# "Topological" (Chern-Simons) quantum mechanics

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We construct quantum-mechanical models that are analogs of three-dimensional, topologically massive as well as Chern-Simons gauge-field theories, and we study the phase-space reductive limiting procedure that takes the former to the latter. The zero-point spectra of operators behave discontinuously in the limit, as a consequence of a nonperturbative quantum-mechanical anomaly. The nature of the limit for wave functions depends on the representation, but is always such that normalization is preserved.

Quantum field theories in three-dimensional (2+1) space-time, especially gauge and gravitational models, whose study was initiated in the early 1980s (Refs. 1 and 2) have now become the focus of widespread research activity, not only for pedagogical and mathematical reasons, but also because of the conjectured role for three-dimensional dynamics in planar condensed-matter (Hall effect, high-$T_c$) and cosmological (strings) settings. These models are particularly interesting since special structures, Chern-Simons terms, that are available in three dimensions (more generally in any odd dimension) give rise to topologically intricate phenomena without even-dimensional analogs.

This paper describes similar dynamical effects in an even simpler, odd-dimensional "field theory" that resides in one (space-) time dimension, i.e., quantum mechanics. The models that we study are, in the general case, governed by a Lagrangian giving rise to Lorentz forces:

$$L=\frac{m}{2}\dot{\mathbf{q}}^2+\frac{e}{c}\dot{\mathbf{q}}\cdot\mathbf{A}(\mathbf{q})-eV(\mathbf{q})\ . \tag{1}$$

A particle of mass $m$ and charge $e$ executes motion in external magnetic and electric fields, $\partial_i A^j-\partial_j A^i$ and $-\partial_i V$, respectively. We consider the simplest case, where motion is two dimensional, $i=1,2$, and rotationally symmetric, $A^i(\mathbf{q})=\epsilon^{ij}q^jA(q)$, $V(\mathbf{q})=V(q)$. Moreover, we simplify further to arrive at an explicitly solvable model, by taking a constant magnetic field, $A(q)=-B/2\le 0$, and a quadratic scalar potential, $V(q)=k\mathbf{q}^2/2\ge 0$. Thus the Lagrangian we study is

$$L=\frac{m}{2}\dot{\mathbf{q}}^2+\frac{B}{2}\mathbf{q}\times\dot{\mathbf{q}}-\frac{k}{2}\mathbf{q}^2\ . \tag{2}$$

(Henceforth, we set $e$, $c$, and $\hbar$ to unity.)

It is clear that (2) is analogous to the Lagrange density $\mathcal{L}$ for three-dimensional, topologically massive electrodynamics[1] in the Weyl ($A^0=0$) gauge:

$$\mathcal{L}=\tfrac{1}{2}\dot{\mathbf{A}}^2+\frac{\mu}{2}\dot{\mathbf{A}}\times\mathbf{A}-\tfrac{1}{2}(\nabla\times\mathbf{A})^2\ . \tag{3}$$

The corresponding dynamical variables are $\mathbf{q}(t)$ and $\mathbf{A}(t,\mathbf{x})$ of (2) and (3), respectively; the kinetic and potential terms in (2) are analogous to Maxwell (first and last) terms in (3); the velocity-dependent, magnetic, Lorentz interaction in (2) models the Chern-Simons term, proportional to $\mu$ in (3). By rescaling $\mathbf{A}\to\sqrt{\kappa/\mu}\,\mathbf{A}$ and setting $\mu\to\infty$, the Maxwell terms disappear from (3), leaving the pure Chern-Simons theory on a reduced phase space:

$$\mathcal{L}_{\rm CS}=\frac{\kappa}{2}\dot{\mathbf{A}}\times\mathbf{A}\ . \tag{4}$$

For the corresponding limit in (2) we set $m$ and $k$ to zero; indeed phase-space reduction is already achieved when only $m$ vanishes:

$$L_0=\frac{B}{2}\mathbf{q}\times\dot{\mathbf{q}}-\frac{k}{2}\mathbf{q}^2\ . \tag{5}$$

As is well known, reduction of phase space alters the symplectic structure.

