Chaotic inflation as an attractor in initial-condition space

J. H. Kung and Robert H. Brandenberger
Physics Department, Brown University, Providence, Rhode Island 02912
(Received 29 March 1990)

We study the evolution of scalar field inhomogeneities in the preinflationary phase of an inflationary universe. We decompose the scalar field configuration in Fourier modes and consider initial conditions in which more than one mode is excited. We find that the long-wavelength modes are stable against perturbations due to short-wavelength excitations and that chaotic inflation results even if at the initial time the short waves contain most of the energy density.

I. INTRODUCTION

The issue of initial conditions has been a stumbling block in the development of consistent inflationary universe models.[1] New inflation,[2] for example, assumed initial conditions which were later[3,4] shown to be inconsistent with the constraints on the model parameters. Chaotic inflation[3] has emerged as the only one of the original inflationary scenarios[5] that has a chance of being self-consistent.

Although there have been many heuristic arguments about the initial conditions required for chaotic inflation, there have only been a couple of attempts at a quantitative analysis (see Sec. II). In Ref. 6, we started a quantitative analysis of this issue. We considered a model in which inflation is generated by a weakly coupled scalar field ϕ (the inflation) with a double-well potential. We considered single-Fourier-mode inhomogeneities in ϕ and classified conditions for which chaotic inflation, dynamical relaxation,[7] or no inflation result. The main result was that for chaotic inflation to occur, the wavelength must be larger than the initial horizon. Here, we consider excitations with more than one Fourier mode. We show that, provided a single long-wavelength mode is excited, it will dominate the late time evolution and give rise to chaotic inflation, even if the energy density in short-wavelength modes is initially larger. In this sense, chaotic inflation is an attractor in initial-condition space.

The outline of this paper is as follows. In Sec. II, we discuss the various aspects of the initial-condition problem and comment on previous work. Next, we describe our method and the approximations we were forced to make. Section IV contains an analysis of two-mode excitations, and Sec. V extends the results to more general initial conditions. Section VI contains our conclusions.

We use units in which $\hbar=c=k_B=1$. t_i is the initial time (which we take to be the Planck time, the earliest time a classical analysis can be justified) and $a(t)$ is the scale factor of the universe [for simplicity a flat Friedmann-Robertson-Walker (FRW) model] normalized such that $a(t_i)=1$. $H(t)$ is the Hubble expansion parameter, $H(t)=\dot{a}(t)/a(t)$. $m_{\rm Pl}$ denotes the Planck mass.

II. PRELIMINARIES

Two of the main claims of success of inflationary universe models[1] are the claims that inflation naturally produces a homogeneous and flat universe out to at least the present Hubble radius. Unfortunately, most analyses of inflation assume homogeneous and flat initial spacetimes at very early times (exceptions are mentioned later). Before claims of having solved the horizon and flatness problems are made, it is important to carefully study the question of initial conditions.

There are two quite separate aspects to the initial-condition problem. The first concerns the inhomogeneities in the matter fields, the second concerns initial inhomogeneities and anisotropies in the metric. In general, the two problems are coupled. In this paper, however, we shall focus on the first aspect.

To illustrate the first problem, consider a toy model which consists of a homogeneous metric $g_{\mu\nu}(t)$, a homogeneous radiation bath with energy-momentum tensor $T_{\mu\nu}(\text{rad})$, and a scalar field $\phi(\mathbf{x},t)$ (the inflaton) with energy-momentum tensor $T_{\mu\nu}(\phi)$. We assume that initially $T_{\mu\nu}(\text{rad})$ dominates. In this case, it is also reasonable to consider $g_{\mu\nu}$ to be homogeneous. We now follow the evolution of $\phi(\mathbf{x},t)$ in this initial preinflationary, radiation-dominated period. This period ends at a critical time t_c when the potential energy in $\phi(\mathbf{x},t)$ and $T_{00}(\text{rad})$ become equal. The first aspect of the initial-condition problem concerns the question, under which circumstances the equation of state of $\phi(\mathbf{x},t)$ at time t_c will lead to inflation. The condition for (generalized) inflation is[5]

$$p(t_c)<-\tfrac{1}{3}\rho(t_c)\ . \tag{2.1}$$

In old inflation,[1] the above problem did not arise. In this model, $\phi(\mathbf{x})$ was assumed to be strongly coupled and $\phi=0$ was a local minimum of the potential energy $V(\phi)$. In this case, $\phi(\mathbf{x},t)$ can be in thermal equilibrium with the thermal bath. Then, finite-temperature effects[8] make $\phi=0$ the global minimum of the finite-temperature effective potential $V_T(\phi)$, ensuring that at t_c $\phi(\mathbf{x},t_c)\simeq 0$ and that, hence [assuming $V(0)>0$],

$$p(t_c)\simeq-\rho(t_c)\ , \tag{2.2}$$

leading to exponential inflation. Old inflation, however, ran into other problems[9] which led to its quick demise.

The problems of old inflation could be successfully overcome in the new inflationary universe.[2] Here, $\phi=0$ is a local maximum of $V(\phi)$. (Fig. 1). It was initially believed that finite-temperature effects would lead to initial

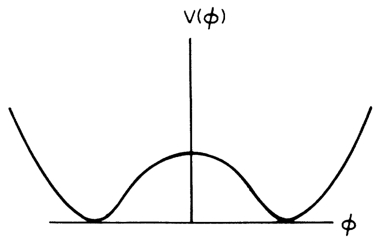

FIG. 1. Sketch of a potential for new inflation.

conditions $\phi(\mathbf{x},t_i)\simeq 0$, similar to old inflation. However, in order not to generate too large density perturbations,[10] the potential $V(\phi)$ must be very flat. For a potential of the form

$$V(\phi)=\tfrac{1}{2}\lambda(\phi^2-\sigma^2)^2 \ , \tag{2.3}$$

$\lambda<10^{-12}$ is required. Constants coupling ϕ to other fields are also constrained to be very small since they induce one-loop contributions to λ. Hence, ϕ will not be in thermal equilibrium with the radiation bath. Therefore[3,4] $\phi(\mathbf{x},t_c)$ is not constrained by thermal forces to be close to $\phi=0$.

