Meson properties in a light-cone quark model

Nihal A. Aboud and John R. Hiller
Department of Physics, University of Minnesota—Duluth, Duluth, Minnesota 55812
(Received 3 April 1989)

A potential with three parameters is used in a light-cone equation for a quark-antiquark pair. Spin dependence is not included. The nonrelativistic limit is the potential of the quark model of Eichten *et al.* Numerical methods are used to solve the eigenvalue problem. The parameters and quark masses are varied to obtain a reasonable fit to masses of low-lying hidden-flavor mesons. Additional masses are calculated for higher states and for flavored mesons. Also, properties of the pion are calculated from a wave function obtained with the numerical method. Spectroscopic assignments are problematic because the light-front formulation breaks rotational invariance. A need for an improved treatment of angular momentum eigenstates is indicated.

I. INTRODUCTION

Relativistic quark models can take many forms.[1,2] Here we consider a confining potential in the context of a light-front bound-state equation.[3–5] The eigenfunctions determined by the equation represent states that are stationary with respect to light-cone "time" $t+z$. Only a single quark-antiquark pair, the valence-meson Fock state, is included.

The equation is solved numerically, with a combination of quadrature[6] and finite-difference[7] methods. Symmetry arguments are used to make a tentative assignment of eigenstates to spectroscopic levels. The quark masses and the three parameters of the potential are then adjusted to fit various meson masses.[8]

Since wave functions are also obtained, properties other than mass can be calculated. We focus on the pion and calculate its electromagnetic form factor, root-mean-square (rms) radius, and decay constant.

An outline of the remaining sections is as follows. In Sec. II we give a brief development of the light-cone equation and, following the introduction of the potential model, we reduce the equation to dimensionless form. Quadrature and finite-difference formulas are then used to convert the equation to a matrix eigenvalue problem which is solved by standard means. In Sec. III we present and discuss the results. These include the determined parameter values, meson spectra, and pion properties. Finally, a short summary and additional remarks are given in Sec. IV.

II. THE LIGHT-FRONT EIGENVALUE PROBLEM

A. General form

We define light-cone coordinates[4] in terms of Cartesian coordinates x^i and time x^0:

$$x^{\pm}=x^0\pm x^3, \quad \mathbf{r}_\perp=(x^1,x^2) . \tag{2.1}$$

Similarly, light-cone components of momentum are defined as

$$p^{\pm}=p^0\pm p^3, \quad \mathbf{p}_\perp=(p^1,p^2) . \tag{2.2}$$

The light-cone time is chosen to be x^+. From the scalar product of position and momentum,

$$px=\tfrac{1}{2}x^+p^-+\tfrac{1}{2}x^-p^+-\mathbf{x}_\perp\cdot\mathbf{p}_\perp , \tag{2.3}$$

we see that p^- may be interpreted as the light-cone energy. On the mass shell it must satisfy

$$p^-=\frac{m^2+\mathbf{p}_\perp^2}{p^+} , \tag{2.4}$$

with m the particle mass.

Given a light-cone Hamiltonian $H_{\rm LC}$, the formal statement of the eigenvalue problem is

$$H_{\rm LC}\Psi(P)=P^-\Psi(P) , \tag{2.5}$$

where Ψ is an eigenstate that depends on the total momentum P. The Hamiltonian is the sum of a kinetic-energy operator K and potential-energy operator V. Although K is diagonal in particle number, V usually is not. Therefore, in general, Eq. (2.5) is a many-body problem.

An explicit representation of the action of K can be written in terms of Fock states of n particles $|n:p_1\ldots p_n\rangle$. The Fock-state expansion for a state Ψ is

$$\Psi=\sum_n\int[dx]_n[d^2k_\perp]_n\psi_n(x_1\cdots x_n,\mathbf{k}_{\perp1}\cdots\mathbf{k}_{\perp n}) \times|n:p_1\cdots p_n\rangle \tag{2.6}$$

with

$$x_i\equiv p_i^+/P^+, \quad \mathbf{k}_{\perp i}\equiv\mathbf{p}_{\perp i}-x_i\mathbf{P}_\perp , \tag{2.7}$$

$$[dx]_n\equiv4\pi\delta\left[1-\sum_{i-1}^n x_i\right]\prod_{i=1}^n\frac{dx_i}{4\pi\sqrt{x_i}} , \tag{2.8}$$

and

$$[d^2k_\perp]_n\equiv4\pi^2\delta\left[\sum_{i=1}^n\mathbf{k}_{\perp i}\right]\prod_{i=1}^n\frac{d^2k_{\perp i}}{4\pi^2} . \tag{2.9}$$

Helicity labels are suppressed. The momentum fractions

x_i and transverse momenta $\mathbf{k}_{\perp i}$ are frame independent.[4] The action of K on this expansion is

$$K\Psi=\frac{1}{P^+}\sum_n\int[dx]_n[d^2k_\perp]_n\left[\sum_{i=1}^n\frac{\mathbf{k}_{\perp i}^2+m_i^2}{x_i}\right]\times\psi_n(x_1\cdots x_n,\mathbf{k}_{\perp 1}\cdots\mathbf{k}_{\perp n})|n:p_1\cdots p_n\rangle+\frac{\mathbf{P}_\perp^2}{P^+}\Psi\ . \tag{2.10}$$

The eigenvalue problem (2.5) can now be written as

$$\sum_n\int[dx]_n[d^2k_\perp]_n\left[\sum_{i=1}^n\frac{\mathbf{k}_{\perp i}^2+m_i^2}{x_i}\right]\times\psi_n(x_1\cdots x_n,\mathbf{k}_{\perp 1}\cdots\mathbf{k}_{\perp n})|n:p_1\cdots p_n\rangle+P^+V\Psi=M^2\Psi\ , \tag{2.11}$$

where $M=\sqrt{P^2}$ is the total mass. The second term in $K\Psi$ has canceled the term of the same form in $P^-\Psi$. The total mass M takes the role of the eigenvalue. The potential operator remains unspecified. An explicit choice typically leads to integral operators acting on the Fock-state wave functions ψ_n.

