3H} + \epsilon\left[-4HRB_\pm - \frac{2RJ_\pm}{a^2} - \frac{R^2 B_\pm}{6H} - \frac{2RB_\pm}{H}(B_+^2 + B_-^2) - \frac{RB_\pm(V-1)}{2Ha^2}\right]\right] , \tag{3.4d}$$

$$\dot{\beta}_\pm = B_\pm , \tag{3.4e}$$

where I have defined the following useful symbols:

$$V = \tfrac{1}{3}\sum_{i=1}^{3}(1 + b_i^4 - 2b_i^{-2}) , \tag{3.5}$$

$$J_+ = \tfrac{1}{6}(b_1^4 + b_2^4 - 2b_3^4 + b_1^{-2} + b_2^{-2} - 2b_3^{-2}) , \tag{3.6}$$

$$J_- = \frac{1}{2\sqrt{3}}(b_1^4 + b_1^{-2} - b_2^4 - b_2^{-2}) , \tag{3.7}$$

$$b_i = e^{\beta_i} . \tag{3.8}$$

V is the anisotropy potential familiar from Hamiltonian cosmology.[9] V and $J_\pm$ are all zero when $\beta_+ = \beta_- = 0$. V is always positive for nonzero β's, while the $J_\pm$ may be either sign.

While these equations are considerably more complicated than those governing a Bianchi type-I universe, they may be handled numerically in the same manner. As before, the contracted field equations provide a theoretically vanishing quantity, whose numerical value provides a check of accuracy. This quantity was never greater than 10^{-5}, and usually was several orders of magnitude less. Typical results are seen in Figs. 5–11, and are described in detail below. Again, an increased anisotropy increases the amount of inflation and eventually decays away.

An interesting case where inflation is prevented is the isotropic one, with $\beta_\pm = 0$, corresponding to the positively curved Robertson-Walker universe. Typical results are shown in Fig. 5. Because V is zero, the $-1/4a^2$ can dominate Eq. (3.4a) while the scale factor is small. This term will make $\dot{H}$ negative, causing rapid recollapse. This behavior is an example of what Barrow termed the "premature recollapse problem."[10] However, a large enough R will prevent recollapse by making a positive contribution to $\dot{H}$. Hence, inflation is not prevented in the isotropic case; rather the R-H phase space which will inflate is decreased compared to the spatially flat case.

The addition of anisotropy in the form of the B terms in (3.4a) might seem further to prevent inflation by giving a negative contribution to $\dot{H}$. In fact this shear contribution aids expansion, as seen in Fig. 6. The reason is that nonzero shear also raises R via (3.4b). This larger R makes a positive contribution to $\dot{H}$ which more than offsets the negative shear term. Further, these B terms will cause the $\beta_\pm$ to assume nonzero values, as seen in (3.4e). V will then also acquire a nonzero value and contribute positively to $\dot{H}$. Hence, the anisotropy plays a significant role in facilitating inflation in a large region of

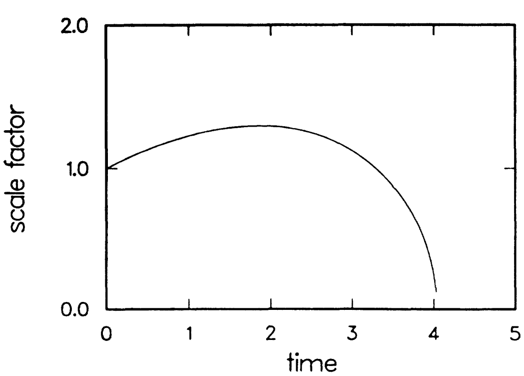

FIG. 5. Scale factor vs time for a positively curved Robertson-Walker (zero-anisotropy limit of Bianchi type-IX universe). Recollapse occurs.