The only classical constraints on $\phi(\mathbf{x},t_i)$ at the initial time come from requiring that the energy density in ϕ does not exceed the Planck density. Two ways to obtain inflation have been proposed, chaotic inflation[3] and dynamical relaxation. The former works also for a quadratic potential $V(\phi)=\frac{1}{2}m^2\phi^2$, the latter requires a double-well potential. Chaotic inflation and dynamical relaxation are compared in Fig. 2 The equation of motion for $\phi(\mathbf{x},t)$ is

$$\ddot{\phi}+3H\dot{\phi}-a^{-2}\nabla^2\phi=-V'(\phi) \ . \tag{2.4}$$

To obtain chaotic inflation, we consider locally homogeneous initial conditions for which $\phi(\mathbf{x},t)$ will move slowly. Thus, in (2.4), $\ddot{\phi}$ and $a^{-2}\nabla^2\phi$ are negligible. ϕ

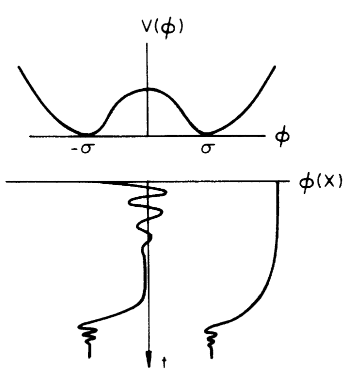

FIG. 2. In the case of a double-well potential (top sketch), different initial conditions can lead either to dynamical relaxation or to chaotic inflation. In the bottom sketch, the evolution of $\phi(\mathbf{x},t)$ at a fixed point $\mathbf{x}$ is shown in two cases. For the curve to the right, $\phi(\mathbf{x})$ has a large initial amplitude and is homogeneous, leading to chaotic inflation. If $\phi(\mathbf{x})$ at the initial time is a plane wave with short wavelength and small amplitude (curve to the left), then dynamical relaxation occurs.

will almost immediately dominate $T_{\mu\nu}$ and inflation will commence. In contrast, if $\phi(\mathbf{x},t_i)$ is a plane-wave excitation with sufficiently large wave number, then, in (2.4), $V'(\phi)$ will be negligible. In this case, $\phi(\mathbf{x},t)$ will oscillate in conformal time $\tau(t)$ with an amplitude which is damped as $a^{-1}(t)$:

$$\phi(\mathbf{x},t)\sim a^{-1}(t)A_0(\mathbf{x})\cos[\omega\tau(t)] \ , \tag{2.5}$$

where ω is the comoving wave number and $\tau(t)$ is defined by

$$dt=a(t)d\tau \ . \tag{2.6}$$

The crucial question is how general is either of the two scenarios.

The second aspect of the initial-condition problem concerns the fate of initial metric inhomogeneities and anisotropies.

There have been several attempts to deal with both aspects of the initial-condition problem from a very general point of view by establishing "no-hair theorems." Such theorems claim to prove that for general initial conditions, in models which can give inflation space-time will exponentially approach de Sitter space.

Most of these "no-hair" theorems are, however, inapplicable to inflationary universe models. A class of theorems which show that anisotropies[11] and inhomogeneities[12] decay exponentially assumes a metric theory which contains an explicit cosmological constant and admits only matter which satisfies the dominant and strong-energy conditions.[13] The strong-energy condition demands that the pressure p be positive and is obviously not satisfied for scalar fields matter. In fact, it is the precise condition $p<0$ required to obtain inflation which renders the "no-hair" theorems inapplicable.

In addition to the above fundamental problem, the proofs contain technical deficiencies. They assume the existence of synchronous coordinates (which could break down if large inhomogeneities develop), and demand that the three-curvature scalar of the spatial section be positive (to prevent a fast recollapse of the universe).

There are improved proofs[14] which show that anisotropies decay exponentially even in a theory without bare cosmological constant but with a scalar field with large amplitude. However, these proofs do not extend to inhomogeneous models, as explicit counterexamples[15,16,6] show.

Some arguments that inflation is generic[17] are based on the claim that in de Sitter space, metric fluctuations decay exponentially. However, the gauge-invariant measure of fluctuations ϕ_H defined by Bardeen[18] remains constant. This means that perturbations are frozen in comoving coordinates, at least when their wavelength exceeds the Hubble radius. This can be seen very nicely in an analysis of scalar field and gravitational perturbations of a chaotic inflation model,[19] and in a nonperturbative Regge calculus model.[16]

In the absence of a general no-hair theorem, an analysis of the initial conditions required for inflation must be based on an analysis of the equations of motion. We shall here mostly focus on the evolution of scalar field inhomogeneities in a homogeneous background metric.

In Ref. 20, the initial conditions required for chaotic inflation were analyzed for a homogeneous scalar field. It was found that, provided $\phi(t_i) > m_{\rm Pl}/\sqrt{4\pi}$, chaotic inflation is generic.