Boundary conditions on ψ_n follow from the requirement that the expectation value of the kinetic energy be finite. The conditions are

$$\psi_n|_{x_i=0}=0,\quad|\mathbf{k}_{\perp i}|\psi_n|_{|\mathbf{k}_{\perp i}|=\infty}=0\ . \tag{2.12}$$

To describe mesons in a simple way, we truncate the Fock-state expansion to only the valence state of a quark-antiquark pair. The single remaining wave function must then satisfy the equation

$$\left[\sum_{i-1}^2\frac{k_{\perp i}^2+m_i^2}{x_i}\right]\psi(x_1,x_2,\mathbf{k}_{\perp 1},\mathbf{k}_{\perp 2})+P^+\int[dx']_2[d^2k'_\perp]_2U(x_1,x_2,\mathbf{k}_{\perp 1},\mathbf{k}_{\perp 2};x'_1,x'_2,\mathbf{k}'_{\perp 1},\mathbf{k}'_{\perp 2})\psi(x'_1,x'_2,\mathbf{k}'_{\perp 1},\mathbf{k}'_{\perp 2})=M^2\psi(x_1,x_2,\mathbf{k}_{\perp 1},\mathbf{k}_{\perp 2})\ . \tag{2.13}$$

The kernel U is derived from a matrix element of the potential-energy operator. Conservation of momentum implies that

$$x_1=1-x_2\equiv x,\quad\mathbf{k}_{\perp 1}=-\mathbf{k}_{\perp 2}\equiv\mathbf{k}_\perp\ . \tag{2.14}$$

Therefore, we can write (2.13) as

$$\left[\frac{k_\perp^2}{x(1-x)}+\frac{m_1^2}{x}+\frac{m_2^2}{1-x}\right]\psi(x,\mathbf{k}_\perp)+P^+\int dx'd^2k'_\perp\tilde{U}(x,\mathbf{k}_\perp;x',\mathbf{k}'_\perp)\psi(x',\mathbf{k}'_\perp)=M^2\psi(x,\mathbf{k}_\perp)\ . \tag{2.15}$$

The kernel $\tilde{U}$ is proportional to U. We now turn to a discussion of a model for this kernel.

B. A model for the potential

For the purposes of this study, the choice of model for the potential is guided by a preference for simplicity and the need for confinement. Here simplicity means ease of calculation and a minimum of parameters. Since only the quark-antiquark Fock state is considered, coupling to multihadron states is not included.

The form chosen for the kernel $\tilde{U}(x-x',\mathbf{k}_\perp-\mathbf{k}'_\perp)$ is

$$\tilde{U}(y,q_\perp)=-\frac{\bar{m}g/\pi^2}{q_\perp^2+(2\bar{m}y)^2}-\frac{2\bar{m}a^2/\pi^2}{[q_\perp^2+(2\bar{m}y)^2]^2}+C\delta(y)\delta(\mathbf{q}_\perp)\ , \tag{2.16}$$

with $\bar{m}=\frac{1}{2}(m_1+m_2)$. In the nonrelativistic limit, Eq. (2.15) then reduces to a momentum-space Schrödinger equation that contains the Fourier transform of the potential of Eichten *et al.*:[9]

$$V=-\frac{g}{r}+a^2r+C\ . \tag{2.17}$$

Thus, in this limit, rotational symmetry is recovered. A potential with one less parameter, but no nonrelativistic rotational symmetry, has also been considered.[10]

The choice of $\tilde{U}$ is not unique. Part of the motivation for the particular form selected is that the Fourier transform from $\mathbf{k}_\perp$ to $\mathbf{r}_\perp$ can be calculated. This provides a means to reduce the three-dimensional integral operator in (2.15) to a one-dimensional operator. The equation then becomes

$$\left[\frac{-\nabla_\perp^2}{x(1-x)}+\frac{m_1^2}{x}+\frac{m_2^2}{1-x}+MC\right]\psi(x,\mathbf{r}_\perp)-P^+\int_0^1dx'\left[\frac{2\bar{m}g}{\pi}K_0(2\bar{m}r_\perp|x-x'|)+\frac{a^2r_\perp}{\pi|x-x'|}K_1(2\bar{m}r_\perp|x-x'|)\right]\psi(x',\mathbf{r}_\perp)=M^2\psi(x,\mathbf{r}_\perp) \tag{2.18}$$

with $\nabla_\perp^2$ the transverse Laplacian and the K_n modified Bessel functions.

Some simplifications of the eigenvalue problem can be made. In the center-of-mass frame, P^+ is equal to M, the eigenvalue, but we assume that

$$P^+ \simeq m_1 + m_2 = 2\bar{m} \ . \tag{2.19}$$

We define a dimensionless coordinate ρ, masses μ_i, and coupling g',

$$\rho = \bar{m} r_\perp, \quad \mu_i = \frac{m_i}{\bar{m}}, \quad g' = \left[\frac{a}{\bar{m}} \right]^2 , \tag{2.20}$$

and a dimensionless eigenvalue

$$\xi = \frac{M^2 - \bar{m}C}{4\bar{m}^2} \ . \tag{2.21}$$

In terms of these new variables, and the azimuthal angle ϕ, we have

$$\left\{ \frac{-1}{x(1-x)} \left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left[\rho \frac{\partial}{\partial \rho} \right] + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \right] + \frac{\mu_1^2}{x} + \frac{\mu_2^2}{1-x} \right\} \psi(x,\rho,\phi)$$
$$- \frac{2}{\pi} \int_0^1 dx' \left[2gK_0(2\rho|x-x'|) + \frac{g'\rho}{|x-x'|} K_1(2\rho|x-x'|) \right] \psi(x',\rho,\phi) = 4\xi \psi(x,\rho,\phi) \ . \tag{2.22}$$

There are then only three dimensionless parameters: g, g', and $\mu_1 = 2 - \mu_2$.