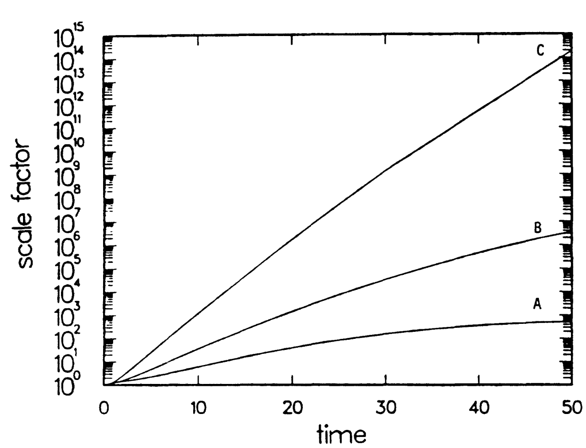

FIG. 6. Scale factor vs time for a Bianchi type-IX universe with initial conditions $B_\pm = 0.35$, 0.50, 1.0 (curves A–C, respectively). In these calculations, $\beta_\pm = 0$ initially.

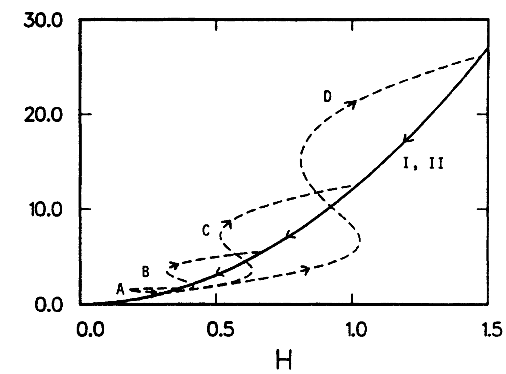

FIG. 7. *R-H* phase space for a Bianchi type-IX universe with initial values of $\beta_{\pm i}=0.2$, 0.3, 0.4, 0.5 (curves A–D, respectively). $B_{\pm}$ is set to zero initially. Curves I and II cannot be distinguished from each other in this figure; they represent $\dot{H}=0$ and $\dot{R}=0$. Curves A–D evolve back towards the origin close to curves I and II.

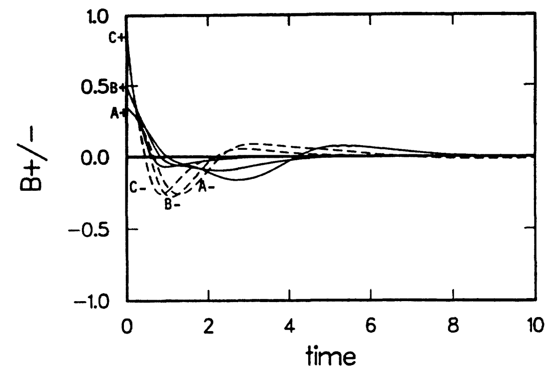

FIG. 9. $B_{\pm}$ vs time for a Bianchi type-IX universe for initial values $B_{\pm i}=0.35$, 0.50, 1.0 (curves A–C, respectively). The solid curves describe B_+, the dashed curves describe B_-. $B_{\pm}$ is set to zero initially.

R-H phase space which would have recollapsed in the isotropic, positively curved case. For example, with $\epsilon=5$, $H_i=0.3$, $R_i=1.08$, and $\beta_{\pm}=0$, an initial value of $Q=B_+^2+B_-^2$ greater than about 0.5 prevented recollapse, while Q_i greater than about 1.3 led to inflation.

The effects of V should be considered in greater detail, because a large enough contribution might seem to decrease the chances of inflation by moving H too far to the right in the *R-H* phase diagram. However, this effect does not occur. To see why inflation is not prevented for the positive V case, consider the $\dot{H}$ and $\dot{R}$ equations, which differ from the Bianchi type-I equations only by the presence of the $(V-1)$ terms. A large positive V does move H to the right in the phase space, but also raises the value of R, as shown in Fig. 7. Hence a large positive V will lead to $R \approx 12H^2$, with larger values of R and H, thus leading to increased expansion (see Fig. 8). For the above values of ϵ, H_i, and R_i, with $Q_i=0$, an initial V greater than about 1.0 prevented recollapse, and when V started at greater than about 10.0 sufficient inflation ensued. As the scale factor increases, the influence of V will wane, as this anisotropy potential is always divided by a^2.