For the theory (2), the symplectic structure is familiar. With

$$p^i=\frac{\partial L}{\partial\dot{q}^i}=m\dot{q}^i-\frac{B}{2}\epsilon^{ij}q^j \tag{6}$$

the Hamiltonian is

$$H(\mathbf{p},\mathbf{q})=\frac{1}{2m}\left(p^i+\frac{B}{2}\epsilon^{ij}q^j\right)\left(p^i+\frac{B}{2}\epsilon^{ik}q^k\right)+\frac{k}{2}q^iq^i \tag{7}$$

so that the Lagrangian, when written in first-order form,

$$\tilde{L}=\mathbf{p}\cdot\dot{\mathbf{q}}-H(\mathbf{p},\mathbf{q}) \tag{8}$$

fixes conventional commutators:

$$[q^i,q^j]=0,\quad [p^i,p^j]=0,\quad [q^i,p^j]=i\delta^{ij}\ . \tag{9}$$

With (7) and (9), the Euler-Lagrange equations of (2) or (8) are regained as Hamiltonian equations:

$$\begin{aligned}\dot{p}^i&=i[H,p^i]=\frac{B}{2m}\epsilon^{ij}\left(p^j+\frac{B}{2}\epsilon^{jk}q^k\right)-kq^i\ ,\\ \dot{q}^i&=i[H,q^i]=\frac{1}{m}\left(p^i+\frac{B}{2}\epsilon^{ij}q^j\right)\ .\end{aligned} \tag{10}$$

 

These are solved by

$$z(t)\equiv q^1(t)+iq^2(t) = e^{-i(B/2m)t}\left[z(0)\cos\Omega t+\frac{p(0)}{m\Omega}\sin\Omega t\right],$$
$$p(t)\equiv p^1(t)+ip^2(t) = e^{-i(B/2m)t}[p(0)\cos\Omega t - m\Omega z(0)\sin\Omega t], \quad (11)$$

$$\Omega=\left[\frac{B^2}{4m^2}+\frac{k}{m}\right]^{1/2}. \quad (12)$$

On the other hand, the reduced theory (5) is already described by a first-order Lagrangian; a Legendre transform is not needed[3] and the Hamiltonian is identified from $L_0$:

$$H_0=\frac{k}{2}\mathbf{q}^2 . \quad (13)$$

With the help of the commutator that is determined by (5),

$$[q^i,q^j]=-\frac{i}{B}\epsilon^{ij} , \quad (14)$$

the Euler-Lagrange equations of (5) are reproduced:

$$\dot q^{\,i}=i[H_0,q^i]=-\frac{k}{B}\epsilon^{ij}q^j . \quad (15)$$

They are solved by

$$z(t)\equiv q^1(t)+iq^2(t)=z(0)e^{i(k/B)t} . \quad (16)$$

The formal steps that effect the reduction on the operator level have already been given.[3] From (6) and (7) we see that taking the limit $m\to 0$ requires constraining to zero the quantities

$$C^i=p^i+\frac{B}{2}\epsilon^{ij}q^j \quad (=m\dot q^{\,i}) . \quad (17)$$

The constraints are second class,

$$[C^i,C^j]=iB\epsilon^{ij}\neq 0 , \quad (18)$$

and lead to Dirac brackets between any pair of operators $O_1$ and $O_2$:

$$[O_1,O_2]_D=[O_1,O_2]-i[O_1,C_i]\frac{\epsilon^{ij}}{B}[C_j,O_2] . \quad (19)$$

In this way the commutator (14) of the reduced theory is recognized as the Dirac brackets in the original but constrained model. Moreover, solving the constraint (17) by setting

$$p^i=-\frac{B}{2}\epsilon^{ij}q^j \quad (20)$$

reduces the first-order Lagrangian (8) to (5). Finally, we note that in the limit $m\to 0$ the solutions (11) to the dynamical equations (10) satisfy the constraint (20) and pass to (16). This is seen, when it is recognized that

$$\Omega \underset{m\to 0}{\sim} \frac{B}{2m}+\frac{k}{B} \quad (21)$$

and that terms oscillating when $m\to 0$ should be dropped.