Standard separation of variables applies to the ϕ dependence. We find

$$\psi(x,\rho,\phi) = \frac{1}{\sqrt{\rho}} u(x,\rho) \frac{1}{\sqrt{2\pi}} e^{im\phi} \tag{2.23}$$

with m an integer. The new function u obeys the equation

$$\left[\frac{1}{x(1-x)} \left[-\frac{\partial^2}{\partial \rho^2} + \frac{m^2 - \frac{1}{4}}{\rho^2} \right] + \frac{\mu_1^2}{x} + \frac{\mu_2^2}{1-x} \right] u(x,\rho)$$
$$- \frac{2}{\pi} \int_0^1 dx' \left[2gK_0(2\rho|x-x'|) + \frac{g'\rho}{|x-x'|} K_1(2\rho|x-x'|) \right] u(x',\rho) = 4\xi u(x,\rho) \ . \tag{2.24}$$

The appropriate boundary conditions are

$$u(0,\rho) = u(1,\rho) = 0, \quad u(x,0) = u(x,\infty) = 0 \ . \tag{2.25}$$

To actually obtain solutions to this eigenvalue problem, numerical techniques are required.

C. A numerical method

There are a variety of numerical methods for momentum-space integral equations.[2,11] Caution is required when the kernel is singular, as it is in this case. The best method that we have found for the problem at hand is the Gauss-Chebyshev-Lobatto (GCL) method.[6] It converts the singular integral over x into an approximate sum and the integro-differential equation (2.24) into a coupled set of differential equations. A standard finite difference formula completes the conversion into a matrix eigenvalue problem where the matrix can be arranged to be symmetric. The matrix problem is readily solved by standard means.

The advantage of the GCL method is that no integrals need be done explicitly. This is because it is based on the following Gauss-Chebyshev (GC) and Lobatto-Chebyshev (LC) quadrature formulas:[6]

$$\int_{-1}^1 \frac{dt' h(t')}{\sqrt{1-t'^2}} \simeq \begin{cases} \dfrac{\pi}{N} \displaystyle\sum_{k=1}^{N} h(t_k), & \text{GC} \ , \\ \dfrac{\pi}{N} \displaystyle\sum_{k=1}^{N-1} h(y_k), & \text{LC}, \quad h(\pm 1) = 0 \ , \end{cases} \tag{2.26}$$

$$\mathcal{P} \int_{-1}^1 \frac{dt'}{\sqrt{1-t'^2}} \frac{h(t')}{t'-t}$$
$$\simeq \begin{cases} \dfrac{\pi}{N} \displaystyle\sum_{k=1}^{N} \frac{h(t_k)}{t_k - t} + S_G(t) h(t), & \text{GC} \ , \\ \dfrac{\pi}{N} \displaystyle\sum_{k=1}^{N-1} \frac{h(y_k)}{y_k - t} + S_L(t) h(t), & \text{LC}, \quad h(\pm 1) = 0 \ , \end{cases} \tag{2.27}$$

where $\mathcal{P}$ denotes the principal value,

$$S_G(t) = \pi \frac{U_{N-1}(t)}{T_N(t)}, \quad S_L(t) = \frac{\pi T_N(t)}{(t^2-1) U_{N-1}(t)} \ , \tag{2.28}$$

with T_N and U_N Chebyshev polynomials of the first and

second kind. The quadrature points

$$t_k=\cos\frac{(2k-1)\pi}{2N},\quad y_k=\cos\frac{k\pi}{N} \tag{2.29}$$

are the roots of T_N and U_{N-1}, respectively. The second set of quadrature formulas, Eq. (2.27), applies to the singular part of K_1; the leading behavior of $K_1(z)$ at small z is $1/z$. Both $K_0(z)$ and $(1/z)K_1(z)$ also contain logarithmic singularities, but these were found to be handled well by the standard formulas, Eq. (2.26), in tests with an analytically integrable function.

To apply these quadrature formulas, we change variables from x to $t=2x-1$, define a new eigenfunction

$$h_m(t,\rho)=\sqrt{1-t^2}u_m\left[\frac{1+t}{2},\rho\right] \tag{2.30}$$

and define the nonsingular part of K_1:

$$\bar{K}_1(z)=K_1(z)-\frac{1}{z}\ . \tag{2.31}$$

The contribution of the singular part of (2.24) can then be written as

$$\int_{-1}^{1}\frac{dt'}{\sqrt{1-t'^2}}\frac{h_m(t',p)}{(t-t')^2}=\frac{d}{dt}\int_{-1}^{1}\frac{dt'}{\sqrt{1-t'^2}}\frac{h_m(t',p)}{t-t'}\ . \tag{2.32}$$

This reveals an indeterminant aspect of the chosen kernel, which we fix by choosing the principal part of the integral on the right. Use of the singular GC formula obtains

$$\int_{-1}^{1}\frac{dt'}{\sqrt{1-t'^2}}\frac{h_m(t',p)}{(t-t')^2}=\frac{\pi}{N}\sum_{k=1}^{N}\frac{h_m(t_k,p)}{(t-t_k)^2}+S_G'(t)h_m(t,p)+S_G(t)\frac{\partial}{\partial t}h_m(t,p)\ . \tag{2.33}$$

The GC formulas convert the complete equation (2.24) to

$$\left[\frac{1}{1-t^2}\left[-\frac{\partial^2}{\partial\rho^2}+\frac{m^2-\frac{1}{4}}{\rho^2}\right]+\frac{\mu_1^2}{2(1+t)}+\frac{\mu_2^2}{2(1-t)}\right]h_m(t,\rho)$$
$$-\frac{\sqrt{1-t^2}}{\pi}\left[g\frac{\pi}{N}\sum_{k=1}^{N}K_0(\rho|t-t_k|)h_m(t_k,\rho)+g'\frac{\pi}{N}\sum_{k=1}^{N}\left[\frac{\rho}{|t-t_k|}\bar{K}_1(\rho|t-t_k|)+\frac{1}{(t-t_k)^2}\right]h_m(t_k,\rho)\right.$$
$$\left.+g'S_G'(t)h_m(t,\rho)+g'S_G(t)\frac{\partial}{\partial t}h_m(t,\rho)\right]=\xi_m h_m(t,\rho)\ . \tag{2.34}$$