In the case of plane-wave scalar field inhomogeneities and a double-well potential (2.3) there are values for the amplitude and wavelength of ϕ for which dynamical relaxation occurs.[7,21] This is true even when including gravitational inhomogeneities.[22] The stability of dynamical relaxation to thermal[23] and gravitational fluctuations[24] has been shown.

At a linearized level, the stability of chaotic inflation to gravitational and scalar field perturbations was shown.[19] Initial pure gravitational perturbations decay exponentially, whereas the scalar field perturbations are frozen in comoving coordinates.

It is crucial to analyze the initial-condition dependence of inflationary universe models beyond linear theory. Linde has argued qualitatively that initial conditions required for chaotic inflation are likely in at least part of the initial spatial hypersurface.[25] He argues that initial conditions will have

$$V(\phi) \sim T(\phi) \sim K(\phi) \sim m_{\rm Pl}^4 , \tag{2.7}$$

where $T(\phi)$ is the tension energy density stemming from spatial gradients in ϕ and $K(\phi)$ is the kinetic energy density. If $V(\phi)$ is only slightly larger than the other two terms, inflation will set in, leading to a sharp decrease of both $T(\phi)$ and $K(\phi)$ due to redshifting, while $V(\phi)$ remains almost constant.

The first attempt to make these arguments more quantitative was made by Piran[26] who showed that certain plane-wave excitations of a scalar field lead to chaotic-type inflation. The full phase space of single-mode scalar field inhomogeneities was investigated in Ref. 6 (see Sec. III). This analysis reveals the connection between Linde's arguments and the explicit examples of Goldwirth and Piran[15] which do not inflate. In this paper, we extend the analysis of inhomogeneities by including multimode excitations.

III. FRAMEWORK

The analysis presented in this paper is based on classical physics alone. It may well be that a quantum theory of gravity would shed light on the initial-condition problem. Recent developments in quantum cosmology[27] address this issue. However, at present, different prescriptions for the "wave function of the universe" give rather different initial classical configurations. In addition, they do not include inhomogeneities beyond linear approximation.

Since our analysis is based on classical physics, it will break down at very early times. The earliest we can apply our methods is to the Planck time $t_{\rm Pl} = t_i$ determined by $T(t_i) \sim m_{\rm Pl}$, where T is the temperature of radiation. In order to be able to apply classical physics, the energy density of ϕ is constrained by

$$\rho(\phi, t_i) \le m_{\rm Pl}^4 . \tag{3.1}$$

Note that this cutoff renders the single-Fourier-mode measure finite.

We take matter to consist of a homogeneous radiation bath plus a scalar field:

$$T_{\mu\nu} = T_{\mu\nu}(\text{rad}) + T_{\mu\nu}(\phi) . \tag{3.2}$$

If the radiation bath has N spin degrees of freedom, we shall require, at the initial time t_i,

$$\rho(\phi) \le \frac{1}{N} \rho(\text{rad}) . \tag{3.3}$$

Our most severe approximation is to constrain the metric to be homogeneous. Matter influences space-time only via the spatial average of the energy-momentum tensor. To be specific, we consider a flat universe and determine the time evolution of the scale factor $a(t)$ via

$$\left[\frac{\dot a}{a} \right]^2 (t) = \frac{8\pi G}{3} \langle T_{00} \rangle , \tag{3.4}$$

where the angular brackets indicate spatial averaging.

Initially, given a homogeneous radiation bath and (3.3), a flat homogeneous metric will be a very good approximation. However, as soon as $T_{\mu\nu}(\phi)$ starts to dominate over $T_{\mu\nu}(\text{rad})$, the approximation becomes questionable.

In the case of single-Fourier-mode scalar field excitations, we can identify situations in which our approximation will be reasonable, and others in which it will break down. If the wavelength λ of the scalar field excitation is much larger than the Hubble radius H^{-1}, the $g_{\mu\nu}$ obtained from (3.4) may be viewed as a local FRW metric valid near the crest of the wave. For $\lambda \ll H^{-1}$, the metric cannot respond to the rapid variations in the matter fields and will evolve according to the time average of $T_{\mu\nu}$ which is equal to the spatial average of $T_{\mu\nu}$. Thus, only for wavelengths $\lambda \sim H^{-1}$ our approximation will be badly wrong. To treat this case, a complete numerical general relativistic treatment is required, such as the code developed by Goldwirth and Piran.[28]

Our model contains the following degrees of freedom: the scale factor $a(t)$, the radiation temperature $T(t)$, and a scalar field $\phi(\mathbf{x}, t)$ with potential $V(\phi)$ given by (2.3). In Ref. 6 we considered single-mode excitations of ϕ at the initial time t_i:

$$\begin{aligned} \phi(\mathbf{x}, t_i) &= \phi_0 \sin \mathbf{k}\cdot\mathbf{x} , \\ \dot\phi(\mathbf{x}, t_i) &= \dot\phi_0 \sin(\mathbf{k}\cdot\mathbf{x} + \alpha) . \end{aligned} \tag{3.5}$$

We here briefly summarize the results before going on in this paper to study multimode excitations.

There are various possible outcomes of the evolution of our system given by initial conditions (3.5). The first possibility is to obtain chaotic inflation. This will occur if $\phi(\mathbf{x}, t)$ is almost static for $t > t_i$. In this case, $T_{\mu\nu}(\text{rad})$ redshifts whereas $T_{00}(\phi)$ remains approximately constant, soon dominates the total $T_{\mu\nu}$, and leads to a period of inflation.

