External sources for topologically massive gauge fields

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We consider the finite-action classical solutions of Euclidean topologically massive gauge theories in the presence of external sources. We study the Abelian case for general sources, as well as the general non-Abelian case for weak sources. We also investigate the solutions within the radial *Ansatz*, both with the usual source coupling and with coupling to gauge-invariant sources. We show that all these solutions correspond to saddle points of the action.

I. INTRODUCTION

Gauge theories in three space-time dimensions[1] have received much attention, not only because they provide an interesting example of a model with a gauge-invariant mass term, the so-called Chern-Simons action, but also because they could be related to the high-temperature limit of the usual four-dimensional Yang-Mills equations,[2] in this respect, one is interested in three-dimensional Euclidean, topologically massive gauge theories. Three-dimensional gauge theories (without the Chern-Simons term) have also been studied in connection with confinement.[3]

The equations for this model are complicated nonlinear equations whose complete set of solutions is not known. Although several important classical solutions have been found,[4] and the quantum-mechanical perturbative treatment has been studied,[1,2] it is still interesting to see how these equations react to external stimuli, that is, what solutions can be found in the presence of external sources. This is done in parallel with similar investigations in four-dimensional Yang-Mills theories[5,6] (which yielded a rich variety of solutions with interesting behavior), and in the hope of gaining some insight into more realistic situations. We will restrict ourselves to classical solutions which are useful as starting points for semiclassical expansions.

Based on these considerations we will study the solutions to the three-dimensional Euclidean gauge theories with a Chern-Simons term included. We will consider both the case of a purely imaginary mass parameter μ and of a purely real μ. This last case is of interest in connection with the high-temperature limit of four-dimensional gauge theories in the presence of a chemical potential.[7] In the spirit of semiclassical expansions we will consider only real solutions to the equations of motion. The restriction to Euclidean space, aside from being physically motivated by the high-temperature limit, presents the possibility of mixing spin and isospin indices, a fact which has produced many interesting results in other situations.[8] This type of problem, but without the inclusion of external sources, has been investigated by D'Hoker and Vinet.[9]

In Euclidean space the global quantity of interest to us will be the action S, which to a certain extent replaces the energy in four-dimensional studies. Infinite-action solutions will be rejected as they are not an adequate starting point for a semiclassical expansion. Since what we have in mind is the coupling to external sources we will not, in general, be able to maintain gauge invariance, the coupling being of the form $J\cdot A$. This could be fixed by coupling the sources to gauge-invariant objects, such as a Wilson loop W. Though this is in general not practical (because of the nonlinearities contained in W), we were able to study such a coupling within a restricted *Ansatz*. This coupling will be labeled "to gauge-invariant sources."

The explicit form of the Lagrangian is[10]

$$\mathcal{L}_0=-\frac{1}{4g^2}F^a_{\mu\nu}F^a_{\mu\nu}+\frac{\mu}{4g^2}\epsilon_{\mu\nu\rho}(F^a_{\mu\nu}A^a_\rho+\tfrac{1}{3}A^a_\mu A^b_\nu A^c_\rho\epsilon_{abc})\ , \tag{1.1}$$

$$\mathcal{L}=\mathcal{L}_0-J^a_\mu A^a_\mu\ ,$$

where J^a_μ is the external source and the field strength is given by

$$F^a_{\mu\nu}\equiv\partial_\mu A^a_\nu-\partial_\nu A^a_\mu-\epsilon_{abc}A^b_\mu A^c_\nu\ . \tag{1.2}$$

The corresponding equations of motion are

$$D^{ab}_\mu F^b_{\mu\nu}+\mu{}^*F^a_\nu=g^2J^a_\nu\ , \tag{1.3}$$

having defined the covariant derivative D and the dual field strength by

$$D^{ab}_\mu\equiv\partial_\mu\delta_{ab}+\epsilon_{abc}A^c_\mu,\quad {}^*F^a_\mu\equiv\tfrac{1}{2}\epsilon_{\mu\nu\rho}F^a_{\nu\rho}\ . \tag{1.4}$$

If we contract the left-hand side of (1.3) with D^{ca}_ν we obtain the well-known consistency condition that the equations have a solution only if the external sources are covariantly conserved:

$$D^{ab}_\mu J^b_\mu=0\ . \tag{1.5}$$

We mentioned above that the situations of interest to us will be when μ is either purely real or imaginary. We can however, generalize and consider the equations of motion for arbitrary complex sources and μ, though A will be kept real. In this case (1.3) separate into

$$\begin{aligned} &\mathrm{Im}\mu({}^*F^a_\nu)=g^2\mathrm{Im}J^a_\nu\ , \\ &D^{ab}_\mu F^b_{\mu\nu}+\mathrm{Re}\mu({}^*F^a_\nu)=g^2\mathrm{Re}J^a_\nu\ . \end{aligned} \tag{1.6}$$

When $\mathrm{Im}\mu\neq 0$ these equations have solutions

$$\begin{aligned} &{}^*F^a_\mu=\left[\frac{g^2}{\mathrm{Im}\mu}\right]\mathrm{Im}J^a_\mu\ , \\ &\mathrm{Re}J^a_\mu=\left[\frac{\mathrm{Re}\mu}{\mathrm{Im}\mu}\right]\mathrm{Im}J^a_\mu-\frac{1}{\mathrm{Im}\mu}\epsilon_{\mu\alpha\beta}D^{ab}_\alpha\mathrm{Im}J^b_\beta\ . \end{aligned} \tag{1.7}$$

Thus the real part of the sources is fixed by the imaginary part. From the above expressions it follows that (1.5) is satisfied; moreover the original second-order equations are replaced by a first-order equation relating the dual curvature to the imaginary source. If $\mathrm{Im}\mu=0$ then one has to deal with the full second-order equations in (1.3). Finally, for $\mathrm{Re}\mu=0$, the only consistent solutions are vanishing sources and pure gauge potentials. This will be explicitly verified in several particular cases below.

In the following sections we will consider first the case where μ is purely real. Then we will study the simpler situation where $\mathrm{Im}\mu\neq 0$.

This is perhaps a good place to comment on the gauge invariance of (1.1), and on the possibility of coupling the sources to gauge-invariant objects. Equation (1.5) implies that, for this type of sources, $\mathcal{L}$ in (1.1) is invariant (up to a total derivative) under infinitesimal gauge transformations. This statement does not extend to arbitrary gauge transformations because of the source term: if this piece of $\mathcal{L}$ is to be invariant under arbitrary gauge transformations, the sources must be covariantly conserved for all A in the gauge orbit of the original gauge field, this is in general impossible for nonvanishing J. Later on, when we consider a solution in the radial *Ansatz*, we shall couple the sources to gauge-invariant objects, specifically we will consider the coupling to Wilson loops. It should also be pointed out that, even in the case of zero sources, gauge transformations with appropriate boundary conditions will change the action by a discrete amount proportional to the winding number of the gauge transformation; this leads to the quantization of μ in the quantum theory.[1]

The plan of the paper is as follows: in Sec. II we consider the classical solutions to the equations of motion; we study the Abelian solutions, the full non-Abelian solutions for weak sources and the radially symmetric solutions with arbitrary sources. In Sec. III we consider the effects of small perturbations of these solutions. In Sec. IV we consider the case of arbitrary complex μ. Finally in Sec. V we make some parting remarks, while some mathematical details are relegated to the Appendix.

II. SOLUTIONS

In this section we will consider several types of solutions to the equations (1.3): we will study the Abelian solutions, also we will consider the full non-Abelian solution in the case where the sources are weak, so that a perturbative expansion is reasonable. Finally, we will study in detail the solutions within the radial *Ansatz* both with the "usual" coupling to sources as in (1.1), and with sources coupled to Wilson loops.

A. Abelian solution

In this case we take

$$J^a_\mu=j_\mu\delta_{a3},\quad A^a_\mu=a_\mu\delta_{a3}\ . \tag{2.1}$$

Then (1.3) becomes

$$\begin{aligned} &\partial_\mu f_{\mu\nu}+\mu{}^*f_\nu=g^2j_\nu\ , \\ &f_{\mu\nu}\equiv\partial_\mu f_\nu-\partial_\nu f_\mu,\quad {}^*f_\alpha\equiv\tfrac{1}{2}\epsilon_{\alpha\mu\nu}f_{\mu\nu}\ , \end{aligned} \tag{2.2}$$

which require j to be conserved for consistency.