The regular and singular parts of K_1 can now be recombined, except for the terms containing S_G. Of these remaining terms, the most troublesome is the one that contains the derivative of h_m. Fortunately, it can be eliminated; this is done by choosing collocation points at the zeros of S_G, which are the zeros of U_{N-1}. Also, S_G' can, with use of various identities, be reduced to a simple form at these collocation points

$$S_G'(y_k)=-\frac{\pi N}{1-y_k^2}\ . \tag{2.35}$$

Equation (2.34) then becomes

$$\left[\frac{1}{1-y_l^2}\left[-\frac{\partial^2}{\partial\rho^2}+\frac{m^2-\frac{1}{4}}{\rho^2}\right]+\frac{\mu_1^2}{2(1+y_l)}+\frac{\mu_2^2}{2(1-y_l)}-g'S_G'(y_l)\frac{(1-y_l^2)^{1/2}}{\pi}\right]h_m(y_l,\rho)$$
$$-\frac{(1-y_l^2)^{1/2}}{\pi}\frac{\pi}{N}\sum_{k=1}^{N}\left[gK_0(\rho|y_l-t_k|)+g'\frac{\rho}{|y_1-t_k|}K_1(\rho|y_l-t_k|)\right]h_m(t_k,\rho)=\xi_m h_m(y_l,\rho)\ . \tag{2.36}$$

Note that the collocation points chosen for the GC formulas are the quadrature points of the LC formulas.

Similar expressions are obtained in working with the LC quadrature formulas. In that case the collocation points are chosen to be the quadrature points of the GC formulas. The equivalent to (2.36) is

$$\left[\frac{1}{1-t_l^2}\left[-\frac{\partial^2}{\partial\rho^2}+\frac{m^2-\frac{1}{4}}{\rho^2}\right]+\frac{\mu_1^2}{2(1+t_l)}+\frac{\mu_2^2}{2(1-t_l)}-g'S_L'(t_l)\frac{(1-t_l^2)^{1/2}}{\pi}\right]h_m(t_l,\rho)$$
$$-\frac{(1-t_l^2)^{1/2}}{\pi}\frac{\pi}{N}\sum_{k=1}^{N}\left[gK_0(\rho|t_l-y_k|)+g'\frac{\rho}{|t_l-y_k|}K_1(\rho|t_l-y_k|)\right]h_m(y_k,\rho)=\xi_m h_m(t_l,\rho)\ . \tag{2.37}$$

Thus the pairing of GC and LC quadrature formulas yields $2N-1$ equations with $2N-1$ unknown functions of ρ at the points

$$z_k\equiv\cos\frac{k\pi}{2N}=\begin{cases}t_{(k+1)/2}, & k \text{ odd}\\ y_{k/2}, & k \text{ even}\ .\end{cases} \tag{2.38}$$

The full set of equations can be collected into a symmetric matrix equation. To obtain the symmetry, define a new function

$$g_m(t,\rho)=(1-t^2)^{-1/4}h_m(t,\rho)=(1-t^2)^{1/4}u_m\left[\frac{1+t}{2},\rho\right] . \tag{2.39}$$

Let the individual functions of ρ form a vector $\mathbf{g}_m(\rho)$ with components

$$[\mathbf{g}_m(\rho)]_k=g_m(z_k,\rho) . \tag{2.40}$$

Then Eqs. (2.36) and (2.37) can be written as

$$A\mathbf{g}_m(\rho)=\xi_m\mathbf{g}_m(\rho) , \tag{2.41}$$

where A is a matrix with the following elements:

$$A_{lk}=\begin{cases}\dfrac{1}{1-z_l^2}\left[-\dfrac{\partial^2}{\partial\rho^2}+\dfrac{m^2-\frac{1}{4}}{\rho^2}\right]+\dfrac{\mu_1^2}{2(1+z_l)}+\dfrac{\mu_2^2}{2(1-z_l)}+\dfrac{Ng'}{(1-z_l^2)^{1/2}}, & l=k , \\ -(1-z_l^2)^{1/4}(1-z_k^2)^{1/4}\left[\dfrac{g}{N}K_0(\rho|z_l-z_k|)+\dfrac{g'}{N}\dfrac{\rho}{|z_l-z_k|}K_1(\rho|z_l-z_k|)\right], & l\neq k, \text{ one odd, the other even} , \\ 0 \quad \text{otherwise} . \end{cases} \tag{2.42}$$

We complete the conversion to a matrix problem by introducing a set of $N_\rho+2$ points ρ_i, at equal separations h, such that

$$\rho_i=ih, \quad i=0,\ldots,N_\rho+1 . \tag{2.43}$$

The second-order derivative is replaced by the central difference approximation[7]

$$\left.\frac{\partial^2}{\partial\rho^2}g_m(z_k,\rho)\right|_{\rho_i}\simeq\frac{1}{h^2}[g_m(z_k,\rho_{i+1})-2g_m(z_i,\rho_i)+g_m(z_k,\rho_{i-1})] . \tag{2.44}$$

The boundary condition (2.25) at infinity is approximated by forcing the wave function to zero at the last point

$$\rho_{\max}\equiv\rho_{N_\rho+1} . \tag{2.45}$$

One need only be careful that $\rho_{\max}$ is sufficiently large.

The eigenfunctions are now approximated by the ordinary eigenvectors

$$[g_m(z_1,\rho_1),\ldots,g_m(z_{2N-1},\rho_1), g_m(z_1,\rho_2),\ldots,g_m(z_{2N-1},\rho_{N_\rho})]^T . \tag{2.46}$$

With this choice of ordering, the full matrix is block diagonal except for two codiagonals from the first and last terms in the finite-difference formula. It is in fact a symmetric banded matrix of bandwidth $2N+1$. Thus the matrix eigenvalue problem is ideal for application of the International Mathematical and Scientific Library (IMSL) routine EVESB (Ref. 12) which was designed for this type of matrix. It was by this routine that the eigenvalues were calculated.

Accuracy of the results was improved by employing extrapolation to large N. The error was found to be roughly quadratic in $1/N$, and two estimates, at $N=5$ and $N=10$, were used as the basis for extrapolation.