The second possible outcome—however only for a double-well potential such as (2.3)—is dynamical relaxation. In this case, it follows from (2.6) that the kinetic energy density and tension energy density decay like radiation, namely, as $a^{-4}(t)$. Thus, given that radiation initially dominates T_{00}, it will keep on dominating T_{00} until $\phi < \sigma$ when $V(\phi)$ is constant and eventually starts to dominate T_{00}, leading to a period of inflation.

Finally, $\phi(\mathbf{x})$ may evolve quickly to its ground-state values $\pm\sigma$ and never give rise to inflation.

Demanding that the initial scalar field energy density obey (3.3) leads to the following *a priori* constraints on the initial conditions ϕ_0, $\dot{\phi}_0$, and k in (3.5):

$$\phi_0 < \lambda^{-1/4} T_i \ , \tag{3.6}$$

$$k\phi_0 < T_i^2 \ , \tag{3.7}$$

$$\dot{\phi}_0 < T_i^2 \ . \tag{3.8}$$

We illustrate these constraints on a ϕ_0/k plot (Fig. 3). For simplicity we assume static initial conditions. The allowed range of initial conditions is below curve 1 and to the left of curve 2.

Minimal conditions for dynamical relaxation result from demanding that with an ansatz of dynamical relaxation for $\phi(\mathbf{x},t)$, the energy density in ϕ remain subdominant over at least one oscillation period of ϕ. The potential energy density gives[6]

$$\frac{\phi_0}{k} < \frac{1}{\alpha_1}\left[\frac{100}{N}\right]^{1/2} \lambda^{-1/4}\frac{m_{\rm Pl}}{T_i} \ , \tag{3.9}$$

where

$$\alpha_1 = 10\left[\frac{8\pi^3}{90}\right]^{1/2} \simeq 16.6 \ ; \tag{3.10}$$

the tension energy gives[6]

$$\phi_0 < \frac{1}{\alpha_1}\left[\frac{100}{N}\right]^{1/2} m_{\rm Pl} \ . \tag{3.11}$$

Thus, dynamical relaxation will occur only for initial conditions in the very bottom of Fig. 3 (below curves 3 and 4).

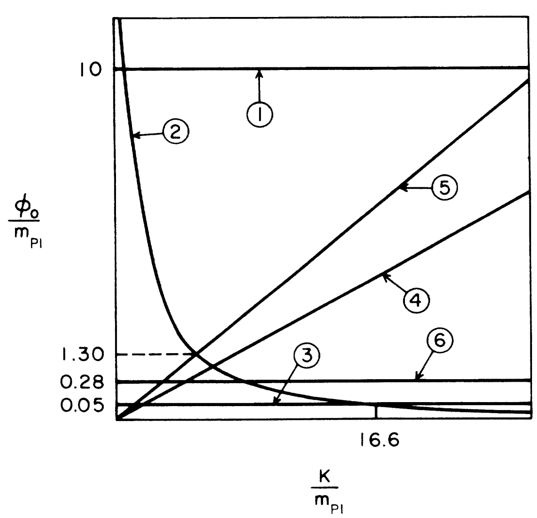

FIG. 3. The single-Fourier-mode initial-condition space (static case). Chaotic inflation occurs for points above line 6, to the left of line 5, below line 1, and to the left of line 2. The latter two conditions come from *a priori* energy density constraints.

Minimal conditions for chaotic inflation to occur are that the scalar field configuration evolve to a configuration dominated by potential energy, and that the slow-rolling ansatz ($\ddot{\phi}$ negligible) be self-consistent. The first condition leads to the requirement[6]

$$\frac{\phi_0}{k} > \frac{2}{\alpha_1}\left[\frac{100}{N}\right]^{1/4} \lambda^{-1/4}\frac{m_{\rm Pl}}{T_i} \ , \tag{3.12}$$

the second to[26]

$$\phi_0 > \left[\frac{1}{4\pi}\right]^{1/2} m_{\rm Pl} \ . \tag{3.13}$$

Thus, the domain for chaotic inflation is to the left of curve 5 and above curve 6.

In the initial-condition space allowed by (3.6) and (3.7) but not included in the region $\mathcal{D}$ of dynamical relaxation or $\mathcal{C}$ of chaotic inflation, the above arguments are too simple to make any definite statements. In Ref. 6 we discussed a subspace of this set in which it can be shown that no inflation results. In models without a double-well potential, region $\mathcal{D}$ becomes a set of initial conditions for which there is no inflation.

At this point it is instructive to discuss previous work in the context of Fig. 3. Linde[25] assumes initial conditions with $V(\phi) \sim T(\phi) \sim m_{\rm Pl}^4$. These initial conditions sit close to point (0.1,10) in Fig. 3 and thus clearly in the region for which chaotic inflation is expected. Goldwirth and Piran[15] consider single-mode excitations and conclude that the "noncausal" wavelength $\lambda > H^{-1}$ is required in order to obtain chaotic inflation. In the light of Fig. 3, this is not surprising. For $\lambda < H^{-1}$ the initial conditions clearly lie outside the region of chaotic inflation. Obviously, in order to draw physical conclusions as to how generic inflation is, we must consider initial conditions with more than one mode excited.

IV. TWO-MODE EXCITATIONS

In the case of realistic initial conditions, many Fourier modes will be excited, with wavelengths both smaller and larger than H^{-1}. We first analyze the simpler case of initial two-mode excitations with (for simplicity) static initial conditions

$$\phi(\mathbf{x},t_i) = A_1 \sin(k_1 x + \alpha_1) + A_2 \sin(k_2 x + \alpha_2) \ . \tag{4.1}$$

In particular, we are interested in determining the late time behavior if the first mode lies in $\mathcal{C}$ and the second in $\mathcal{D}$ (see Fig. 3).