In terms of a (2.2) imply

$$\Box a_\beta-\partial_\beta(\partial\cdot a)=g^2\frac{\Box}{\Box+\mu^2}\left[j_\beta-\frac{\mu}{\Box}\epsilon_{\beta\mu\nu}\partial_\mu j_\nu\right]\ . \tag{2.3}$$

Writing $a_\beta=\tilde{a}_\beta+\partial_\beta\alpha$, $\Box\alpha=\partial\cdot a$ we obtain that the right-hand side of (2.3) equals $\Box\tilde{a}_\beta$. This provides a complete solution for the *Ansatz* (2.1) with α arbitrary.

Substituting (2.3) into the action we get

$$S=-\frac{g^2}{2}\int d^3x j_\mu\left[\delta_{\mu\nu}+\frac{\mu}{\Box}\epsilon_{\mu\nu\alpha}\partial_\alpha\right]\left[\frac{1}{\Box+\mu^2}j_\nu\right]\ , \tag{2.4}$$

having ignored surface terms. Consider the following <u>*Ansatz*</u> for the sources $j_\mu=f(R)(-y,x,0)$, $R\equiv\sqrt{x^2+y^2}$, then the condition $\partial\cdot j=0$ is immediately satisfied [the more symmetric *Ansatz* $j_\mu=\hat{r}_\mu f(r)$ cannot satisfy (1.5)]. For this cylindrically symmetric source the potential is given by

$$\begin{aligned} &(\Box+\mu^2)\tilde{a}_\mu=g^2(-yf,xf,-\mu F)\ , \\ &F(R)\equiv-\int_R^\infty sf(s)ds\ . \end{aligned} \tag{2.5}$$

Note that for imaginary μ we get a complex potential as expected, though in this case the action per unit length, $\int d^2x\tilde{a}\cdot j$, is real. For real μ one must specify a prescription for dealing with the pole at $\Box=-\mu^2$. We shall define the integrals by their principal value, this being the only case where the potentials are real for a generic source. As a specific example consider the case $f(R)=Q\delta(R-R_0)$, and define $R_<\equiv\min(R,R_0)$, $R_>\equiv\max(R,R_0)$ then $\tilde{a}=(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3)$ with

$$\begin{aligned} &\tilde{a}_1=-\frac{\pi g^2}{2}QR_0^2\frac{y}{R}J_1(\mu R_<)N_1(|\mu|R_>)\ , \\ &\tilde{a}_2=\frac{\pi g^2}{2}QR_0^2\frac{x}{R}J_1(\mu R_<)N_1(|\mu|R_>)\ , \\ &\tilde{a}_3=-\frac{\pi g^2}{2}QR_0^2\frac{\mu}{|\mu|}J_n(\mu R_<)N_{1-n}(|\mu|R_>)\ , \end{aligned} \tag{2.6}$$

where J,N denote the usual Bessel functions and $n=1$ for $R>R_0$ and $n=0$ for $R<R_0$. There is a discontinuity in

$\tilde{a}$ product of the singular nature of j. The action per unit length equals

$$\int \mathcal{L} d^2x = -\frac{\pi}{4} g^2 Q^2 R_0^3 J_1(\mu R_0) N_1(|\mu| R_0) . \quad (2.7)$$

B. Non-Abelian solution, perturbative treatment

Here we consider the full non-Abelian solution to (1.3) in the case of weak sources. To be precise, we assume the source is of order Q and we expand the potentials in powers of Q. To the lowest order in Q it is clear that $F^a_{\mu\nu}=0$, i.e., the potential is a pure gauge. Therefore we write

$$A^a_\mu = -i\,\mathrm{tr}(\sigma^a U^\dagger \partial_\mu U) + Q a^a_\mu + O(Q^2) , \quad (2.8)$$

where σ_a denote the usual Pauli matrices. Clearly it is convenient to make a gauge transformation so as to get rid of the first term in the expansion of A. Denoting the gauge-transformed quantities by a prime we get

$$A'^a_\mu = Q a'^a_\mu + O(Q^2) , \qquad J'^a_\mu = \tfrac{1}{2}\mathrm{tr}(\sigma_a U^\dagger \sigma_b U) J^b_\mu . \quad (2.9)$$

Writing $J'^a_\mu = Q\rho^a_\mu / g^2$ the equations of motion become

$$a'^a_\mu = \left[g_{\mu\nu} + \frac{\mu}{\Box}\epsilon_{\mu\nu\alpha}\partial_\alpha \right] \frac{1}{\Box+\mu^2}\rho^a_\nu + \partial_\mu \alpha^a , \qquad \equiv \tilde{a}^a_\mu + \partial_\mu\alpha^a , \quad (2.10)$$

for some functions α^a. These functions are further restricted by the consistency condition (1.5). Indeed, expanding in powers of Q, the covariant divergence of J' is

$$\partial_\mu J'^a_\mu + Q\epsilon_{abc} a'^b_\mu J'^c_\mu + O(Q^3) = 0 ; \quad (2.11)$$

to lowest order this implies that J' is conserved; to next order it requires $a'^b_\mu J'^c_\mu$ to be symmetric under $b \leftrightarrow c$. This gives, using the definition of $\tilde{a}$,

$$\epsilon_{abc}\partial_\sigma(\tilde{a}^b_\mu \partial_\sigma \tilde{a}^c_\mu - \tfrac{1}{2}\mu\epsilon_{\mu\nu\sigma}\tilde{a}^b_\mu \tilde{a}^c_\nu + \alpha^b\rho^c_\sigma) = 0 . \quad (2.12)$$

The above equation determines the isospin components of α orthogonal to ρ (in the region where $\rho \neq 0$) in terms of ρ, $\tilde{a}$ and the curl of an arbitrary isospinor.

The action for the solutions (2.8)–(2.11) equals

$$S = -\frac{g^2}{2}\int d^3x J'^a_\mu \left[\delta_{\mu\nu} + \frac{\mu}{\Box}\epsilon_{\mu\nu\alpha}\partial_\alpha \right] \left[\frac{1}{\Box+\mu^2} J'^a_\nu \right] + \frac{4\pi\mu}{g^2} w(U) + O(Q^3) . \quad (2.13)$$

where $w(U)$ is the winding number of the gauge transformation U. Note that if the boundary condition $U \to$const as $r\to\infty$ is not imposed, $w(U)$ need not be an integer.[2] In (2.13), a prescription must be specified to deal with the pole in the case where μ is real [unless J' is of the form $(\Box+\mu^2)J''$]. The freedom in the choice of U allows us to consider sources which would be unacceptable within the Abelian *Ansatz;* for example, $J'^a_\mu = (Q/2r_0)\epsilon_{\mu ab}\hat{r}^b\delta(r-r_0)$ (where $r^2 \equiv x_\mu x_\mu$, $\hat{r}_\mu \equiv x_\mu / r$, and the normalization is chosen for ease of comparison with the results from the radial *Ansatz*). This does satisfy $\partial_\mu J'^a_\mu = 0$ and gives rise to

$$A'^a_\mu = \frac{Qg^2}{2r_0}\left[\epsilon_{\mu ab}\hat{r}^b F(r) - (\delta_{a\mu} - \hat{r}_a\hat{r}_\mu)\frac{(rG)'}{r} - \frac{2G}{r}\hat{r}_a\hat{r}_\mu \right] + Q\partial_\mu\alpha^a , \quad (2.14)$$

where a prime denotes differentiation with respect to r. The functions F,G are defined through $\hat{r}^b\delta(r-r_0) = (\Box+\mu^2)(\hat{r}^b F)$, and $\mu\hat{r}^b\delta(r-r_0) = \Box(\Box+\mu^2)(\hat{r}^b G)$. Note that, in contrast, a source $j_\mu = \hat{r}_\mu\delta(r-r_0)$ cannot be used in the radial *Ansatz* since it violates (1.5).