When the quark masses are equal, the size of the matrix can be reduced. In this case, we have

$$m_1=m_2=\bar{m}, \quad \mu_i=1 . \tag{2.47}$$

The eigenfunctions can be chosen to be eigenstates of reflection in t:

$$g_m(-t,\rho)=\pm g_m(t,\rho) . \tag{2.48}$$

Because of this symmetry, roughly half of the entries in the eigenvector (2.46) may be discarded. The diagonal blocks are only of dimension N for the even case or $N-1$ for the odd. However, in the even case, $g_m(0,\rho_i)$ must be replaced by $g_m(0,\rho_i)/\sqrt{2}$ to maintain the symmetry of the matrix.

III. DISCUSSION AND RESULTS

A. Determination of parameters

A complete classification of states according to angular momentum cannot be done. We interpret m as the z component of orbital angular momentum, but states with definite particle number cannot, in general, be eigenstates of total angular momentum. Assignment of states to experimentally observed particles is therefore problematic.

Assignment of states is least difficult when the quark masses are equal. Reflection in $t=2x_1-1$ corresponds to the interchange of momentum fractions x_1 and $x_2=1-x_1$. Thus the symmetry (2.48) and the structure of the ϕ dependence in (2.23) imply that the complete eigenfunction represents an eigenstate of particle interchange:

$$\psi_\pm(x_2,\rho,\phi+\pi)=\pm(-1)^m\psi_\pm(x_1,\rho,\phi) . \tag{3.1}$$

We assume that mixing of low-lying states with different

total angular momentum l is small. The assigned l value must not be less than m. Also, since particle interchange is equivalent to inversion of the ordinary relative momentum, the parity $(-1)^l$ should be equal to $\pm(-1)^m$. Therefore, the lowest even, $m=0$ state is treated as the $1S$ state. The lowest odd, $m=0$ and even, $m=\pm1$ states should be $1P$. Assignment of excited states is more difficult since, for example, some even, $m=0$ states are D states rather than S. For the strange mesons, the inequality in quark masses means that (2.48) is not directly available to aid in assigning P and D states, but because the quark masses turn out to be nearly equal, the eigenfunctions themselves can be classified as approximately odd or even. Assignment of charmed and bottom mesons is not attempted, except for the ground states.

Some dimensionless eigenvalues are presented in Fig. 1. They are values appropriate for a description of charmonium. The figure illustrates the assumed spectroscopic assignments. The $m=0$, $1D$ state was assigned based on the known charmonium spectrum and nearness to other $1D$ states.

Note that states with different m but the same l are not degenerate. To obtain a single value as the eigenvalue for that l, we calculate an average weighted by the degeneracy at each m.

A fit to meson masses[13] of levels calculated in this way has been done. Since states other than $1S$ are quite suspect, a precise optimization of parameters was not attempted. The momentum dependence of the coupling constant g was modeled via the parametrization[14]

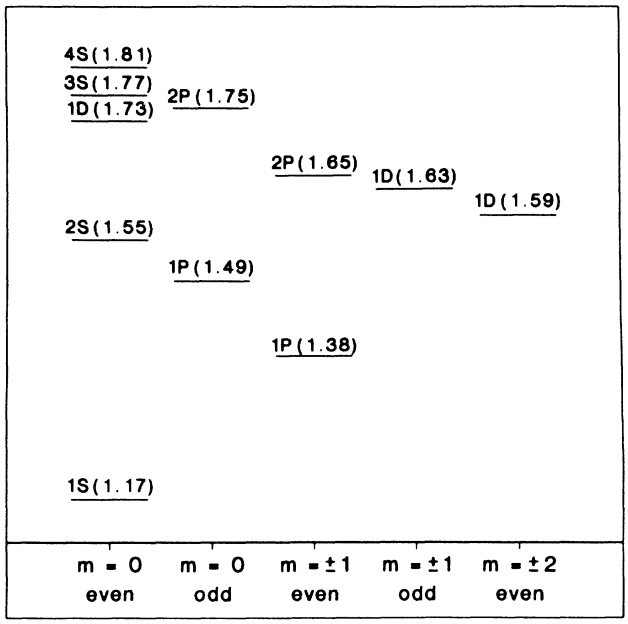

FIG. 1. Spectroscopic assignments for various low-lying states. The eigenvalue ξ is indicated by the position of the line and by the number written in parentheses. The values are obtained from numerical solution of Eq. (2.24) of the text at values of g, g', and μ_i appropriate for charmonium. The labels "odd" and "even" refer to the symmetry under interchange of momentum fractions, and m is the z component of angular momentum.

$$g=\frac{g_0}{2\ln(M/\Lambda)}\ , \tag{3.2}$$

with g held constant for any one system and M taken to be the $1S$ mass. The mass scale Λ was chosen somewhat arbitrarily to be 0.5 GeV. The values of g_0, a, and m_c were adjusted to give a close fit to the $2S$-$1S$ and $1P$-$1S$ mass differences in the $c\bar{c}$ system. The constant C was fixed by the J/ψ mass. The other quark masses were determined by fits to the appropriate hidden-flavor $1S$ mass. The light up and down quarks were treated as identical. In the case of the $b\bar{b}$ system, where the $1S$ mass is unknown, an estimate[15] of 35 MeV for the $\Upsilon-\eta_b$ splitting was used to give 9.451 GeV as the $1S$ mass. The resulting parameter values are

$$\begin{aligned} &m_u=m_d=0.277\ \text{GeV},\quad m_s=0.410\ \text{GeV}\ ,\\ &m_c=1.651\ \text{GeV},\quad m_b=4.933\ \text{GeV}\ ,\\ &g_0=3.670,\quad \Lambda=0.500\ \text{GeV}\ ,\\ &a=0.442\ \text{GeV},\quad C=-1.024\ \text{GeV}\ . \end{aligned} \tag{3.3}$$

The quark masses obtained are reasonably consistent with accepted values.[16] The coupling constant g for charmonium is 1.01. For comparison, the parameters of the model of Eichten *et al.*[9] are $g=0.52$, $a=0.427$ GeV, and $m_c=1.84$ GeV. Therefore, the short-distance behavior of the potential is changed but not the long-distance behavior.