We consider weakly self-coupled scalar field models, the only ones which do not give rise to fluctuations in excess of those observed.[29] In this case, we expect both Fourier modes to evolve independently. Since the energy density of the second mode decays as $a^{-4}(t)$ whereas that of the first remains almost constant, we expect the first mode to dominate the late time behavior. Note that this is essentially the argument given by Linde[25] in support of the claim that inflation is generic.

The above intuition could be proved wrong by the following effects: (a) nonlinearities in the equation of motion of ϕ could lead to a significant change in the evolution of lution of the first mode and prevent the onset of inflation; (b) the presence of the second mode will change the expansion rate of the universe and can thereby effect the evolution of the first mode. This is a gravitational back-reaction effect.

We do not expect either effect to be important. Nonlinearities should be negligible because the self-coupling λ is very small. Concerning the gravitational back reaction: adding the additional scalar field excitation will add more energy density, will thus increase the Hubble damping constant H in the equation of motion for the "homogeneous" mode, and thus should not significantly effect the onset of inflation.

In the following, we shall demonstrate that the above conjectures are indeed true. In particular, we shall show that inflation occurs even if the energy density in the homogeneous mode is much smaller than the density in the inhomogeneous mode. [We are calling the mode $(A_1,k_1)\in\mathcal{C}$ the "homogeneous" mode, the mode $(A_2,k_2)\in\mathcal{D}$ the "inhomogeneous" mode.]

In the case of two-mode excitations it is easy to address (a). The extra force term in the equation of motion for $(A_1,k_1)=\phi_1$ due to $(A_2,k_2)=\phi_2$ is

$$\delta F=-2\lambda|\phi_2|^2\phi_1 \ . \tag{4.2}$$

For any $\phi_1\in\mathcal{C}$ and $\phi_2\in\mathcal{D}$, δF is smaller than the force F from ϕ_1 alone:

$$F=-2\lambda|\phi_1|^2\phi_1 \ . \tag{4.3}$$

Hence, nonlinearities do not disrupt the evolution of ϕ_1.

Let us now address (b). If the energy density is dominated at t_i by the homogeneous mode ϕ_1, there is nothing to prove since ϕ_2 will not change the gravitational back reaction. Thus, we now assume that at t_i the energy in ϕ is dominated by the inhomogeneous mode ϕ_2. In this case, the universe is radiation dominated and the gravitational back reaction differs. In the evolution equation for ϕ_1, H is no longer constant but decreases as t^{-1}. Hence, $\phi_1(t)$ will be affected. We now show that nevertheless, at late times $\phi_1(t)$ will dominate $T_{\mu\nu}$.

We first solve the slow-rolling equation for ϕ_1:

$$3H\dot{\phi}_1=-2\lambda\phi_1^3 \tag{4.4}$$

for $H(t)=H_i t_i/t$. The solution is

$$\phi_1(t)=\phi_1(t_1)\left[1+\frac{2\lambda}{3H_i t_i}(t^2-t_i^2)\phi_1^2(t_i)\right]^{-1/2} . \tag{4.5}$$

The onset of inflation can be prevented if the slow-rolling approximation breaks down before ϕ_1 dominates the energy density. Using $H_i=(2t_i)^{-1}$, the condition for breakdown becomes

$$\lambda t^2\phi_1(t_i)^2\sim 1 \ . \tag{4.6}$$

If $\phi_1(t_i)$ is α times the maximal amplitude, i.e.,

$$\phi_1(t_i)=\alpha\lambda^{-1/4}m_{\mathrm{Pl}} \ , \tag{4.7}$$

then radiation domination will end at

$$\left[\frac{t}{t_i}\right]=\alpha^{-2} \ . \tag{4.8}$$

Using $t_i^{-1}=(32\pi/3)m_{\mathrm{Pl}}$, (4.7) and (4.8), the condition (4.6) becomes

$$\alpha<\left[\frac{3}{32\pi}\right]^{1/2}\lambda^{1/4} \ , \tag{4.9}$$

which implies a value of $\phi_1(t_i)$, which conflicts with the criterion (3.13) for $\phi_1\in\mathcal{C}$. Hence, gravitational back reaction from a single mode $\phi_2\in\mathcal{D}$ cannot disrupt the onset of chaotic inflation due to a mode ϕ_1 in $\mathcal{C}$, even if initially the energy density in ϕ_2 dominates. In this sense, chaotic inflation is an attractor in initial-condition space.

The arguments made above are based on various approximations and must be checked numerically. We have solved the equation of (2.4) for a real scalar field with the double-well potential (2.3) numerically in an expanding universe. The expansion rate is given by (3.4); and $T_{\mu\nu}$ contains both the contribution from ϕ and from the homogeneous radiation bath with temperature $T(t)\sim a^{-1}(t)$. The initial time t_i in the numerical runs is the Planck time, as in the preceding discussion. We assume planar symmetry [this reduces the problem to a $(1+1)$-dimensional problem]. Since computer time increases as the coupling constant λ decreases we choose a fairly large value of λ (by the standards of inflationary universe models), namely, $\lambda=10^{-4}$. We also chose $\sigma=0$ (without loss of generality in a study of chaotic inflation).

In one set of simulations we chose two modes $\phi_1\in\mathcal{C}$ and $\phi_2\in\mathcal{D}$ with equal energy densities and determined the value a_c of the scale factor at which inflation commences (i.e., at which pressure p becomes smaller than $-1/3\rho$). We then reduced the amplitude of the homogeneous modes in steps from $\lambda^{-1/4}m_{\mathrm{Pl}}$ to $0.4\lambda^{-1/4}m_{\mathrm{Pl}}$, the lowest value for which we get a substantial amount of inflation in the absence of the inhomogeneous mode. We kept the second mode unchanged and determined a_c as a function of the amplitude of the first mode.