The equations defining F and G are equivalent to

$$G'' + \frac{2}{r}G' - \frac{2}{r^2}G = \mu F , \qquad F'' + \frac{2}{r}F' - \frac{2}{r^2}F + \mu^2 F = \delta(r-r_0) . \quad (2.15)$$

In solving (2.15) it is convenient to define $x \equiv \mu r/2$ and

$$\xi_0 \equiv \frac{1}{\sqrt{2\pi}}\left[\cos 2x - \frac{\sin 2x}{2x} \right] , \qquad \eta_0 \equiv \frac{1}{\sqrt{2\pi}}\left[\sin 2x + \frac{\cos 2x}{2x} \right] . \quad (2.16)$$

Then F and G are given by $(x_0 \equiv \mu r_0/2)$

$$F = \frac{2c_F}{x\mu}\xi_0(x) + \frac{\pi r_0}{x}\begin{cases}\eta_0(x_0)\xi_0(x) & \text{if } x \le x_0, \\ \xi_0(x_0)\eta_0(x) & \text{if } x > x_0,\end{cases}$$
$$G = \frac{2c_G}{x\mu^2}\xi_0(x) - \frac{F}{\mu} - \frac{2}{3\mu^2}\begin{cases} x & \text{if } x \le x_0, \\ \dfrac{x_0^3}{x^2} & \text{if } x > x_0,\end{cases} \quad (2.17)$$

which in particular implies

$$\left.\frac{d(xG)}{dx}\right|_{x=x_0} = -\frac{2\pi x_0}{\mu^3}\eta_0(x_0)\xi_0'(x_0) - \frac{2c_G}{\mu^3}\xi_0'(x_0) - \frac{4x_0}{3\mu^3} . \quad (2.18)$$

The undetermined constants $c_{F,G}$ multiply the zero eigenvectors of the operator $\Box+\mu^2$ and for this reason are arbitrary when μ is real. On the other hand, for μ purely imaginary, they should be chosen so that F and G decrease exponentially as $r\to\infty$. This is easily accomplished: when $\mu = im$, the appropriate replacements are $x\to ix$, $\eta_0 \to -i\eta_0$, then we need $c_F = \pi m r_0 \xi_0(imr_0/2)/2$, $c_G = 0$. In this case, as can be seen from (2.17), G is purely imaginary, so that A is complex. At least for this choice for J' there is no real solution for the potentials; this will be generalized in Sec. III for arbitrary source strength within the radial *Ansatz*. This agrees with the general argument made below (1.5).

The condition (2.12) implies, taking $\alpha^a = v(r)\hat{r}^a$,

$$v(r_0)=\frac{\mu g^2}{2r_0}\frac{d(rG)}{dr}\bigg|_{r=r_0}, \quad (2.19)$$

which can be satisfied, for example, by either choosing $c_G=c_F=0$ and v according to (2.19), or $v\equiv 0$ and c_F, c_G so that (2.18) vanishes.

The action (2.13) corresponding to this source is given by

$$S=-\pi Q^2 g^2 F(r_0)+\frac{4\pi\mu}{g}w(U)+O(Q^3),$$

which suffers from the same ambiguity as F due to the freedom in dealing with the pole in (2.13) at $\Box=-\mu^2$. This solution will be compared to the radially symmetric one in Sec. II C.

C. Radially symmetric solutions

As is well known,[11] the most general radially symmetric *Ansatz* for the potentials can be parametrized as follows

$$A_\mu^a=\frac{\phi_1+1}{r}\epsilon_{\mu ab}\hat{r}_b+\frac{\phi_2}{r}(\delta_{\mu a}-\hat{r}_\mu\hat{r}_a)+A\hat{r}_\mu\hat{r}_a, \quad (2.20)$$

where ϕ_1, ϕ_2, and A are functions of r only.

Within this *Ansatz* the potentials change, under a radial gauge transformation specified by $U=\exp[i\sigma\cdot\hat{r}\theta(r)/2]$, to

$$(\phi_1+i\phi_2)\to\exp(-i\theta)(\phi_1+i\phi_2), \quad A\to A+\frac{d\theta}{dr}. \quad (2.21)$$

It is convenient to define a complex scalar field Φ and a covariant derivative D by

$$\Phi\equiv\phi_1+i\phi_2, \quad D\equiv\partial_r+iA. \quad (2.22)$$

In terms of these variables the Lagrangian $\mathcal{L}_0$ in (1.1) becomes

$$4\pi r^2\mathcal{L}_0\equiv L_0=-\frac{4\pi}{g^2}\left[|D\Phi|^2+\frac{1}{2r^2}(|\Phi|^2-1)^2+\mu\left[\mathrm{Im}(\Phi^*D\Phi)-A+\frac{1}{2}\frac{d\phi_2}{dr}\right]\right], \quad (2.23)$$

which changes by a total derivative under the transformation (2.21).

Within this *Ansatz* the fields are given by

$${}^*F_\mu^a=\frac{1}{r}\left[(\delta_{a\mu}-\hat{r}_a\hat{r}_\mu)(\mathrm{Re}D\Phi)+\epsilon_{a\mu b}\hat{r}_b(\mathrm{Im}D\Phi)+\hat{r}_a\hat{r}_\mu\frac{1-|\Phi|^2}{r}\right]. \quad (2.24)$$

We shall be interested only in finite-action solutions. This requires that

$$|\Phi(r=0)|=1, \quad \frac{d}{dr}|\Phi(r)|\bigg|_{r=0}=0. \quad (2.25)$$

Moreover, we will need to consider the boundary conditions as $r\to\infty$. In particular, for μ real, the nonsingular solutions will oscillate at infinity with constant amplitude. Therefore the action integral will not be absolutely convergent. To evaluate such integrals we will multiply the integrand by $\exp(-\eta r)$, perform the integration, and then let $\eta\to 0^+$. It is in this sense that the solutions with μ real will have finite action. The boundary conditions imposed above are different from the ones used in Ref. 9.

Having specified the boundary conditions we turn to the various sources that we will consider. First there is the coupling of the potential in the "usual" manner to external sources. In the same spirit of radial symmetry we will consider sources

$$J_\mu^a=\frac{1}{2r}j_1(r)\epsilon_{\mu ab}\hat{r}_b+\frac{1}{2r}j_2(r)(\delta_{\mu a}-\hat{r}_\mu\hat{r}_a)+\frac{1}{r^2}J(r)\hat{r}_\mu\hat{r}_a, \quad (2.26)$$

for which the Lagrangian becomes, dropping a total derivative,

$$L\equiv L_0-4\pi r^2 J_\mu^a A_\mu^a=-\frac{4\pi}{g^2}\left[|D\Phi|^2+\frac{1}{2r^2}(|\Phi|^2-1)^2+\mu[\mathrm{Im}(\Phi^*D\Phi)-A]-g^2(j_1\phi_1+j_2\phi_2+JA)\right]. \quad (2.27)$$

The equations of motion derived from (2.27) are

$$-D^2\Phi+\frac{1}{r^2}(|\Phi|^2-1)\Phi-i\mu D\Phi=\tfrac{1}{2}g^2 j, \quad j\equi j_1+ij_2; \quad (2.28)$$

$$\mathrm{Im}(\Phi^*D\Phi)+\tfrac{1}{2}\mu(|\Phi|^2-1)=\tfrac{1}{2}g^2 J,$$

which can also be obtained by direct substitution of the *Ansätze* in (1.3). Similarly the consistency equation (1.5) reads

$$\frac{dJ}{dr}=\mathrm{Im}j^*\Phi. \quad (2.29)$$

Another possibility is to couple the sources to gauge-invariant function(al)s of the fields. In particular one can consider the Wilson W loop corresponding to the circle $r(\cos(s),\sin(s),0)$ in the 1-2 plane. Then the two relevant invariants are $\det W$ and $\mathrm{tr}W$. We obtain $\det W=1$ and

$$\mathrm{tr}W=-2\cos(\pi|\Phi|). \quad (2.30)$$

Based on this we will consider another coupling, namely, one of the form

$$L_{GI}=L_0+\frac{\pi\mu^2}{g^2}\mathcal{T}(r)|\Phi(r)|^2, \quad (2.31)$$

where $\mathcal{T}$ is assumed real. As mentioned before, we call this type of coupling "gauge invariant."

The equations of motion which follow from (2.31) are

$$-D^2\Phi+\frac{1}{r^2}(|\Phi|^2-1)\Phi-i\mu D\Phi=\frac{\mu^2}{4}\mathcal{T}\Phi\ ,$$
$$\mathrm{Im}(\Phi^* D\Phi)=\tfrac{1}{2}\mu(1-|\Phi|^2)\ . \qquad (2.32)$$

Note that there is no condition corresponding to (1.5) for this type of sources since these couple, within the *Ansatz*, to gauge-invariant objects. Equations (2.32) are, as expected, the same as those obtained when coupling the gauge fields to scalar triplets, which are then assumed infinitely massive, so that they act as external sources, (see Ref. 9).

Below we will study the solutions to the equations of motion (2.28) and (2.32).