Examining Eq. (3.4d) shows why anisotropy always decreases, as seen in Fig. 9. All but the terms involving $(V-1)$ or $J_{\pm}$ enter with sign opposite $B_{\pm}$, and will hence drive $B_{\pm}$ towards zero. The $(V-1)$ term acts to increase the shear only for cases near isotropy, where its effect will be minimal. More anisotropic initial conditions will result in the potential also contributing to (3.4d) with a sign

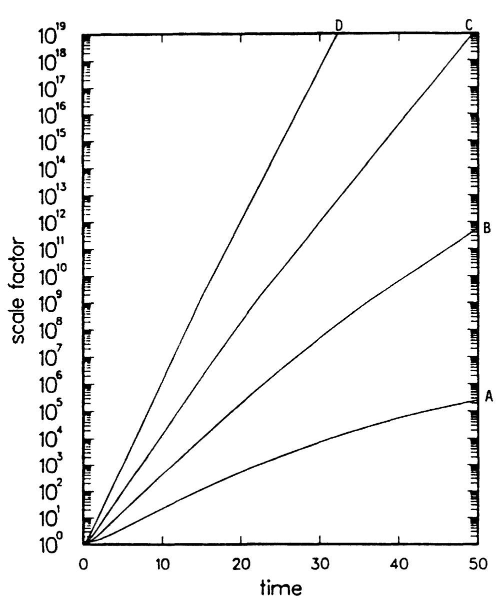

FIG. 8. Scale factor vs time for a Bianchi type-IX universe with initial values of $\beta_{\pm i}=0.2, 0.3, 0.4, 0.5$ (curves A–D, respectively). $B_{\pm}$ is set to zero initially in these calculations.

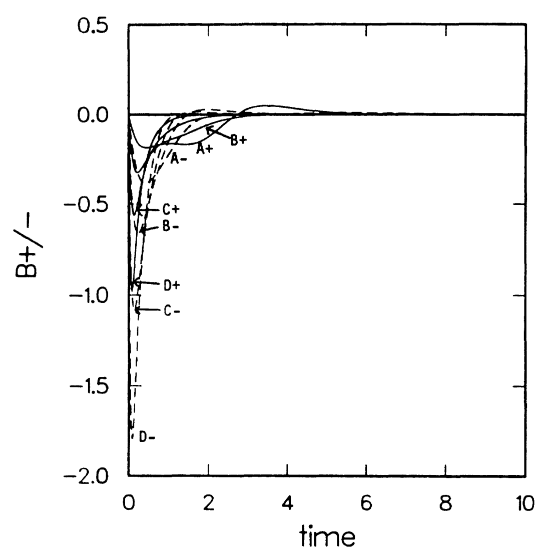

FIG. 10. $B_{\pm}$ vs time for a Bianchi type-IX universe with $B_{\pm}=0$ initially. The contribution of the $\beta_{\pm}$ terms makes the $B_{\pm}$ nonzero, but competing effects quickly drive $B_{\pm}$ back to zero. The four cases are $\beta_{\pm i}=0.2$, 0.3, 0.4, 0.5 (curves A–D, respectively). The solid curves describe B_+, the dashed curves describe B_-.

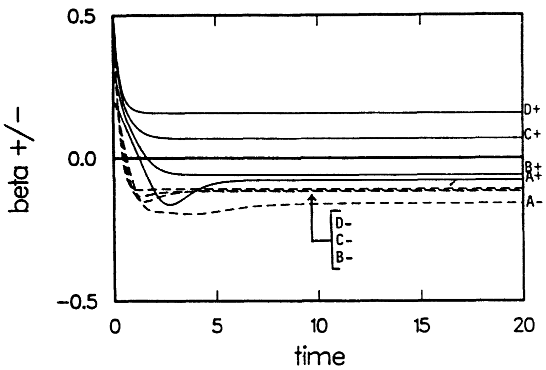

FIG. 11. $\beta_\pm$ vs time for a Bianchi type-IX universe with initial values $\beta_{\pm i}=0.2$, 0.3, 0.4, 0.5 (curves A–D, respectively). As $B_\pm \to 0$ (Fig. 10), the $\beta_\pm$ become frozen at nonzero values. The solid curves describe β_+, the dashed curves describe β_-.

which decreases the shear. Large negative values of $J_\pm$ might appear to cause an increase in shear, because they would enter (3.4d) with sign opposite the rest of the terms. However, corresponding large values of V will always occur when $J_\pm$ is large. The V term will act oppositely to the $J_\pm$ terms, mitigating their effects (see Fig. 10). Also note that any increase in $B_\pm$ will be compensated for by the large negative contribution to $\dot{B}_\pm$ from the $B_\pm$ terms. Furthermore, as discussed above, either a large V or large shear will result in increased expansion. This expansion minimizes the effects of W and $J_\pm$, which are always divided by the scale factor squared. Hence, the occurrence of inflation always leads to vanishing shear.