While the operator structures behave smoothly in the limit $m\to 0$, the eigenvalues and eigenfunctions possess interesting behavior, and it is our aim to describe this. The states that we consider are the simultaneous eigenstates of the Hamiltonian and of the rotational symmetry generator that commutes with it—the angular momentum.

For the full theory, the angular momentum, obtained by Noether's theorem from (2) or (8),

$$M=\mathbf{q}\times\mathbf{p} , \quad (22)$$

generates rotations upon commutation, with the help of (9):

$$i[M,q^i]=-\epsilon^{ij}q^j . \quad (23)$$

In the reduced theory, the angular momentum operator, obtained either by Noether's theorem from (5) or by imposing the constraint (20) on (22),

$$M_0=\frac{B}{2}\mathbf{q}^2 , \quad (24)$$

also generates rotations, with the commutator (14):

$$i[M_0,q^i]=-\epsilon^{ij}q^j . \quad (25)$$

The simultaneous eigenstates of $M$ and $H$,

$$M|N,n\rangle=n|N,n\rangle , \quad (26a)$$

$$H|N,n\rangle=E(N,n)|N,n\rangle , \quad (26b)$$

$$E(N,n)=\Omega(2N+|n|+1)-\frac{B}{2m}n , \quad (27)$$

are given in the coordinate representation by normalized wave functions:

$$\langle\mathbf{q}|N,n\rangle=\left[\frac{N!}{\pi(N+|n|)!}\right]^{1/2}(m\Omega)^{(1+|n|)/2} \times r^{|n|}e^{in\theta}e^{-(m/2)\Omega r^2}L_N^{|n|}(m\Omega r^2) ,$$
$$\mathbf{q}\equiv(r\cos\theta, r\sin\theta) . \quad (28)$$

Here $L_N^{|n|}$ is the associated Laguerre polynomial, satisfying the differential equation

$$w\frac{d^2}{dw^2}L_N^{|n|}(w)+(|n|+1-w)\frac{d}{dw}L_N^{|n|}(w)+NL_N^{|n|}(w)=0 . \quad (29)$$

$N$ is a non-negative integer to ensure normalizability. Moreover, single valuedness of the wave function (28) requires $n$, the angular momentum eigenvalue, to be any integer.

In the reduced theory $H_0$ and $M_0$, (13) and (24), essentially coincide. From the commutator (14) and the quadratic expression for the Hamiltonian (13), we recognize that $H_0$ has the structure of a one-dimensional harmonic-oscillator Hamiltonian. (Identify $q^1$ and $-Bq^2$ with a canonically conjugate pair of coordinate and momentum.) Hence the energy spectrum is

$$H_0|n\rangle = E_0(n)|n\rangle\ ,$$
$$E_0(n) = \frac{k}{B}\left[n+\frac{1}{2}\right],\quad n=0,1,\ldots\ . \tag{30}$$

Moreover, the angular momentum spectrum consists of positive half-integers:

$$M_0|n\rangle = (n+\tfrac{1}{2})|n\rangle\ . \tag{31}$$

To describe these eigenstates of the reduced theory by wave functions, a polarization must be chosen to select on which *one* of the *two* noncommuting coordinates the wave function is to depend. For our purposes, it is convenient to use the holomorphic representation.[4] We form the non-Hermitian combination