1. Gauge-invariant source within the radial Ansatz

When μ is real, (2.32) can be simplified by defining new variables ρ, Θ, by

$$\Phi\equiv\rho\exp\left[-i\int_0^r A(r')dr'-i\frac{\mu}{2}r+i\Theta\right]\ , \qquad (2.33)$$

which must satisfy

$$-\rho''+\frac{1}{\rho^3}+\left[\frac{\rho^2-1-x^2}{x^2}\right]\rho=\mathcal{T}\rho\ ,$$
$$\Theta'=\rho^{-2}\ , \qquad (2.34)$$

where $x\equiv\mu r/2$ and ρ, Θ, and $\mathcal{T}$ are regarded as functions of x. In obtaining (2.34) we used the fact that, substituting (2.33) in (2.32) the imaginary part of the first equation is redundant when the second equation is included. Note that we can always take $\Theta(0)=0$.

The general features of the solutions are studied in the Appendix, here we only present a short summary of the results. The solutions have three possible kinds of behavior as $x\to\infty$: they can oscillate, which corresponds to finite-action solutions; they can diverge at a finite value of x; or they diverge linearly in x when $x\to\infty$. The solutions never vanish and are analytic for $x\to 0$. In the case $\mathcal{T}=Q\delta(x-a)$, for each a there is a critical value of Q, $Q_c(a)$, above which the solutions diverge for finite values of x, as $a\to\infty$, $Q_c\to\infty$ also. Some examples of these solutions are presented in Fig. 1. The graphs of the critical charge versus a can be found in the Appendix.

For the case where $\mu=im$ with m real it is easy to see that there are no interesting solutions. Indeed, from the second equation in (2.32) it immediately follows that Φ and A must correspond to a pure gauge solution $\Phi=\exp i\theta$, $A=\theta'$ and then the first equation implies that $\mathcal{T}=0$ is the only consistent possibility. The solutions are pure gauges as expected from the general argument following (1.5).

It is now a simple matter to obtain the field strengths by substituting in (2.24). Though the explicit expressions are not illuminating, it is worth pointing out that as $x\to 0$, the dominant contribution to $^*F$ comes from the terms in $\mathrm{Re}D\Phi$ while for large x, $^*F$ behaves as an oscillatory function whose amplitude decreases as $1/x$. All coefficient functions in $^*F$ are analytic.

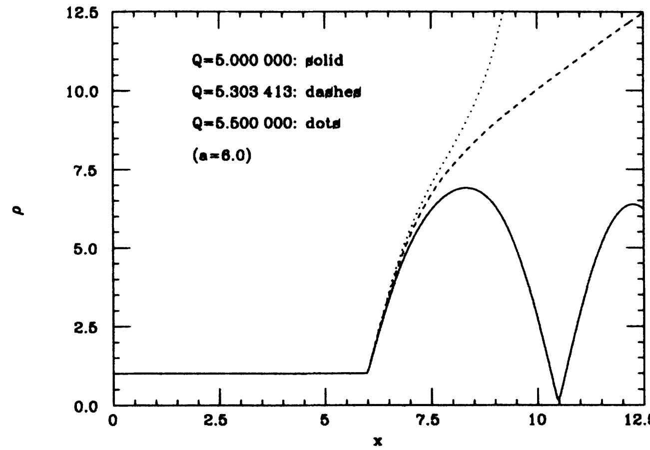

FIG. 1. Solutions to the equations of motion in the radial *Ansatz* for gauge-invariant sources. The dashed curve corresponds to the critical charge for $a=6$.

Choosing $\Theta(0)=0$, the action for solutions to (2.34) is given by

$$S_{GI}=-\frac{2\pi\mu}{g^2}\int_0^\infty dx\left[\frac{1-\rho^4}{2x^2}-\frac{4A}{\mu}\right]\ . \qquad (2.35)$$

We have evaluated the action as a function of Q for a delta function source and $A=0$. The results are presented in Fig. 2.

2. "Usual" sources within the radial Ansatz

Consider (2.28) for the case where μ is real and take, for simplicity,

$$j=q\delta(r-r_0)\exp(i\gamma)\ . \qquad (2.36)$$

Now defining new (real) variables χ, λ, by [cf. (2.33)]

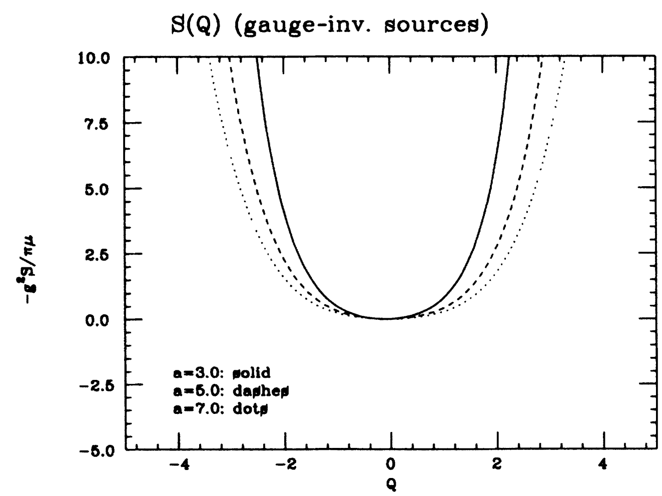

FIG. 2. Action for the solutions with gauge-invariant sources in the radial *Ansatz*. The curves need not be symmetric under $Q\to -Q$ since there is no such symmetry in (2.32). We assumed $A=0$.

$$\Phi \equiv \chi \exp \left[-i \int_{r_0}^{r} A(r')dr' - i\frac{\mu}{2}(r - r_0) + i[\lambda(r) - \lambda(r_0)] + i\gamma \right], \quad (2.37)$$

the equations (2.28) reduce to

$$-\chi'' + \lambda'^2\chi + \frac{\chi^2 - 1 - x^2}{x^2}\chi = \tilde{j} ,$$
$$\lambda''\chi + 2\lambda'\chi' = 0 , \quad (2.38)$$
$$\lambda'\chi^2 = \tilde{J} + 1 ,$$

where a prime denotes differentiation with respect to $x \equiv \mu r/2$, and $\tilde{j} \equiv (2g^2/\mu^2)q\delta(r - r_0)$, $\tilde{J} \equiv (g^2/\mu)J$.

The second equation in (2.38) can be integrated to yield

$$\lambda'(x) = \frac{l}{\chi(x)^2}, \quad l \equiv \lambda'(0) , \quad (2.39)$$

where we have used the boundary condition $\chi(0) = 1$, which follows from (2.25). Then the third equation in (2.38) implies

$$\tilde{J} + 1 = l = \text{const} \quad (2.40)$$

so that (1.3) and (1.5) allow the *Ansatz* (2.20) and (2.26) only if J is a constant. Finally, using (2.39), the first of Eqs. (2.38) becomes

$$-\chi'' + \frac{l^2}{\chi^3} + \frac{\chi^2 - 1 - x^2}{x^2}\chi = \tilde{j} , \quad (2.41)$$

where, in terms of x, $\tilde{j} = Q\delta(x - a)$, $a \equiv \mu r_0/2$, $Q \equiv g^2q/\mu$. Note that for $j = 0$ we have $\chi = 1$, and this requires $l^2 = 1$ and $\lambda'(x) = \pm 1$.

The qualitative behavior of this equation is studied in the Appendix and is found to be basically the same as for (2.34). The only difference is that in this case we allow $l \neq 1$. Several examples of these functions are given in Fig. 3. Ignoring surface terms, the action for solutions to (2.38) is given by (2.35) under the replacements $\rho \to \chi$, $A \to lA$. We have evaluated the action as a function of Q for this case, taking again $A = 0$; the results are given in Fig. 4.

For the case $\text{Re}(\mu) = 0$ and with sources (2.36), it is not hard to show that there are no nontrivial solutions. This is in accord with the argument following (1.5), and with the results obtained in Sec. I for weak sources.