In Figs. 4 and 5 we show the results of a simulation with initial amplitudes and wave numbers $(A_1,k_1)=(10m_{\mathrm{Pl}},0.1m_{\mathrm{Pl}})$ and $(A_2,k_2)=(0.05m_{\mathrm{Pl}},20m_{\mathrm{Pl}})$. In Fig. 4 we plot how the two modes lie in initial-condition space (A,k) as a function of the varying A_1. Note that if A_1 is reduced by a factor α, the energy density in ϕ_1 decreases by a factor α^2. Thus, at the end, our homogeneous mode contains only 16% of the energy density of the inhomogeneous mode. In Fig. 5(a) we plot a_c as a function of A_1. Since a_c is determined by when the potential energy starts to dominate, we expect

$$a_c\sim A_1^{-1} \ , \tag{4.10}$$

which is satisfied to good accuracy. In Fig. 5(b) we plot

$$R(A_1)=\frac{a_c(A_1)}{a_c^{|0|}(A_1)}-1 \ , \tag{4.11}$$

where $a_c^{|0|}$ is the critical value of a without the inhomogeneous mode. The striking result is how minor the

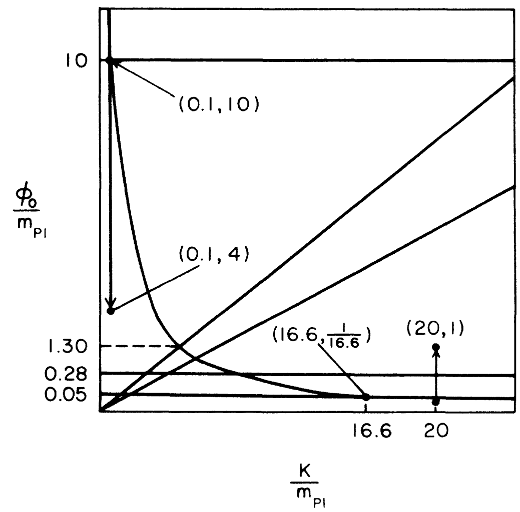

FIG. 4. A representation of the two Fourier mode runs on the initial-condition space of Fig. 3. The two lines with arrows show how the initial data are varied. In one set of runs, the amplitude of the homogeneous mode is reduced, keeping the inhomogeneous mode constant. In the second, the amplitude of the inhomogeneous mode is increased, keeping the homogeneous mode fixed.

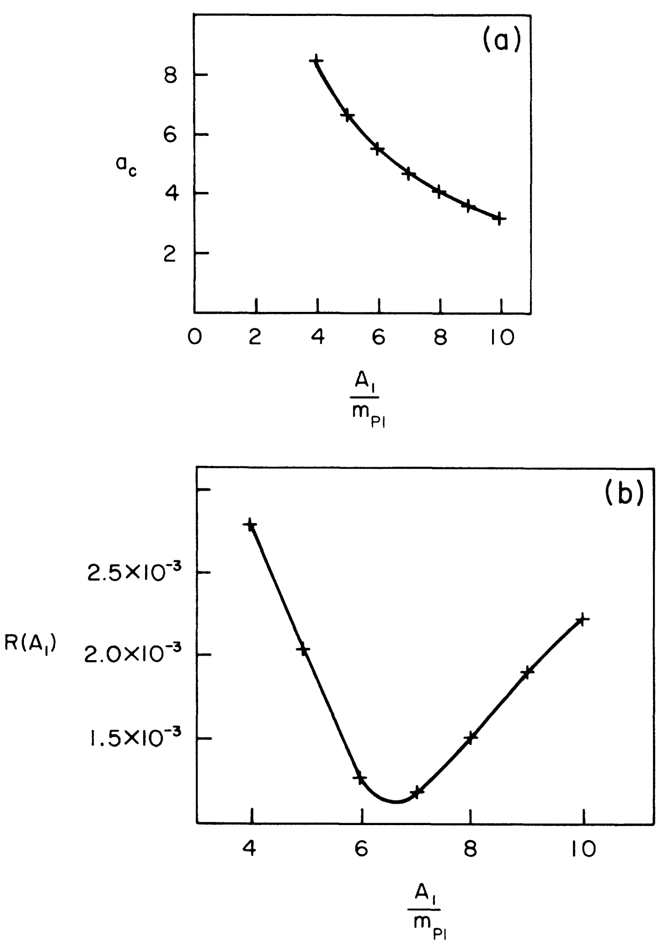

FIG. 5. The scale factor a_c when inflation begins as a function of the initial amplitude of the homogeneous mode for the first series of runs (a). In (b), the relative change in a_c as a function of A_1 is shown.

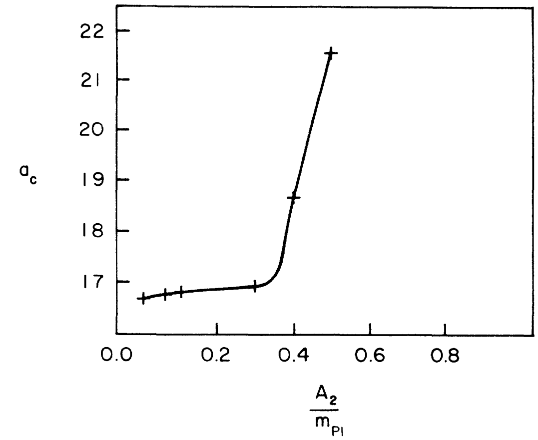

FIG. 6. The critical value a_c as a function of A_2 in the second series of runs.

effect of inhomogeneities is on the onset of inflation.