Note that when the shear vanishes the values of $\beta_\pm$ will be frozen at values not necessarily zero, implying anisotropy (Fig. 11). This anisotropy manifests itself in the equations by V and $J_\pm$ assuming values not equal to the positively curved Robertson-Walker ones. However, as the universe expands, the influence of these terms diminishes, as they are always divided by the square of the scale factor. This reduction of influence is reflected in the vanishing of the three-curvature as the scale factor increases, namely,

$$ {}^{(3)}R=\frac{3(1-V)}{2a^2} . \tag{3.9}$$

The exponential expansion of the Universe also explains why the oscillatory behavior of standard general relativity is not seen. These oscillations result from the Universe bouncing off the potential V in β_+-β_- space.[7] However, the equipotential lines of V move further apart with increasing scale factor. Once inflation occurs, the potential will be pushed out to such large values of phase space that it will lose all importance.

So far I have only considered diagonal Bianchi type-IX equations. However, I will now argue that these results should hold in general. As in the Bianchi type-I case, I start by writing the metric as[11]

$$ds^2=-dt^2+(\sigma^1)^2+(\sigma^2)^2+(\sigma^3)^2 , \tag{3.10}$$

with the same relations for σ^i, b_{ij}, and g_{ij} as before. The differential forms σ^i thus obey

$$d\sigma^i=k_{ij}dt\wedge\sigma^j+\tfrac{1}{2}d^i{}_{jk}\sigma^j\wedge\sigma^k , \tag{3.11a}$$

with

$$d^i{}_{jk}=b_{is}\epsilon_{stu}b_{tj}^{-1}b_{uk}^{-1} . \tag{3.11b}$$

K, **L**, and **M** are defined as before.

For the vacuum case the $(0i)$ field equations give

$$l_{st}d^t_{si}[1+2\epsilon(2\dot{l}_{kk}+l_{ku}l_{ku}+l_{kk}l_{uu} -\tfrac{1}{4}d^j_{ku}d^j_{ku}-\tfrac{1}{2}d^k_{ju}d^u_{jk})]=0 . \tag{3.12a}$$

In general, this is solved by

$$l_{st}d^t_{si}=0 . \tag{3.12b}$$

By proper choice of coordinates, I may take **B** to be diagonal at some time t_0,

$$\mathbf{B}=a(t)\mathrm{diag}(e^{\beta_{11}},e^{\beta_{22}},e^{\beta_{33}}) . \tag{3.13}$$

Then (3.12b) implies

$$l_{23}\sinh(\beta_{22}-\beta_{33})=0, \text{ et cyc.}, \tag{3.14}$$

at $t=t_0$. In general $\beta_{22}\neq\beta_{33}$, so the general solution of (3.14) is $l_{23}=0$, and hence **L** is diagonal at t_0. The relationship

$$e^\beta\mathbf{L}-\mathbf{L}e^\beta=e^\beta\mathbf{M}+\mathbf{M}e^\beta , \tag{3.15}$$

then implies that $\mathbf{M}=0$ at t_0. The vacuum (ij) equations yield

$$(\dot{l}_{ij}-\tfrac{1}{2}d^s_{it}d^s_{jt}-\tfrac{1}{2}d^s_{it}d^t_{js}+\tfrac{1}{4}d^i_{st}d^j_{st})(1+2\epsilon R)=0 , \quad i\neq j,\ t=t_0 . \tag{3.16}$$

Because R depends on time and t_0 is chosen arbitrarily, the relationship in the first set of parentheses must be zero. For $i\neq j$ the d terms automatically vanish, giving $\dot{l}_{ij}=0$, and hence $\mathbf{L}(t_0)$ is diagonal. Writing

$$\dot{\mathbf{B}}=\mathbf{LB}+\mathbf{MB} \tag{3.17}$$

shows that $\dot{\mathbf{B}}(t_0)$ is diagonal. Similar arguments show that arbitrarily many time derivatives of **L** and **B** remain diagonal at t_0, while arbitrarily many times derivatives of **M** are zero at t_0. By constructing a Taylor series about t_0, **B** is seen to remain diagonal for all times, and hence the diagonal Bianchi type-IX metric is the most general that need be considered for the vacuum case.