To make contact with the perturbative calculation consider the case $\gamma = 0$, $l = 1$ which corresponds to the source used at the end of Sec. II B. Expanding χ in powers of Q we have $\chi = 1 + Q\delta\chi$, which satisfies

$$\left[-\frac{d^2}{dx^2} - 4 + \frac{2}{x^2} \right] \delta\chi = \frac{g^2}{\mu}\delta(x - x_0) . \quad (2.42)$$

But from (2.15) it follows that this is essentially the equation satisfied by rF. With the appropriate normalization we have

$$\delta\chi = -\frac{g^2}{2}\frac{r}{r_0}F . \quad (2.43)$$

On the other hand, (2.14) corresponds to

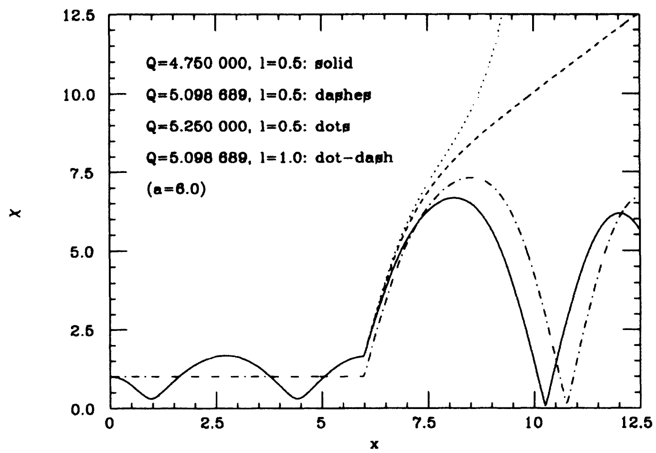

FIG. 3. Solutions to the equations of motion in the radial *Ansatz* for the usual sources. $a \equiv \mu r_0/2$ denotes the position of the source, and $Q \equiv g^2q/\mu$ its strength. The dashed curve corresponds to the critical charge for $l = 0.5$, $a = 6.0$.

$A = -(Qg^2/r_0)(G/r)$ and $\Phi = -1 + (Qg^2/2r_0)[rF - i(rG)']$ (a prime denotes differentiation with respect to r here). Making a gauge transformation (2.21) with $\theta = \pi + (Qg^2/2r_0)(rG)'$, and using (2.15) we obtain

$$\Phi \to 1 - \frac{Qg^2}{2}\frac{r}{r_0}F, \quad A \to \mu\frac{Qg^2}{2}\frac{r}{r_0}F . \quad (2.44)$$

[Under this gauge transformation the source merely changes sign due to (1.5).] Since this gives precisely the same result as (2.43) we conclude that, as expected, in the $Q \to 0$ limit the radial *Ansatz* goes smoothly into the perturbative calculation of Sec. I.

We would like to stress that most of the features described above are independent of the specific form as-

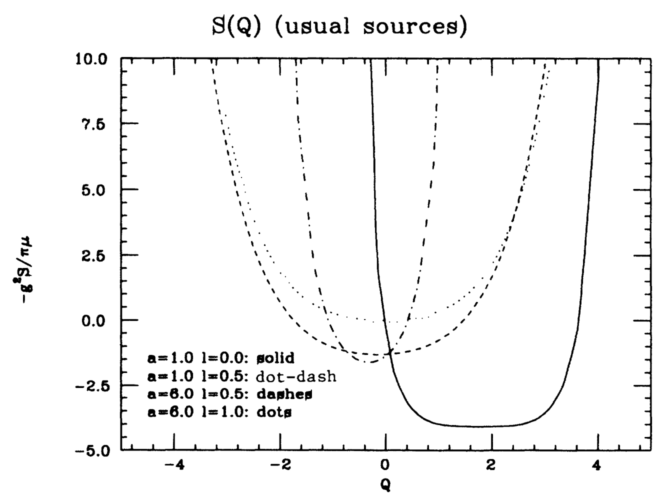

FIG. 4. Action for the solutions with usual sources in the radial *Ansatz*. The curves are not symmetric under $Q \to -Q$ [taking also $\chi \to -\chi$, see (2.41)] because of the boundary conditions imposed on χ. We assumed $A = 0$.

sumed for j, and will remain valid as long as j is localized. Note however that in this more general case, (2.39) and (2.40) will hold only in the region where $j=0$. Moreover one can no longer dispose of the field A by a change of variables, a given form for A will modify the sources for χ and λ [see (2.37)] and alter the character of the solution.

We conclude this section with a brief comment on the behavior of the fields as given by (2.24). In general $^*F$ has the same behavior as for the gauge-invariant sources, with two differences: first, for $l\neq 1$, the coefficient functions are no longer analytic as is clear from (A4) and (A5). And second, the dominant terms for small x are those in $\mathrm{Re}D\Phi$ and in $\mathrm{Im}D\Phi$ [so that in general $D\Phi$ $(x=0)$ is a nonzero complex number].

III. PERTURBATIONS AROUND A CLASSICAL SOLUTION

In this section we consider small perturbations around a solution to equations (1.3). We shall label a solution stable if it is a local minimum of the action. We shall see that, in fact, none of the solutions studied in Sec. II correspond to local minima; the corresponding destabilizing modes are described in detail below. We shall only consider the case where μ is real.

Given a solution to the classical equations of motion $\bar{A}^a_\mu$, we will study the action when the potentials are given by $A^a_\mu \equiv \bar{A}^a_\mu + \delta A^a_\mu$, where only terms up to quadratic order in the perturbation will be retained. In expanding the action, the zeroth-order term $\bar{S}$ is just the action for the solution $\bar{A}^a_\mu$; the first-order terms are ignored as they can be reduced to surface contributions by using the fact that $\bar{A}^a_\mu$ solves the equations of motion. After a straightforward computation we obtain

$$S=\bar{S}+\frac{1}{g^2}\int d^3x\,\delta A^a_\mu \Upsilon^{ab}_{\mu\nu}\delta A^b_\nu + \text{surface terms} + O((\delta A)^3)\,,$$
$$\Upsilon^{ab}_{\mu\nu}\equiv(\bar{D}_\alpha\bar{D}_\alpha\delta_{\mu\nu}-\bar{D}_\mu\bar{D}_\nu)^{ab}+2\epsilon_{abc}\bar{F}^{\,c}_{\mu\nu}-\mu\epsilon_{\mu\nu\alpha}\bar{D}^{\,ab}_\alpha\,, \quad (3.1)$$

where $\bar{D}$ is the covariant derivative associated with $\bar{A}^a_\mu$. The perturbation analysis therefore reduces to the study of the eigenvalues of Υ. In the following such a problem will be considered for the solutions obtained in Sec. II.

A. Abelian solution

Using the fact that $\bar{D}^{ab}_\mu=\partial_\mu\delta^{ab}-\epsilon_{a3b}a_\mu$, it follows that the nonvanishing elements of Υ are

$$\Upsilon^{11}_{\mu\nu}=\Upsilon^{22}_{\mu\nu}=(\Box-a_\alpha a_\alpha)\delta_{\mu\nu}-(\partial_\mu\partial_\nu-a_\mu a_\nu)-\mu\epsilon_{\mu\nu\alpha}\partial_\alpha\,,$$
$$\Upsilon^{33}_{\mu\nu}=(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu)-\mu\epsilon_{\mu\nu\alpha}\partial_\alpha\,, \quad (3.2)$$
$$\Upsilon^{12}_{\mu\nu}=-\Upsilon^{21}_{\mu\nu}=(\partial_\alpha a_\alpha+2a_\alpha\partial_\alpha)\delta_{\mu\nu}+(a_\mu\partial_\nu+a_\nu\partial_\mu)+(\partial_\mu a_\nu)-2(\partial_\nu a_\mu)-\mu\epsilon_{\mu\nu\alpha}a_\alpha\,.$$

Consider, in particular, the Fourier transform of Υ^{33} (which we denote by a tilde):

$$\tilde{\Upsilon}^{33}_{\mu\nu}(k)=-k^2\delta_{\mu\nu}+k_\mu k_\nu+i\mu\epsilon_{\mu\nu\alpha}k_\alpha\,. \quad (3.3)$$

This matrix has eigenvalues 0, $k^2\pm|\mu\sqrt{k^2}|$, so that unstable modes appear for $k^2<\mu^2$. The three zero modes of $\tilde{\Upsilon}^{33}$ are $z_1(k)=k_\mu f(k^2)$, $z_2(k)=\delta(|k|-\mu)(\hat{e}_1-i\hat{e}_2)_\mu$, $z_3(k)=\delta(|k|+\mu)(\hat{e}_1+i\hat{e}_2)_\mu$ (where $\hat{e}_1$, $\hat{e}_2$, $\hat{k}$ form an ordered orthonormal basis); the term in f is clearly related to gauge transformations. The fact that there is more than one zero mode for Υ implies that its lowest eigenvalue is negative, this is explicitly verified for Υ^{33}: modes of the form $g(k^2)\theta(\mu^2-k^2)(\hat{e}_1\mp i\hat{e}_2)$ correspond to negative eigenvalues for $\pm\mu>0$.