In another set of runs we considered initial modes $(A_1,k_1)=(2m_{Pl},0.1m_{Pl})$ and $(A_2,k_2)=(\frac{1}{16.6}m_{Pl}, 16.6m_{Pl})$ and gradually raised the amplitude of A_2 to $0.5m_{Pl}$. With $A_2=\frac{1}{16.6}m_{Pl}$, the inhomogeneous mode contains 25 times the energy density of the homogeneous mode; with $A_2=0.5m_{Pl}$, the ratio is about 2×10^3. However, in all case the homogeneous mode prevails at late times. The onset of inflation changes from $a_c\simeq17$ to $a_c\simeq22$ (Fig. 6).

We conclude that a single inhomogeneous mode cannot disrupt the onset of chaotic inflation due to a mode in $\mathcal{C}$, even if it contains initially much more energy.

V. MULTIMODE EXCITATIONS

We now extend the stability considerations of Sec. IV to the more realistic case in which one "homogeneous" mode ϕ_1 and many (n) inhomogeneous modes are excited. To make an analytical analysis possible we assume that the n short-wavelength modes have random phases. We denote the modes by

$$\phi_i=\phi(\mathbf{k}_i)=(A_i,\mathbf{k}_i)\ , \tag{5.1}$$

where as before A_i and $\mathbf{k}_i$ are amplitude and wave vector of the mode.

For simplicity, we take all n inhomogeneous modes to lie in a rather narrow range of wave numbers. Hence it is not unreasonable to take their amplitudes to be equal. Using $k_i\gtrsim m_{Pl}$ and demanding that the tension energy density not exceed m_{Pl}^4, we obtain the following condition on the amplitude A_i:

$$A_i<n^{-1/2}m_{Pl}\ . \tag{5.2}$$

With these assumptions, it is easy to show that once again the evolution of the inhomogeneous mode is not disrupted by the nonlinear coupling to the other modes via the force term $V'(\phi)$ in the equation of motion. The force term in the equation of motion for ϕ_1 is

$$V'(\phi_1)=2\lambda|\phi_1|^2\phi_1+2\lambda\sum_{\mathbf{k}_i}|\phi_{\mathbf{k}_i}|^2\phi_1 + 2\lambda\sum_{\mathbf{k}_1,\mathbf{k}_2\neq\mathbf{k}}\phi_{\mathbf{k}_1}\phi_{\mathbf{k}_2}\phi_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2}\,, \tag{5.3}$$

where the sums run over all inhomogeneous modes. By (5.2), the second term is bounded by

$$2\lambda\sum_{\mathbf{k}_i}|\phi_{\mathbf{k}_i}|^2\phi_1<2\lambda m_{\mathrm{Pl}}^2\phi_1\,, \tag{5.4}$$

and is thus negligible compared to the self-coupling of ϕ_1. In the last term in (5.3), all modes are of short wavelength. Hence,

$$2\lambda\sum_{\mathbf{k}_1\mathbf{k}_2\neq\mathbf{k}}\phi_{\mathbf{k}_1}\phi_{\mathbf{k}_2}\phi_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2}<2\lambda\eta^{-1/2}m_{\mathrm{Pl}}^3\,, \tag{5.5}$$

which is smaller than the other two terms. Thus, the evolution of ϕ_1 is not disrupted via nonlinear matter couplings to short-wavelength modes.

The second issue to check is the effect of many inhomogeneous modes via gravitational back reaction. However, the back-reaction analysis of Sec. IV [Eqs. (4.4)–(4.9)] depended only on the energy density in inhomogeneous modes and applies equally well to the case discussed in this section. We conclude that, provided the initial energy density in inhomogeneous modes is bounded by m_{Pl}^4, gravitational back reaction cannot prevent the onset of chaotic inflation.

VI. DISCUSSION AND CONCLUSIONS

We have analyzed the evolution of classical scalar field configurations in a model which initially is dominated by a homogeneous radiation bath. It is convenient to Fourier transform the initial scalar field configuration.

We considered initial conditions in which a single large wavelength ($\lambda > H^{-1}$) mode is excited with an amplitude sufficient to, on its own, lead to chaotic inflation. We have then shown that this behavior is not disrupted when adding short-wavelength excitations, even if the energy density of the short waves exceeds that of the "homogeneous" mode. The energy density in short waves decays as radiation, and the homogeneous mode determines the late time evolution and the onset of inflation. In this case, chaotic inflation is an attractor in initial-condition space.

We have not addressed the question of how likely it is to have a mode in $\mathcal{C}$ excited. It is important not to forget that these modes have a wavelength larger than the initial Hubble radius. However, our view is that, after Fourier transforming some general initial scalar field configuration, it is unlikely not to have some long-wave length modes excited. However, we do not expect such a mode to dominate the energy density—and hence our results are important in that they vastly extend the space of initial conditions for which chaotic inflation has been shown to arise.

The main deficiency in our analysis is that we neglect gravitational perturbations. In particular, we assume that the metric is initially homogeneous and flat. Large initial inhomogeneities and deviations from flatness can prevent any inflation. Since chaotic inflation is stable to linear gravitational perturbations,[19] we do not expect our conclusions to be qualitatively changed by including gravitational perturbations at a later stage. Goldwirth and Piran[28] have developed a code to follow inhomogeneous gravitational and scalar field configurations. Using this code, they have checked[30] some of the examples we analyzed in Sec. IV.