B. Spectra

Calculated masses for various quark-antiquark systems are listed in Tables I–III. Experimental values are also listed if available.[13] In most instances, the calculated meson masses are not unreasonable, given the primitive nature of the model. The worst deviations occur for the excited states of the $b\bar{b}$ system where the differences between the calculated and experimental values grow to almost 600 MeV. However, it is excited states that are particularly difficult to interpret. Clearly, a better understanding of angular momentum eigenstates is needed.

C. Pion properties

With an approximate pion wave function now in hand, various properties can be calculated. We consider the form factor, the rms radius and the pion decay constant. Similar calculations with model wave functions have been done by Dziembowski[17] and others.[18]

The electromagnetic form factor is given by[4]

$$F(\mathbf{q}_\perp^2=Q^2) = \int_0^1 \frac{dx}{4\pi}\int \frac{d^2k_\perp}{4\pi^2}\psi^*(x,\mathbf{k}_\perp+(1-x)\mathbf{q}_\perp))\psi(x,\mathbf{k}_\perp)\ , \tag{3.4}$$

where $\mathbf{q}_\perp$ is the transverse photon momentum. The Fourier transform from $\mathbf{k}_\perp$ to $\mathbf{r}_\perp$ yields

$$F(Q^2)=\frac{1}{4\pi}\int_0^1 dx\int d^2r_\perp e^{-ix\mathbf{q}_\perp\cdot\mathbf{r}_\perp}|\psi(x,\mathbf{r}_\perp)|^2\ . \tag{3.5}$$

TABLE I. Calculated meson masses compared with experimental values for hidden-flavor states of l (up and down) and s quarks. Only isospin-zero $l\bar{l}$ mesons are listed.

	$l\bar{l}$ systems				$s\bar{s}$ systems			
			Mass of centroid (MeV)				Mass of centroid (MeV)	
State	Particle	Mass (MeV)[a]	Experiment	Model	Particle	Mass (MeV)[a]	Experiment	Model
$1\,^1S_0$	η	548.8±0.6	724[b]	724	$\eta'(958)$	957.50±0.24	1004[b]	1004
$1\,^3S_1$	$\omega(783)$	782.0±0.1			$\phi(1020)$	1019.41±0.01		
$2S$				1320				1589
$1\,^1P_1$	$h_1(1170)$	1170±40						
$1\,^3P_0$	$f_0(1400)$	≃1400			$f_0(975)$	976±3		
$1\,^3P_1$	$f_1(1285)$	1283±5	1291	1221	$f_1(1420)$	1422±10	1430	1443
$1\,^3P_2$	$f_2(1270)$	1274±5			$f_2'(1525)$	1525±5		
$1\,^3D_1$								
$1\,^3D_2$				1511				1744
$1\,^3D_3$	$\omega_3(1670)$	1668±5						

[a]Reference 13.
[b]Used in fit.

Substitution of (2.23) and use of (2.39) reduces this to

$$F(Q^2)=\frac{1}{8\pi}\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}} \times\int_0^\infty d\rho\, J_0\left[\frac{1+t}{2}\frac{Q}{m}\rho\right][g_0(t,\rho)]^2 , \tag{3.6}$$

where J_0 is an ordinary Bessel function. The LC quadrature formula (2.26), with all the z_k as quadrature points, and the trapezoidal rule, lead to the approximation

$$F(Q^2)=\frac{h}{16N}\sum_{k=1}^{2N-1}\sum_{i=1}^{N_\rho} J_0\left[\frac{1+z_k}{2}\frac{Q}{m}\rho_i\right][g_0(z_k,\rho_i)]^2 . \tag{3.7}$$

Correct normalization of g_0 is determined by requiring $F(0)=1$. The numerical wave function can be used directly here to obtain an approximate form factor. The size of h used for this calculation was 0.1; this limits Q to being less than 3 GeV. A plot of $Q^2F(Q^2)$ in this momentum range is given in Fig. 2. Data[19] are also included for comparison.

The calculated form factor yields a reasonable approximation to the data. This does not mean, however, that perturbative quantum chromodynamics does not apply in the same region. The hard processes included in perturbative calculations[4] may well be represented in the model potential. Note, though, that the large-Q^2 behavior of $Q^2F(Q^2)$ is not constant as it is for the perturbative result.[4]

There is no natural definition of the rms radius $r_{\rm rms}$ in the light-cone formulation. We have chosen two possible

TABLE II. Same as Table I, but for c and b quarks.

	$c\bar{c}$ systems				$b\bar{b}$ systems			
			Mass of centroid (MeV)				Mass of centroid (MeV)	
State	Particle	Mass (MeV)[a]	Experiment	Model	Particle	Mass (MeV)[a]	Experiment	Model
$1\,^1S_0$	$\eta_c(2980)$	2979.6±1.7	3068[b]	3068				9451
$1\,^3S_1$	$\psi(3097)$	3096.9±0.1			$\Upsilon(9460)$	9460.3±0.2		
$2\,^1S_0$	$\eta_c(3590)$	3594.0±5.0	3663[b]	3680				9829
$2\,^3S_1$	$\psi(3685)$	3686.0±0.1			$\Upsilon(10023)$	10023.3±0.3		
$3\,^1S_0$				3986				9930
$3\,^3S_1$	$\psi(4040)$	4040.0±10.0			$\Upsilon(10355)$	10355.3±0.5		
$4\,^1S_0$				4039				10007
$4\,^3S_1$	$\psi(4415)$	4415.0±6.0			$\Upsilon(10575)$	10580.0±3.5		
$1\,^3P_0$	$\chi_0(3415)$	3415.1±1.0			$\chi_{b0}(9860)$	9859.8±1.3		
$1\,^3P_1$	$\chi_1(3510)$	3510.6±0.5	3525[b]	3476	$\chi_{b1}(9895)$	9891.9±0.7	9900	9781
$1\,^3P_2$	$\chi_2(3555)$	3556.3±0.4			$\chi_{b2}(9915)$	9913.2±0.6		
$2\,^3P_0$					$\chi_{b0}(10235)$	10235.3±1.1		
$2\,^3P_1$				3872	$\chi_{b1}(10255)$	10255.2±0.4	10261	9807
$2\,^3P_2$					$\chi_{b2}(10270)$	10269.0±0.7		
$1\,^3D_1$	$\psi(3770)$	3769.9±2.5						
$1\,^3D_2$				3800				9877
$1\,^3D_3$								

[a]Reference 13.
[b]Used in fit.