$$a \equiv \sqrt{B/2}(q^1 - iq^2)\ , \tag{32a}$$

which together with its conjugate

$$a^\dagger \equiv \sqrt{B/2}(q^1 + iq^2) \tag{32b}$$

satisfies

$$[a,a^\dagger]=1\ . \tag{33}$$

Therefore, if we choose states $\langle\alpha|$ such that

$$\langle\alpha|a^\dagger = \langle\alpha|\alpha = \langle\alpha|\sqrt{B/2}re^{i\theta}\ , \tag{34}$$

then states $|\psi\rangle$ can be described by wave functions that tions that depend on $\alpha$:

$$\langle\alpha|\psi\rangle = \psi(\alpha)\ . \tag{35}$$

The operator $a^\dagger$ acts on these functions by multiplication, $a$ by differentiation; the adjoint relationship between the two being maintained by virtue of a nontrivial measure:

$$\langle\alpha|a^\dagger|\psi\rangle = \alpha\psi(\alpha)\ ,$$
$$\langle\alpha|a|\psi\rangle = \frac{d}{d\alpha}\psi(\alpha)\ , \tag{36}$$

$$\frac{1}{2\pi i}\int d\alpha^* d\alpha |\alpha\rangle\langle\alpha| e^{-|\alpha|^2} = I\ ,$$
$$\frac{1}{2\pi i}d\alpha^* d\alpha \equiv \frac{B}{2\pi}dq^1 dq^2 = \frac{B}{2\pi} r\,dr\,d\theta\ . \tag{37}$$

Within the holomorphic representation, we have

$$\langle\alpha|n\rangle = \frac{\alpha^n}{\sqrt{n!}}\ . \tag{38}$$

Other polarizations are available—e.g., one may let the wave functions depend on $q^1$, now renamed as $\sqrt{k}x/B$, and realize $q^2$ as $(i/\sqrt{k})(d/dx)$. The energy and angular momentum eigenfunctions are then the harmonic-oscillator ones, involving Hermite polynomials $H_n$, and frequency $k/B$:

$$u_n(x) = \left[\left[\frac{k}{\pi B}\right]^{1/2}\frac{1}{2^n n!}\right]^{1/2} \times H_n(\sqrt{k/B}\,x)e^{-(k/2B)x^2}\ . \tag{39}$$

We now examine how the various quantities of the complete theory behave as $m\to 0$ and whether the corresponding quantities of the reduced theory are attained in the limit.

The energy spectrum (27) diverges as $m\to 0$:

$$E(N,n) \underset{m\to 0}{\sim} \frac{B}{2m}(2N+|n|-n+1) + \frac{k}{B}(2N+|n|+1)\ . \tag{40a}$$

However, a universal subtraction can remove the infinity in the $N=0$, $n\geq 0$ states, for which

$$E(0,|n|) \underset{m\to 0}{\sim} \frac{B}{2m} + \frac{k}{B}(|n|+1)\ . \tag{40b}$$

Thus in the limit $m\to 0$ all states with $N>0$, as well as those with $N=0$ and $n<0$ are separated by an infinite gap and decouple from the remaining $N=0$, $n\geq 0$ states. The surviving states, whose angular momentum is aligned with the external magnetic field, are in one-to-one correspondence with the states of the reduced theory. They carry finite energy, provided a subtraction is performed on the Hamiltonian. The infinite part of the subtraction evidently is $B/2m$; also a finite subtraction of $k/2B$ must be made to obtain agreement with the energy spectrum (30) of the reduced theory. While we have no *a priori* determination of this finite subtraction, we certainly can accommodate it since an infinite subtraction is already required.

A similar half-integer discrepancy is present in the angular momentum spectrum: for the complete theory the spectrum of $M$ comprises all integers; in the reduced theory $M_0$ possesses only positive half-odd-integers in its spectrum. Here no subtraction is called for; indeed we shall show below that this discrepancy reflects an important difference in the action of the rotation symmetry for the two theories.