For the rest of the operators Υ, it is convenient to define $\hat{\Upsilon}\equiv\Upsilon^{11}-i\Upsilon^{12}$ and $v\equiv\delta A^1+i\delta A^2$; then the remaining eigenvalues λ are obtained from

$$\hat{\Upsilon}v=\lambda v\,. \quad (3.4)$$

The solutions of this eigenvalue problem depend on the explicit form of a and will not be studied further.

Because of the presence of destabilizing modes we conclude that the extremum of the action represented by the solution (2.3) is actually a saddle point. This can be seen directly from the action: if we consider perturbations only in the same isospin direction as the source's then the Lagrangian is of the form $(\partial A)^2+\mu A\partial A$ (since the potentially stabilizing terms in $\mathcal{L}$, behaving as $[A,A]^2$, vanish identically). Therefore we expect the long-wavelength (i.e., for $k<\mu$) modes to be unstable directions for the action functional. Other directions will be stable, provided the F^2 term in $\mathcal{L}$ dominates for large fields irrespective of the wavelength of the mode considered. Unfortunately, as we will see below, it is difficult to restate this condition so that one can easily decide whether or not a solution will be stable.

B. Non-Abelian solution

Working to the same order as in Sec. II B we can use the solutions (2.10) and expand Υ in powers of Q. The result is

$$\Upsilon^{ab}_{\mu\nu}=(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu-\mu\epsilon_{\mu\nu\alpha}\partial_\alpha)\delta^{ab}+Q\epsilon_{abc}[(\partial_\mu a'^c_\nu)-2(\partial_\nu a'^c_\mu)+(\Box\alpha^c)\delta_{\mu\nu}+2a'^c_\alpha\partial\alpha\delta_{\mu\nu}-a'^c_\nu\partial_\mu-a'^c_\mu\partial_\nu-\mu\epsilon_{\mu\nu\alpha}a'^c_\alpha]+O(Q^2)\,. \quad (3.5)$$

The lowest order in Q can be treated in the same way as (3.2). We find an instability for modes of momentum magnitude lower than $|\mu|$ and also a zero mode associated with gauge invariance to this order. In this case the instability is present because, to lowest order in Q, the potential is a pure gauge and $\mathcal{L}$ behaves as A^3. If the amplitude of a destablizing mode becomes of order $1/Q$, then the expansion in powers of Q is invalid, and no conclusions regarding the behavior of the action are available.

A new feature of (3.5) [as compared with (3.2)] is the degeneracy in the isospin indices. Therefore we will study the next-order corrections to see how this degeneracy is broken, and to investigate whether there is any indication of the higher orders tending to stabilize the zeroth-order unstable modes.

Denote by $\Upsilon^{(1)}$ the order Q term in (3.5); notice that $\Upsilon^{(1)}=\epsilon_{abc}\Psi^c$, for an operator Ψ whose form will be of no relevance. Consider the eigenmodes v_λ of the zeroth-order term of (3.5) corresponding to eigenvalue λ, and define $i\psi_\lambda^c \equiv \langle v_\lambda | \Psi^c | v_\lambda \rangle$ (with ψ real) so that the matrix that determines the degeneracy breaking is $i\epsilon_{abc}\psi_\lambda^c$. But this generically has one positive, one negative, and one zero eigenvalues; therefore $\Upsilon^{(1)}$ does break the degeneracy present in the zeroth-order approximation to Υ; however only one of the originally unstable modes tends to get stabilized. We conclude that, at least to this order, the extremum of the action is also a saddle point. Some of the comments regarding the Abelian zero modes apply in this case also: Υ has a zero mode due to the gauge freedom of the model, but it also has other zero modes, and this forces the lowest eigenvalue to be negative. On the other hand, we argued above that outside the purely Abelian situation we should get stable solutions; this does not occur in our calculations because of the approximations made: the stability is produced by the commutator terms in F and these are neglected in the calculations of this subsection.

C. Radially symmetric solutions

In studying the stability of the solutions to perturbations within the radial *Ansatz*, we shall concentrate on the more familiar case of the "usual" coupling to sources; the coupling to gauge-invariant sources is briefly dealt with at the end.

Instead of directly substituting (2.20) into Υ, it proved easier to study (2.27) directly, and make the replacements $A \to A + \mu a/2$, and $\chi \to \chi + (u+iv)$, in (2.37). We assume χ, λ, and A solve (2.28), and that u, v, and a are small, real perturbations. Since in general $v \neq 0$, this replacement is the most general variation allowed within the *Ansatz*. After use is made of the equations of motion (2.38), and since u and v vanish at $x=0,\infty$ so as to preserve the boundary conditions, the change in the action is given by

$$\Delta S = -\frac{\pi\mu^2}{g^2}\int_0^\infty dx \left[u\mathcal{O}u + \left|\frac{\tilde{j}}{\chi}\right| v^2 + \left[a\chi + v' - \frac{\chi'}{\chi}v + 2u\lambda' \right]^2 \right], \tag{3.6}$$

where

$$\begin{aligned}\mathcal{O} &\equiv -\frac{d^2}{dx^2} + \frac{3\chi^2 - 1 - x^2}{x^2} - \frac{3l^2}{\chi^4} \\ &= -\frac{d^2}{dx^2} + \mathcal{V} ,\end{aligned} \tag{3.7}$$

which shows that the destabilizing modes are characterized by the negative eigenvalues of $\mathcal{O}$. If these are absent, then the solutions will be stable against radial perturbations; we shall see, however, that negative eigenvalues *do* appear in general so that the solutions will be unstable even against the restricted class of radially symmetric perturbations.

The potential $\mathcal{V}$ in the Schrödinger operator (3.7) diverges as $2/x^2$ when $x \to 0$ and becomes negative for large enough x: $\mathcal{V} \leq -1$ for sufficiently large x. Some examples of $\mathcal{V}$ are presented in Fig. 5. The regularity conditions imposed on the eigenfunctions are that, as $x \to \infty$, they should either vanish or, at worst, oscillate with a bounded amplitude. As $x \to 0$, the eigenfunctions vanish as x^2. Since the potential becomes negative at infinity, $\mathcal{O}$ will accept zero modes for specific values of the charge; such eigenfunctions will be denoted by Ξ_0. More importantly, quite generally $\mathcal{O}$ will have negative eigenvalues, so that even within the radial *Ansatz* there are destabilizing modes present: the solution represents a saddle point of the action.

Note that the above considerations are still true even for $j=0$, i.e., for pure gauge solutions; in this simple case we can easily obtain the spectrum of $\mathcal{O}$ exactly. Indeed, for $j=0$, $\chi = l^2 = 1$, and

$$\mathcal{O} = -\frac{d^2}{dx^2} + \frac{2}{x^2} - 4 \quad (\tilde{j}=0) , \tag{3.8}$$

which has the eigenfunctions and eigenvalues

$$\xi_\lambda = \cos(x\sqrt{\lambda+4}) - \frac{\sin(x\sqrt{\lambda+4})}{x\sqrt{\lambda+4}} \quad (\lambda \geq -4) . \tag{3.9}$$

The eigenfunctions corresponding to $\lambda < -4$ increase exponentially at infinity, while the other solutions to the Schrödinger-like equation are ill defined at the origin. Hence (3.9) gives the only acceptable eigenfunctions. ξ_0 has been used previously in (2.16).

Within the radial *Ansatz*, a perturbation corresponding to a gauge transformation is given by $u = -\theta^2\chi/2$, $v = \theta\chi$, $a = -\theta'$. When these expressions are substituted in (3.6) we obtain $\Delta S = 0$ (having used the equations of motion and boundary conditions), i.e., gauge transformations correspond to a zero mode. On the other hand, if $\mathcal{O}$ has a zero mode Ξ_0, then we can choose $u = \Xi_0$, $v = 0$, $a = 2\Xi_0\lambda'/\chi$ which, when substituted in (3.6), also give

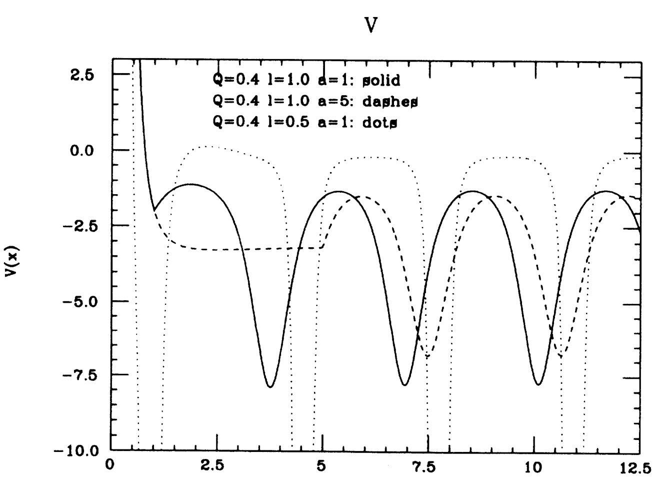

FIG. 5. Potential for the perturbations within the radial *Ansatz*.