TABLE III. Same as Table I, but for flavored mesons.

System	State	Particle	Mass (MeV)[a]	Mass of centroid (MeV) Experiment	Model
$\bar{s}l$	$1\,^1S_0$	K	493.67–497.72	793	784
	$1\,^3S_1$	$K^*(892)$	892.1±0.3		
	$2S$				1430
	$1\,^1P_1$	$K_1(1270)$	1270±10		
	$1\,^3P_0$	$K_0^*(1430)$	1429±7		
	$1\,^3P_1$	$K_1(1400)$	1401±10	1418	1300
	$1\,^3P_2$	$K_2^*(1430)$	1426±2		
	$1\,^3D_1$				
	$1\,^3D_2$	$K_2(1770)$	≃1770		1611
	$1\,^3D_3$	$K_3^*(1780)$	1776±4		
$c\bar{l}$	$1\,^1S_0$	D	1864.5±0.6–1869.3±0.6	1974	2046
	$1\,^3S_1$	$D^*(2010)$	2007.1±1.4–2010.1±0.6		
$c\bar{s}$	$1\,^1S_0$	D_s	1969.3±1.1	2077	2094
	$1\,^3S_1$	$D_s^*(2110)$	2112.7±2.3		
$\bar{b}l$	$1\,^1S_0$	B	5277.6±1.4–5279.4±1.5	5318	5346
	$1\,^3S_1$	$B^*(5325)$	5331.3±4.7		

[a]Reference 13.

definitions. One uses the expectation value of $r_\perp$,

$$r_{\rm rms}^{(1)}=\langle r_\perp^2\rangle^{1/2}\simeq\frac{h}{16N}\sum_{k=1}^{2N-1}\sum_{i=1}^{N_\rho}\left[\frac{\rho_i}{\bar{m}}\right]^2[g_0(z_k,\rho_i)]^2 \quad (3.8)$$

and the other, the expansion[20]

$$F(Q^2)=1-\tfrac{1}{6}(Qr_{\rm rms})^2+O(Q^4),\quad Qr_{\rm rms}\ll 1\ , \quad (3.9)$$

which yields

$$r_{\rm rms}^{(2)}=\lim_{Q\to 0}\frac{1}{Q}\sqrt{6[1-F(Q^2)]}\ . \quad (3.10)$$

Of course, the numerical calculation cannot resolve $F(Q^2)$ at very small Q, but in graphical form the limit in (3.10) is obvious to better than two-place accuracy. The results obtained from these definitions are

$$r_{\rm rms}^{(1)}=0.82\ \text{fm},\quad r_{\rm rms}^{(2)}=0.54\ \text{fm}\ . \quad (3.11)$$

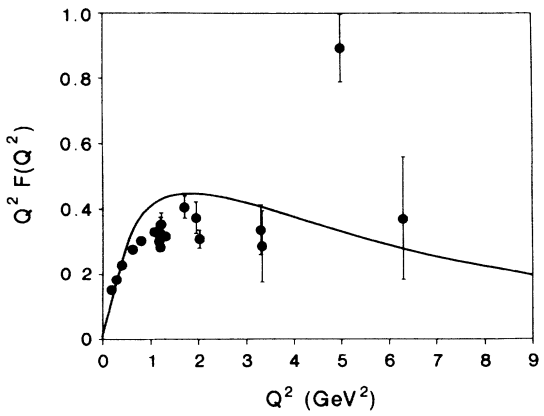

FIG. 2. Pion form factor $F(Q^2)$ multiplied by Q^2. The line was obtained from Eq. (3.7) of the text. The data are from Ref. 19.

The experimental value[21] is 0.66 fm.

The pion decay constant f_π, defined by the matrix element of the weak current,[22]

$$\langle 0|\bar{u}\gamma^+(1-\gamma_5)d|\pi^+\rangle=\sqrt{2}f_\pi p^+\ , \quad (3.12)$$

can also be expressed in terms of the numerical wave function. We find

$$f_\pi=\frac{\sqrt{3}}{2\pi}\int_0^1\psi(x,0)dx$$
$$=\frac{1}{2\pi}\left[\frac{3}{2\pi}\right]^{1/2}\int_0^1 dx\frac{1}{\sqrt{\rho}}g_0(x,\rho)\Big|_{\rho=0}\ . \quad (3.13)$$

To approximate the value of $(1/\sqrt{\rho})g_0(x,\rho)$ at $\rho=0$, we combine different Taylor expansions about $\rho=h$ to obtain

$$\frac{1}{\sqrt{\rho}}g_0(x,\rho)\Big|_{\rho=0}\simeq\frac{3}{\sqrt{h}}g_0(x,h)-\frac{3}{\sqrt{2h}}g_0(x,2h)$$
$$+\frac{1}{\sqrt{3h}}g_0(x,3h)\ . \quad (3.14)$$

The integral is done with the LC quadrature formula (2.26). The result is $f_\pi=57$ MeV, which is to be compared with an experimental value of 93 MeV.

Inclusion of spin wave functions of the type used by Dziembowski[17] does not improve the results. In fact, the results become much worse. This is in spite of considerable similarity between the Fourier transform of the model wave functions in Ref. 17 and the dynamically determined wave functions used here. A repetition of the calculations described in Ref. 17 indicates considerable sensitivity to the model wave-function parameters.