The wave functions (28) of the surviving states become, in the zero-mass limit,

$$\langle \mathbf{q}|0,|n|\rangle \underset{m\to 0}{\to} \left[\frac{B}{2\pi}\right]^{1/2}\left[\frac{B}{2}\right]^{|n|/2} \times \frac{1}{\sqrt{|n|!}} r^{|n|} e^{i|n|\theta} e^{-(B/4)r^2}$$
$$= \left[\frac{B}{2\pi}\right]^{1/2}\frac{1}{\sqrt{|n|!}}\alpha^{|n|}e^{-\alpha^*\alpha/2}$$
$$= \left[\frac{B}{2\pi}\right]^{1/2}\langle\alpha|n\rangle e^{-\alpha^*\alpha/2}\ . \tag{41a}$$

Thus the complete wave functions *do not* approach those of the reduced theory. It could not be otherwise: the former depend on *two* variables $q^1$ and $q^2$ (or $\alpha$ and $\alpha^*$) while the latter on only *one*, determined by the choice of polarization. However, since wave functions are normalized, their dependence on one of their arguments cannot disappear. Note nevertheless that in the holomorphic polarization the normalization densities are related properly:

$$d^2\mathbf{q}|\langle \mathbf{q}|0,|n|\rangle|^2 \underset{m\to 0}{\longrightarrow} dq^1 dq^2 \frac{B}{2\pi}\frac{1}{n!}(\alpha\alpha^*)^{|n|}e^{-\alpha^*\alpha}$$
$$= \frac{d\alpha\, d\alpha^*}{2\pi i}|\langle \alpha|n\rangle|^2 e^{-\alpha^*\alpha} . \quad (41b)$$

Of course the limiting relations between wave functions take different forms in different representations. As is well known, the Hamiltonian (7) is equivalent to two decoupled one-dimensional harmonic oscillators, described by the canonical pairs $(p_\pm, q_\pm)$ and frequencies $\omega_\pm$:

$$p_\pm = \left[\frac{\omega_\pm}{2m\Omega}\right]^{1/2} p^1 \pm \left[\frac{m\Omega\omega_\pm}{2}\right]^{1/2} q^2 ,$$
$$q_\pm = \left[\frac{m\Omega}{2\omega_\pm}\right]^{1/2} q^1 \mp \frac{1}{\sqrt{2m\Omega\omega_\pm}} p^2 , \quad (42)$$
$$\omega_\pm = \Omega \pm \frac{B}{2m} .$$

Thus wave functions of the complete problem may be presented in a "modified coordinate" representation:

$$\langle q_\pm|N,n\rangle = u^+_{n_+}(q_+)u^-_{n_-}(q_-) ,$$
$$n_\pm = N + \frac{|n|\mp n}{2} , \quad (43)$$

where $u^\pm_n$ are the harmonic-oscillator wave functions (39) with frequency $\omega_\pm$. In the zero-mass limit, $\omega_+ \sim B/m + k/B$, $\omega_- \to k/B$; only the minus oscillator survives, and the limiting relation now connects the wave functions (43) with (39), in a way consistent with the preservation of norms:

$$\int_{-\infty}^{\infty} dq_+ |\langle q_\pm|0,|n|\rangle|^2 \underset{m\to 0}{\longrightarrow} |u_{|n|}(q_-)|^2 . \quad (44)$$

Alternatively, one may use holomorphic versions of (42),

$$a_\pm = \frac{1}{\sqrt{2}}\left[\sqrt{\omega_\pm}q_\pm + \frac{i}{\sqrt{\omega_\pm}}p_\pm\right] , \quad (45a)$$
$$\langle \alpha_\pm|a^\dagger_\pm = \langle \alpha_\pm|\alpha_\pm , \quad (45b)$$

and the wave functions are simply

$$\langle \alpha_\pm|N,n\rangle = \frac{\alpha_+^{n_+}}{\sqrt{n_+!}}\frac{\alpha_-^{n_-}}{\sqrt{n_-!}} . \quad (46)$$

Since

$$\langle \alpha_\pm|0,|n|\rangle = \frac{\alpha_-^{|n|}}{\sqrt{|n|!}} ,$$

we see that the relation between (46) and (38) is direct, and proper normalization is assured by the holomorphic measure factor.