$\Delta S=0$; this second zero mode is unrelated to gauge transformations, that is, one can check that in this case δA_μ cannot be written in the form $\bar{D}_\mu \sigma$ for some functions σ.

The reason for the instability is the following. The terms that are quartic in the gauge potential are of the form $|\Phi|^2 A^2$ or $|\Phi|^4/x^2$; the first quantity can be eliminated with a gauge transformation (which sets $A=0$); the second quantity is gauge invariant, but can be (and in fact is) smaller than the quadratic terms for sufficiently low wavelength modes, because of the factor of $1/x^2$. This illustrates the previously mentioned fact that even if the "stabilizing" terms $[A,A]^2$ in $\mathcal{L}$ do not vanish, it does not follow that the corresponding solution is stable; thus precluding a simple stability test.

For gauge-invariant sources we obtain the same form for ΔS as above; the only difference is that in this case $\mathcal{O}\to\mathcal{O}-\mathcal{T}$. Thus, for localized sources, we obtain the same qualitative behavior of the solutions as for the previous case.

In the above calculations we considered perturbations which vanished as $|x|\to\infty$. This will not be the case if, for example, we consider gauge transformations U which do not satisfy $U\to$const for large x. Under this type of transformations we have

$$\mathcal{L}_0\to\mathcal{L}_0+\frac{\mu}{2g^2}[\tfrac{1}{3}\epsilon_{\mu\nu\alpha}\mathrm{tr}\,U^\dagger\partial_\mu U U^\dagger\partial_\nu U U^\dagger \partial_\alpha U + 2\epsilon_{\mu\nu\alpha}\partial_\mu(\mathrm{tr}\,A_\nu\partial_\alpha U^\dagger)] . \qquad (3.10)$$

The second term on the right-hand side gives rise to the winding number when appropriate boundary conditions are imposed on U; in this case its integral is $(4\pi^2\mu/g^2)\times$integer. If no such boundary conditions are imposed this last term, which can be written as a surface term,[2] can take arbitrary values. In either case a gauge transformation can change the action by an arbitrarily large (in absolute value) quantity without, of course, affecting the equations of motion. This is true even for pure gauge solutions.

IV. COMPLEX μ

When $\mathrm{Im}\mu\neq 0$ the equations of motion simplify to a set of consistency conditions on the real part of the sources and to a first-order differential equation which determines the curvature, as seen from (1.7). In this case, the action is complex. We will now study the three types of solutions considered above.

1. *Abelian solutions*

Taking the same *Ansatz* as in (2.1), we find that (1.7) imply

$$^*f_u=\frac{g}{\mathrm{Im}\mu}\mathrm{Im}j_\mu \Longrightarrow a_\mu=-\left[\frac{g^2}{\mathrm{Im}\mu}\right]\frac{1}{\Box}\epsilon_{\mu\nu\beta}\partial_\nu \mathrm{Im}j_\beta+\partial_\mu\alpha , \qquad (4.1)$$

which provides the complete answer in this case. For the specific example $\mathrm{Im}j_\mu=f(R)(-y,x,0)$, $R\equiv\sqrt{x^2+y^2}$ we get

$$a_\mu-\partial_\mu\alpha=\left[\frac{g^2}{\mathrm{Im}\mu}\int_R^\infty sf(s)ds\right](0,0,1) ,$$
$$\mathrm{Re}j_\mu=\frac{\mathrm{Re}\mu}{\mathrm{Im}\mu}f(R)\left[-y,x,\frac{2+Rf'/f}{\mathrm{Re}\mu}\right] . \qquad (4.2)$$

2. *Non-Abelian solution, weak sources*

As in Sec. II we assume the source is of strength Q, to zeroth order the potential is a pure gauge and we go to the gauge where it vanishes. In this new gauge the quantities are denoted by primes. Thus we have $A_\mu^{a'}=Qa_\mu^{a'}+Q^2b_\mu^{a'}+\cdots$, which, upon substituting in (1.7) and solving, gives

$$a_\mu^{a'}=-\frac{g^2}{\mathrm{Im}\mu}\frac{1}{\Box}\epsilon_{\mu\nu\beta}\partial_\nu\mathrm{Im}J_\beta^{a'}+\partial_\mu\alpha^a ,$$
$$b_\mu^{a'}=\frac{1}{\Box}\epsilon_{abc}(a_\nu^{b'}a_\nu^{c'})+\partial_\mu\beta^a \qquad (4.3)$$

where α^a, β^a are arbitrary functions.

3. *Radial Ansatz*

We will consider only the usual sources. Then, substituting (2.24) in (1.7), we get, using (2.26),

$$D\Phi=-ipj_I,\quad 2p\,\mathrm{Im}J=1-|\Phi|^2 , \qquad (4.4)$$

where $p\equiv g^2/2\,\mathrm{Im}\mu$, $j_I\equiv\mathrm{Im}j_1+i\,\mathrm{Im}j_2$. The general solution to (4.4) is

$$\Phi(r)=\Phi_0-ip\mathcal{A}(r)\int_0^r\mathcal{A}(r')^*j_I(r')dr' ,$$
$$\mathcal{A}(r)\equiv\exp\left[-i\int_0^r A(r')dr'\right] . \qquad (4.5)$$

Note that a general feature is that A, being a gauge artifact, is completely free. For finite-action solutions we must have $|\Phi_0|=1$ and $\mathrm{Im}[j_I(0)\Phi_0^*]=0$. In the specific case where $j_I=Q\delta(r-r_0)$, we have $\Phi=\Phi_0-ipQ\mathcal{A}(r)\theta(r-r_0)/\mathcal{A}(r_0)$.

This concludes the description of the solutions when μ is not purely real.

V. CONCLUSIONS

In the above calculations we obtained a complete characterization of several classes of finite-action solutions to the classical equations of motion. Since we work in Euclidean space, the general behavior of the solutions at infinity is oscillatory (as opposed to exponentially damped, for a Minkowskian metric) and this decreases to certain degree the diversity of the solutions when compared to the case of the Yang-Mills equations in 3+1 dimensions.

In the Abelian case we studied sources with cylindrical symmetry and showed that they give rise to finite-action, well-behaved solutions; while spherically symmetric sources cannot satisfy the consistency condition (1.5). This contrasts with the non-Abelian case where radially symmetric sources are allowed. In the case of the radial *Ansatz* we were able to study the complete set of solutions both for the usual source couplings as well as for coupling to gauge-invariant objects. It should be pointed out that within this *Ansatz* there is no equivalent of the type II solutions of Ref. 5; this is due to the boundary conditions and the Euclidean nature of the equations which require them to oscillate at infinity with a fixed amplitude. We showed that the radial solutions reduce to the perturbative non-Abelian case when the sources are weak.

A peculiar feature of the solutions studied is that they all correspond to saddle points of the action, this contrasts with the previously studied cases where the classical solutions are (in general) minima of the energy. Moreover, stability is recovered when μ vanishes. A qualitative explanation of this result is that the terms proportional to μ are odd in A and contain at most one derivative, so they can dominate provided we look at long wavelength modes, and if the quartic terms are not dominant. This happens rather trivially in the Abelian case and perturbative non-Abelian case to lowest order in Q: the quartic terms vanish identically. A more interesting situation occurs within the radial *Ansatz*, where the quartic terms are nonzero but can be subdominant. A more profound and systematic study of the stability properties of the action functional along the lines of Ref. 12 is currently under investigation.

When μ is purely imaginary we found that there are no solutions with real potentials. While for μ an arbitrary complex number the equations become of first order when the solutions are required to be real; general solutions were obtained in this case also.

Finally, we make some brief remarks concerning the differences with our approach and the one considered in Ref. 9. The basic difference lies in the choice of boundary conditions: while we require finite action, Ref. 9 requires the solutions to become pure gauges at infinity, and this radically alters the character of the solutions.