IV. SUMMARY

To summarize, we list some of the key steps and results. The form of the potential is specified by Eq. (2.16). The wave function is factored according to (2.23); the

function u must then satisfy the dimensionless equation (2.24). The eigenvalues ξ in this equation are related to meson masses M by (2.21). Equation (2.24) is solved by a numerical method based on Gauss-Chebyshev and Lobatto-Chebyshev quadrature formulas (2.26) and (2.27). The values determined for the parameters of the potential and for the quark masses are listed in (3.3). Results for meson spectra are given in Tables I–III, and an indication of spectroscopic assignments is given by Fig. 1. The model is successful in that reasonable quark masses and coupling constants were obtained. Reasonable results for pion properties were also obtained; the form factor is shown in Fig. 2.

The major difficulty in the use of this model has been the spectroscopic assignments. The light-front approach breaks rotational invariance, except about the z axis. Enough symmetry is retained to suggest assignments of low-lying states. Unfortunately, states of equal l are not degenerate, and averaging must be used to obtain a single mass value. A better treatment of angular momentum is needed.

ACKNOWLEDGMENTS

This material is based upon work supported by the National Science Foundation under Grant No. PHY-8611751 and by the Graduate School and the Theoretical Physics Institute of the University of Minnesota. We thank S. J. Brodsky and Z. Dziembowski for useful discussions.

[1]For some recent examples, see H. W. Crater and P. Van Alstine, Phys. Rev. Lett. **53**, 1527 (1984); M. Bander *et al.*, Phys. Rev. D **29**, 2038 (1984); S. Godfrey and N. Isgur, *ibid.* **32**, 189 (1985); T. Biswas and F. Rohrlich, Nuovo. Cimento **A88**, 125 (1985); T. Biswas, *ibid.* **A88**, 145 (1985); A. Mittal and A. N. Mitra, Phys. Rev. Lett. **57**, 290 (1986); S. Jacobs, M. G. Olsson, and C. J. Suchyta III, Phys. Rev. D **35**, 2448 (1987); O. C. Jacob and L. S. Kisslinger, in *Nuclear and Particle Physics on the Light Cone,* proceedings of the Workshop, Los Alamos, New Mexico, 1988, edited by M. B. Johnson and L. S. Kisslinger (World Scientific, Singapore, 1989); A. Gara, B. Durand, L. Durand, and L. J. Nickisch, Phys. Rev. D **40**, 843 (1989).

[2]D. Eyre and J. P. Vary, Phys. Rev. D **34**, 3467 (1986); J. R. Spence and J. P. Vary, *ibid.* **35**, 2191 (1987); report, 1988 (unpublished).

[3]P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).

[4]For reviews, see S. J. Brodsky, T. Huang, and G. P. Lepage, in *Quarks and Nuclear Forces,* edited by D. C. Fries and B. Zeitnitz (Springer Tracts in Modern Physics, Vol. 100), (Springer, Berlin, 1982); G. P. Lepage, S. J. Brodsky, P. B. Mackenzie, and T. Huang, in *Particles and Fields-2,* proceedings of the Banff Summer Institute on Particle Physics, Banff, Canada, 1981, edited by A. Z. Capri and A. N. Kamal (Plenum, New York, 1983).

[5]For calculations of bound-state spectra in light-cone scalar-exchange models, see M. Sawicki, Phys. Rev. D **32**, 2666 (1985); **33**, 1103 (1986); C.-R. Ji and R. J. Furnstahl, Phys. Lett. **167B**, 11 (1986); C.-R. Ji, *ibid.* **167B**, 16 (1986).

[6]N. I. Ioakimidis and P. S. Theocaris, Q. Appl. Math. **37**, 325 (1979). See also F. Erdogan and G. D. Gupta, *ibid.* **29**, 525 (1972); M. M. Chawla and T. R. Ramakrishnan, BIT **14**, 14 (1974); P. S. Theocaris, Serdica (Bulgaria) **2**, 252 (1976); P. S. Theocaris and G. Tsamasphyros, Appl. Math. Comput. **15**, 47 (1984); Y. Ma and J. R. Hiller, J. Comput. Phys. **82**, 229 (1989).

[7]See, for example, R. L. Burden and J. D. Faires, *Numerical Analysis,* 3rd ed. (PWS, Boston, 1985).

[8]For a brief presentation of some of these results, see J. R. Hiller, in *Nuclear and Particle Physics on the Light Cone* (Ref. 1).

[9]E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, and T.-M. Yan, Phys. Rev. D **17**, 3090 (1978); **21**, 203 (1980).

[10]N. A. Aboud and J. R. Hiller (unpublished).

[11]G. Hardekopf and J. Sucher, Phys. Rev. A **30**, 703 (1984); D. P. Heddle, Y. R. Kwon, and F. Tabakin, Comput. Phys. Commun. **38**, 71 (1985); R. J. Luce and F. Tabakin, *ibid.* **46**, 193 (1987).

[12]*International Mathematical and Scientific Library User's Manual* (IMSL, Houston, 1987).

[13]Particle Data Group, G. P. Yost *et al.*, Phys. Lett. B **204**, 1 (1988).

[14]W. J. Marciano, Phys. Rev. D **29**, 580 (1984).

[15]B. Diekmann, Phys. Rep. **159**, 99 (1988).

[16]For a summary of quark masses, see D. B. Lichtenberg, University of Washington report (unpublished).

[17]Z. Dziembowski, Phys. Rev. D **37**, 778 (1988).

[18]For example, see N. Isgur and C. H. Llewellyn Smith, Phys. Rev. Lett. **52**, 1080 (1984); O. C. Jacob and L. S. Kisslinger, *ibid.* **56**, 225 (1986); P. L. Chung, F. Coester, and W. Polyzou, Phys. Lett. B **205**, 545 (1988).

[19]C. N. Brown *et al.*, Phys. Rev. D **8**, 92 (1973); C. J. Bebek *et al.*, *ibid.* **9**, 1229 (1974); **13**, 25 (1976); **17**, 1693 (1978).

[20]This expansion follows from an expansion of the Bessel function in the ordinary nonrelativistic expression for the form factor. One could also use such an expansion in Eq. (3.6).

[21]E. B. Dalley *et al.*, Phys. Rev. Lett. **48**, 375 (1982).

[22]G. P. Lepage and S. J. Brodsky, Phys. Rev. D **22**, 2157 (1980).