One important property of the field-theoretic Lagrange densities $\mathcal{L}$ and $\mathcal{L}_{CS}$, (3) and (4), is not modeled by our quantum-mechanical example: the field theory is gauge invariant; equations following from (3) and (4) must be supplemented by a subsidiary condition (Gauss's law), which requires that physical states be annihilated by the generator of the symmetry that is gauged. By extending our quantum-mechanical model, something similar can be achieved.

Observe that (2) and (5) are rotation symmetric—invariant under "global" rotations:

$$\delta q^i(t) = -\epsilon^{ij}q^j(t)\lambda, \quad \lambda \text{ time independent} . \quad (47)$$

These transformations are generated by the angular momenta of the two models, $M$ and $M_0$, respectively—see (22)–(25). Moreover, the "global" symmetry (47) may be promoted to a "local" gauge symmetry with time-varying $\lambda$, provided a "gauge potential" $a(t)$ is introduced, transforming inhomogeneously under time-dependent rotations, thus rendering covariant the time derivative:

$$Dq^i \equiv \dot q^i + a\epsilon^{ij}q^j , \quad (48)$$
$$\delta q^i(t) = -\epsilon^{ij}q^j(t)\lambda(t) , \quad (49a)$$
$$\delta a(t) = \dot\lambda(t) . \quad (49b)$$

In this way we arrive at a U(1) or SO(2) quantum-mechanical gauge theory, with $a(t)$ playing the role of the time component of a gauge potential—a Lagrange multiplier for the rotation generator.[5,6]

Once a one-dimensional (odd-dimensional) gauge theory is under discussion, a further Chern-Simons term may be added to the Lagrangian. In the present context this term is linear in the potential. (The Chern-Pontryagin term in one higher dimension, i.e., in two dimensions, is a two-form; hence, the Chern-Simons term in one dimension is a one-form.) Thus we are led to the following generalization of (2) and (5):

$$L^\nu = \frac{m}{2}D\mathbf{q}\cdot D\mathbf{q} + \frac{B}{2}\mathbf{q}\times D\mathbf{q} - \frac{k}{2}\mathbf{q}^2 + \nu a$$
$$= \frac{m}{2}(\dot q^i + a\epsilon^{ij}q^j)(\dot q^i + a\epsilon^{ik}q^k)$$
$$+ \frac{B}{2}\epsilon^{ij}q^i(\dot q^j + a\epsilon^{jk}q^k) - \frac{k}{2}q^iq^i + \nu a , \quad (50)$$

$$L_0^\nu = \frac{B}{2}\mathbf{q}\times D\mathbf{q} + \nu a$$
$$= \frac{B}{2}\epsilon^{ij}q^i(\dot q^j + a\epsilon^{jk}q^k) + \nu a . \quad (51)$$

[In the reduced Lagrangian, the harmonic coupling $k$ is absorbed in a redefinition of $a$; hence, it does not appear in (51).]

$L^\nu$ and $L_0^\nu$ exemplify the "topological" (Chern-Simons) quantum mechanics of our title. In the Weyl ($a=0$) gauge, they coincide with the Lagrangians (2) and (5), respectively, supplemented by subsidiary conditions:

$$\mathbf{q}\times\mathbf{p} = M = \nu , \quad (52)$$
$$\frac{B}{2}\mathbf{q}^2 = M_0 = \nu . \quad (53)$$

Since $\Pi_1(U(1)) = Z$ we expect that in the quantum theory the Chern-Simons coupling strength $\nu$ must be quantized. From (52) and (53) we see that this quantization is also demanded by the quantization of the symmetry generator, i.e., of angular momentum.