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APPENDIX

In this appendix we consider the qualitative behavior of the radially symmetric solutions for both kinds of couplings. To this end we consider the solutions to the equation [see (2.34) and (2.38)]

$$-\rho''+\frac{l^2}{\rho^3}+\left[\frac{\rho^2-1-x^2}{x^2}\right]\rho = \begin{cases}\mathcal{T}\rho & \text{for gauge-invariant sources}\\ \tilde{j} & \text{for usual sources}\end{cases} \quad \text{(A1)}$$

where $l=\text{const}$.

We study the qualitative behavior of ρ for solutions of (A1) when the sources are localized.

The solution ρ cannot vanish for $l\neq 0$. Indeed, if $\rho(x_0)=0$, then near x_0, $\rho\simeq\alpha y$, $y\equiv x-x_0$ (if ρ vanishes as a higher power of y similar results hold). But then (A1) reads

$$\frac{l^2}{\alpha^2 y^3}+O(1/y^2)=\mathcal{T}\rho=O(y)\ , \quad \text{(A2)}$$

which is impossible. When $l=0$ the proof fails and the solutions can vanish. In fact this case ($l=0$) allows solutions with arbitrarily large source strengths for the first type of coupling in (A1). For if we consider (A1) with $l=0$ and $\mathcal{T}=0$ the solution satisfying $\rho(0)=1$, $\rho'(0)=0$ will vanish at a set of values X_n. Now if we consider sources

$$\mathcal{T}=\sum_n Q_n\delta(x-X_n) \quad \text{(A3)}$$

then the same ρ will solve (A1) for arbitrary Q_n. The first few values of X_n are $X_1\simeq 0.876$, $X_2\simeq 4.555$, $X_3\simeq 7.769$,

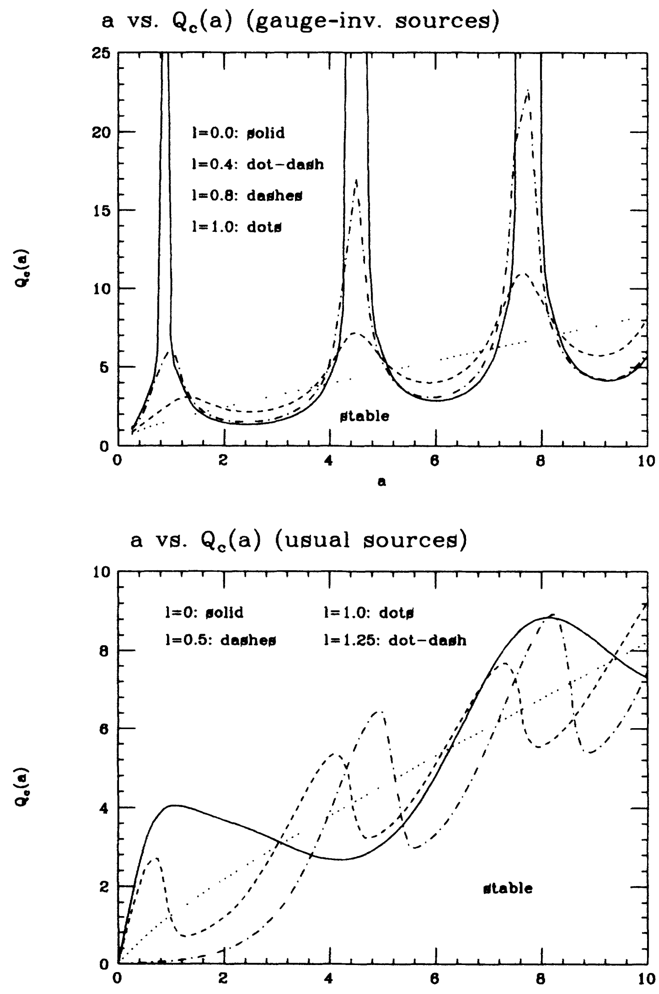

FIG. 6. Graph of the critical charge for a delta function source as function of a, for various values of l. The connected region above the a axis corresponds to stable solutions, the rest of the a-Q plane corresponds to unstable solutions. (a) Gauge-invariant sources; (b) usual sources.

$X_4 \simeq 10.942$, $X_5 \simeq 14.102$. Note however that this peculiar situation will not hold for (2.34) since it corresponds to $l=1$. Note that this is relevant only for the gauge-invariant couplings, for only then we have a factor of ρ multiplying the source.

For $x \to 0$ the solutions of (A1) behave as

$$\rho = 1 + \sum_{n=\text{even}\geq 2} [a_n + b_n \ln x + c_n (\ln x)^2 + d_n (\ln x)^3 + \cdots] x^n , \tag{A4}$$

where the first few values of the constants are

$$\begin{aligned} &a_2: \text{ arbitrary} , \\ &a_4 = \tfrac{1}{375}(3l^2 - 1 - 165 a_2) + \tfrac{1}{10} a_2 (3a_2 - 4) , \\ &b_2 = \tfrac{1}{3}(l^2 - 1) , \\ &b_4 = -\tfrac{1}{75}(l^2-1)(11 l^2 - 1 - 15 a_2) , \\ &c_2 = 0 , \\ &c_4 = \tfrac{1}{30}(l^2-1)^2 , \\ &d_2 = 0 , \\ &d_4 = 0 . \end{aligned} \tag{A5}$$

The first nonzero value of the d_n is $d_6 = (l^2-1)^3/270$. It is a general feature of this expansion that the first nonzero term proportional to $(\ln x)^k$ behaves as x^{2k}. Note that all terms in powers of $\ln x$ disappear when $l=1$: only in this case the solution is analytic for small x.

The value of a_2 is fixed so that the solution is well-behaved as $x \to \infty$. Because of the nonlinear nature of (A1) in general a_2 can be found only numerically.

If $\lim_{x\to\infty} \rho/x = 0$, then (A1) becomes in the limit

$$\rho'' = -\frac{dV(\rho)}{d\rho}, \quad V(\rho) \equiv \frac{1}{2}\frac{l^2}{\rho^2} + \tfrac{1}{2}\rho^2 \tag{A6}$$

with the solution

$$\rho = [\sqrt{E^2 - l^2}\cos(2x + 2\nu) + E]^{1/2} \quad (x \to \infty) . \tag{A7}$$

If $\lim_{x\to\infty}(\rho/x) = \infty$ but $|\rho(x)| < \infty$ for finite x, then for x large and localized $\mathcal{T}$, (A1) is equivalent to

$$-\rho'' + \rho^3/x^2 \simeq 0 . \tag{A8}$$

If we now replace $\rho(x) = \tau(\ln x)$ we obtain

$$\tau'' - \tau' - \tau^3 \simeq 0 , \tag{A9}$$

which can be interpreted as a particle moving in a $-\tau^4/4$ potential with negative friction. Therefore we conclude that τ and hence ρ will diverge for *finite* x; i.e., the assumed behavior of ρ is not allowed.

If ρ diverges as $x \to x_0$ for finite x_0, then near x_0 (A1) is equivalent to

$$-\rho'' + \frac{1}{x_0^2}\rho^3 \simeq 0 \tag{A10}$$

with solution $\rho \simeq \sqrt{2} x_0/(x - x_0)$ for $x \to x_0 - 0$.

Finally, if $\lim_{x\to\infty}(\rho/x) = a_{-1} < \infty$, then we can find an asymptotic solution for (A1) as $x \to \infty$ by expanding ρ in powers of $1/x$. The result is

$$\rho = \pm \left[x + \frac{1}{2x} + \frac{3 - 4l^2}{8x^3} + \frac{29 - 28 l^2}{16 x^5} + \frac{3203 - 3064 l^2 - 144 l^4}{128 x^7} + \frac{172\,239 - 163\,448 l^2 - 8784 l^4}{256 x^9} + O(1/x^{11}) \right] . \tag{A11}$$

The above six points cover the main qualitative behavior of solutions. These either diverge at a finite x, they grow like x as $x \to \infty$ or they oscillate at infinity. These last solutions are the ones that will give rise to a finite action.

It follows from the above arguments that for both types of coupling there will be a critical chage $Q_c(l,a)$ such that for $Q > Q_c$ the solutions will diverge at some finite value of x. Q_c can be obtained numerically as a function of its arguments, we present the plots for delta sources of the form $\tilde{j}$, $\mathcal{T} = Q\delta(x-a)$ in Fig. 6.

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