Finally, consider the addition of matter. The energy density ρ affects the Bianchi type-IX evolution via Eqs. (3.4b) and (3.4d). In the $\dot{\mathbf{R}}$ equation, the presence of matter makes $\dot{\mathbf{R}}$ larger and thus only helps inflation. Meanwhile, matter makes a contribution opposite in sign of $B_\pm$ and thus only serves to hasten the vanishing of shear. For the diagonal case, the $(0i)$ equations imply that the spatial components u_i of the matter four-velocity are zero. Then, the covariant conservation of the stress-energy tensor yields the same behavior for matter as in the Bianchi type-I universe, given by Eq. (2.20). Again, as the Universe expands the matter contribution to the modified field equation diminishes rapidly. While the full Bianchi type-IX case will be considerably more complicated, this same trend of the matter providing a boost to $\dot{\mathbf{R}}$ and then vanishing should hold. Once the matter becomes negligible, a suitable coordinate choice will diagonalize the metric, which will remain diagonal as shown above.

IV. CONCLUSIONS

For an inflationary mechanism to be successful in explaining the observed flatness and isotropy of the real Universe, it must not use these properties as initial data. The R^2 theory of inflation does indeed pass the anisotropy test, as far as Bianchi types-I and -IX models are concerned. No matter what the initial anisotropy, inflation still occurs in these models for a wide range of initial values of R and H. Indeed, the anisotropy even increases the amount of expansion. The exponential increase of the scale factor drives the anisotropy to zero, and hence could explain the observed isotropy of the Universe. Of course, to discuss the likelihood of inflation a measure in R-H phase space is needed. Page[12] has attempted such an analysis for isotropic universes, but the results are ambiguous.

Exceptions can occur in a positive-curvature Robertson-Walker universe or a slightly anisotropic Bianchi type-IX model. In these cases, the positive curvature of the spatial sections can cause the Universe to recollapse. Inflation will still be realized if the Universe exits the Planck era with a value of R sufficiently large compared to the Hubble parameter H, but the amount of phase space which leads to inflation is restricted compared to the Bianchi type-I case. Thus, the ability of R^2 inflation to explain spatial flatness is weakened.

The Bianchi type-IX case is of particular interest, because the result of Maeda[4] will not always be applicable. Wald's result[5] is only valid if the cosmological constant is initially greater than $\frac{1}{2}$ of the isotropic $^{(3)}R$ at fixed proper volume. Maeda showed that the Lagrangian density with added R^2 term is conformally equivalent to standard general relativity plus a scalar field with a potential equal to $1/8\epsilon$ initially. For initial values such that

$$\frac{1}{8\epsilon} < \frac{3}{4a^2(1+2\epsilon R)} , \tag{4.1}$$

inflation is not guaranteed. In fact, for the initial values I chose, with $\epsilon=5$, $R_i=1.08$, and $a_i=1$, this condition is not met. For fixed R, I found that both recollapse and inflation are possible. Recollapse occurs for small anisotropy, while larger anisotropy can cause inflation by raising the value of R so that Wald's criterion holds at a later time.

One final problem of R^2 inflation deserves comment. A very large value of the R^2 coupling constant $\epsilon > 10^{11} l_P^2$, where l_P is the Planck length, is required to yield sufficient perturbations for galaxy formation.[3] This limit immediately poses a fine-tuning problem: This value seems unnaturally large. Furthermore, once an R^2 term is added, consideration of higher powers of R in the action seems reasonable. Indeed, these terms do arise as higher-order effects in most renormalization schemes. Yet if ϵ is so large, then it is not unreasonable to suppose that the coupling constants for the higher-order terms will also be large. Hence, these terms will contribute significantly in a regime just after the Planck era. Although Mijić *et al.*[3] give several arguments for why only an R^2 term may be added, none of them are convincing.

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