In the full theory where $M$ possesses integer eigenvalues, $\nu$ must be an integer. Gauge invariance leads to the same conclusion. Under a gauge transformation, the action

$$I^{\nu}=\int dt\, L^{\nu} \tag{54}$$

transforms as

$$I^{\nu}\rightarrow I^{\nu}+\nu\Delta\lambda\ , \tag{55}$$

where $\Delta\lambda$ is the change of the gauge function over the time interval, $\Delta\lambda=\int dt\,(d/dt)\lambda$. We classify gauge transformations by requiring $\Delta\lambda$ to be $2\pi\mathcal{N}$, $\mathcal{N}$ being the winding number of the U(1) gauge transformation. Hence $e^{iI^{\nu}}$ is gauge invariant when $\nu$ is an integer. Upon setting $\nu=n$, the spectrum of the theory governed by (50) consists of a multiplet of states with fixed angular momentum $n$, but still varying $N$.

In the reduced theory, the angular momentum eigenvalues are positive half-integers, and therefore the Chern-Simons coefficient $\nu$ in (51) must take a value in that set. This appears to be at variance with the gauge-invariance argument for quantizing $\nu$, which superficially follows the analysis presented above for the complete theory. However, a quantum anomaly modifies the naive argument. It would appear that apart from the Chern-Simons term, the Lagrangian in (51) is gauge invariant. But let us calculate the effective action by functionally integrating over $q^{i}(t)$. To effect the integration it is useful to pass to the complex coordinates $z$ and $z^{\dagger}$. Then $L_0^{\nu}$ reads (apart from a total time derivative)

$$L_0^{\nu}=-\frac{B}{2}z^{\dagger}\left[i\frac{d}{dt}+a\right]z+\nu a \tag{56}$$

and the effective quantum action is

$$\Gamma_0^{\nu}=-i\ln\det\left[i\frac{d}{dt}+a\right]+\nu\int dt\, a\ . \tag{57}$$

The determinant has been computed[5] and one finds that it is not invariant against gauge transformations with nontrivial winding number: under a gauge transformation the determinant acquires the sign $(-1)^{\mathcal{N}}$. Hence gauge invariance of $\exp(i\Gamma_0^{\nu})$ requires the half-integral quantization of $\nu$. Upon choosing $\nu=|n|+\frac{1}{2}$ in (51), the Hilbert space becomes one dimensional, with the wave function given, e.g., by (38) or (39).

[The above calculation of the effective action for the *reduced* theory shows that the effective action of the *complete* theory is gauge invariant, as anticipated by the formal argument. The point is that in terms of variables (42) the determinant of the complete theory is seen to be a product of two anomalously behaving determinants; but the factor $(-1)^{\mathcal{N}}$ disappears from the product.]

In conclusion,[7] we see that the simple quantum-mechanical models discussed in this paper illustrate the change in symplectic structure that occurs when the vanishing of a parameter takes a second-order Lagrangian into a first-order one. Being exactly solvable, these quantum-mechanical examples allow detailed analysis of this limiting procedure at the level of eigenvalues and eigenfunctions. This analysis probes beyond the formal operator level, where intricacies are not at all evident.

In particular, the limiting procedure for these (0+1)-dimensional "field theories" mimics some of the properties of the analogous limit in (2+1)-dimensional gauge theories involving Chern-Simons terms. When $\mu\rightarrow\infty$, the (second-order) Lagrange density (3) of the topologically massive gauge theory (in the Weyl, $A_0=0$, gauge) formally becomes the (first-order) Chern-Simons Lagrange density (4), and the limiting behavior of field-theoretic wave functionals is exactly the same as that of quantum-mechanical wave functions. The wave functional of the Abelian topologically massive gauge theory is[8]

$$\Psi(A_1,A_2)=\exp\left[i\frac{\mu}{2}\int_{\mathbf{x}}(\partial_2A_1-\partial_1A_2)\nabla^{-1}\cdot\mathbf{A}\right]\times\exp\left[-\tfrac{1}{2}\int_{\mathbf{x}}A_T^i\sqrt{-\nabla^2+\mu^2}A_T^i\right], \tag{58}$$