Our analysis requires some severe approximations. The most important one is our crude way of including gravitational back reaction. We have commented on this in Sec. III. Nevertheless, we hope to have pointed out the main mechanisms which make chaotic inflation an attractor in initial-condition space.

ACKNOWLEDGMENTS

We are grateful to Dalia Goldwirth and Tsvi Piran for many interesting discussions and for checking some of our examples with their code. We also acknowledge a useful discussion with Varun Sahni. One of us (R.B.) would like to thank Dalia Goldwirth and Tsvi Piran for their hospitality during a visit to Israel. This work was supported in part by the U.S. DOE under Grant No. DE-AC02-76ER03130 Tasks K&A, and by the Alfred P. Sloan Foundation.

[1] A. Guth, Phys. Rev. D **23**, 347 (1981).
[2] A. Linde, Phys. Lett. **108B**, 389 (1982); A. Albrecht and P. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982).
[3] A. Linde, Phys. Lett. **129B**, 177 (1983).
[4] G. Mazenko, W. Unruh, and R. Wald, Phys. Rev. D **31**, 273 (1985).
[5] For recent reviews, see, e.g., A Linde, *Inflation and Quantum Cosmology* (Academic, Boston, 1990); K. Olive, Phys. Rep. (to be published); R. Brandenberger, in Proceedings of the Scottish Universities Summer School in Physics, 1989 (unpublished).
[6] J. Kung and R. Brandenberger, Phys. Rev. D **40**, 2532 (1989).
[7] A. Albrecht and R. Brandenberger, Phys. Rev. D **31**, 1225 (1985).
[8] For reviews, see, e.g., A. Linde, Rep. Prog. Phys. **42**, 389 (1979); R. Brandenberger, Rev. Mod. Phys. **57**, 1 (1985).
[9] A. Guth and E. Weinberg, Nucl. Phys. **B212**, 321 (1983).
[10] A. Guth and S.-Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982); S. Hawking, Phys. Lett. **115B**, 295 (1982); A. Starobinsky, *ibid.* **117B**, 175 (1982); J. Bardeen, P. Steinhardt, and M. Turner, Phys. Rev. D **28**, 679 (1983).
[11] A. Starobinsky, Pis'ma Zh. Eksp. Teor. Fiz. **37**, 55 (1983) [JETP Lett **37**, 66 (1983)]; R. Wald, Phys. Rev. D **28**, 2118 (1983); L. Jensen, and J. Stein-Schabes, *ibid.* **34**, 931 (1986); E.

Martinez-Gonzalez and B. Jones, Phys. Lett. **167B**, 37 (1986); M. Turner and L. Widrow, Fermilab Report No. PUB-86/49-A, 1986 (unpublished).

[12]L. Jensen and J. Stein-Schabes, Phys. Rev. D **35**, 1146 (1987); J. Stein-Schabes, *ibid.* **35**, 2345 (1987).

[13]S. Hawking and G. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973).

[14]I. Moss and V. Sahni, Phys. Lett. B **178**, 159 (1986); T. Rothman and G. Ellis, *ibid.* **180**, 19 (1986).

[15]D. Goldwirth and T. Piran, Phys. Rev. Lett. **64**, 2852 (1990).

[16]T. Piran and R. Williams, Phys. Lett. **163B**, 331 (1985).

[17]J. Frieman and C. Will, Astrophys. J. **259**, 437 (1982).

[18]J. Bardeen, Phys. Rev. D **22**, 1882 (1980).

[19]H. Feldman and R. Brandenberger, Phys. Lett. B **227**, 359 (1989).

[20]V. Belinsky, L. Grishchuk, I. Khalatnikov, and Ya. Zel'dovich, Phys. Lett. **155B**, 2332 (1985); L. Kofman, A. Linde, and A. Starobinsky, *ibid.* **157B**, 36 (1985); T. Piran and R. Williams, *ibid.* **163B**, 331 (1985).

[21]A. Albrecht, R. Brandenberger, and R. Matzner, Phys. Rev. D **32**, 1280 (1985); **35**, 429 (1987).

[22]H. Kurki-Suonio, R. Matzner, J. Centrella, and J. Wilson, Phys. Rev. D **35**, 435 (1987).

[23]R. Brandenberger, H. Feldman, and J. MacGibbon, Phys. Rev. D **37**, 2071 (1988); H. Feldman, *ibid.* **38**, 459 (1988).

[24]R. Brandenberger and H. Feldman, Phys. Lett. B **220**, 361 (1989).

[25]A. Linde, *Inflation and Quantum Cosmology* (Ref. 5); Mod. Phys. Lett. A **1**, 81 (1985); Phys. Lett. **162B**, 281 (1985).

[26]T. Piran, Phys. Lett. B **181**, 238 (1986).

[27]J. Hartle and S. Hawking, Phys. Rev. D **28**, 2960 (1983); A. Vilenkin, *ibid.* **30**, 509 (1984); A. Linde, Zh. Eksp. Teor. Fiz. **87**, 369 (1984) [Sov. Phys. JETP **60**, 211 (1984)].

[28]D. Goldwirth and T. Piran, Phys. Rev. D **40**, 3263 (1989).

[29]A. Guth and S.-Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982); S. Hawking, Phys. Lett. **115B**, 295 (1982); A. Starobinsky, *ibid.* **117B**, 175 (1982); J. Bardeen, P. Steinhardt, and M. Turner, Phys. Rev. D **28**, 679 (1983).

[30]D. Goldwirth and T. Piran (unpublished).