where $A_T^i$ is the transverse part of $A^i$, and $\nabla^{-1}\equiv\nabla/\nabla^2$. To effect the desired limit, first the fields are rescaled, then the wave functional (58) tends as $\mu\rightarrow\infty$ to

$$\Psi(A_1,A_2)\underset{\mu\rightarrow\infty}{\rightarrow}\exp\left[i\frac{\kappa}{2}\int_{\mathbf{x}}(\partial_2A_1-\partial_1A_2)\nabla^{-1}\cdot\mathbf{A}\right]\times\exp\left[-\frac{\kappa}{2}\int_{\mathbf{x}}A_T^iA_T^i\right]. \tag{59}$$

Equation (59) can be written in terms of holomorphic fields $A_\pm=(1/\sqrt{2})(A_1\pm iA_2)$, as

$$\Psi(A_+,A_-)\underset{\mu\rightarrow\infty}{=}\exp\left[\frac{\kappa}{2}\int_{\mathbf{x}}A_-\frac{\partial_+}{\partial_-}A_-\right]\times\exp\left[-\frac{\kappa}{2}\int_{\mathbf{x}}A_+A_-\right]. \tag{60}$$

On the other hand, the wave functional of the (non-Abelian) Chern-Simons theory is[6]

$$\Psi(A_-)=\exp\left[-\kappa\int_{\mathbf{x}}\mathrm{tr}(A_-h^{-1}\partial_+h)-i8\pi^2\kappa\int_{\mathbf{x}}\omega^0(h)\right], \tag{61a}$$

where $h$ is defined through

$$A_-=h^{-1}\partial_-h \tag{61b}$$

and $\omega^0$ is the time-component of the three-vector whose divergence is the winding number density:

$$\partial_\mu\omega^\mu=\frac{1}{24\pi^2}\epsilon^{\alpha\beta\gamma}\mathrm{tr}(g^{-1}\partial_\alpha g g^{-1}\partial_\beta g g^{-1}\partial_\gamma g)\ . \tag{61c}$$

In the Abelian case (61) reduces to

$$\Psi(A_-)=\exp\left[\frac{\kappa}{2}\int_{\mathbf{x}}A_-\frac{\partial_+}{\partial_-}A_-\right]. \tag{62}$$

Therefore from (60) we obtain

$$\Psi(A_+,A_-) \underset{\mu\to\infty}{\longrightarrow} \Psi(A_-)\exp\left[-\frac{\kappa}{2}\int_x A_+A_-\right] . \qquad (63)$$

Thus, just as in the quantum-mechanical case, the wave functional of the topologically massive theory passes in the limit to that of the pure Chern-Simons theory *times* the square root of the holomorphic measure factor.[9] Consequently, the normalization density possesses the correct limit as in (41b).

The shift in the zero-point eigenvalues is another example of the noncommutativity between phase-space reduction and quantization, and can be formulated as an ordering ambiguity in terms of the operators (42) that diagonalize the Hamiltonian (7) (Ref. 10). The field-theoretic analog of this phenomenon gives rise to a dependence of the Wilson loop expectation value in the Chern-Simons theory on the method of calculation: results differ depending on whether they are calculated directly or as limits of the topologically massive model.[11]

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[9]This fact is also known to A. Polyakov—we thank him for private discussions.

[10]We have already noted such behavior in Ref. 6 and it has been again emphasized in the present context by P. Gerbert, who suggested using the variables (42)—we are grateful for his advice.

[11]That the expectation value of the Wilson loop operator in the Abelian theory varies in this fashion is shown in Ref. 6; G. Zemba, MIT Report No. CTP 1727 (unpublished), Int. J. Mod. Phys. A (to be published); and (private communications); T. Hansson, A. Karlhede, and M. Roček, Phys. Lett. B **225**, 92 (1989).