PHYSICAL REVIEW D VOLUME 41, NUMBER 2 15 JANUARY 1990

Anomalies from stochastic quantization and their path-integral interpretation

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(Received 1 June 1989; revised manuscript received 18 September 1989)

The regularized generating functional in Euclidean stochastic quantization of Dirac theory in a background field is derived in both Markovian and non-Markovian stochastic regularization schemes to show the existence of the equilibrium limit under the same condition as in the unregularized theory. Contrary to the latter case, however, the equilibrium limit depends on an arbitrary kernel in the fermion Langevin equation, which confirms our previous result on the kernel dependence of anomalies in an external gauge field. In addition to the known kernels yielding the consistent and covariant anomalies, a new kernel is proposed, which leads to a consistent form in $D=2$ dimensions and a new form in $D=4$. The relation to Fujikawa's path-integral method is also discussed, where the kernel introduces, in general, non-Hermitian operators which are treated by the plane-wave fermion measure and the regularized transformations of Bern *et al.*, now depending on the kernel.

I. INTRODUCTION

In the Parisi-Wu stochastic quantization method[1] of field theory the regularization is formulated at the Langevin level. There have been proposed so far two kinds of stochastic regularizations[2–4] which modify the original correlation formula of the noise field(s) in either the coordinate-space or the extra-time directions. We shall call the former the Markovian stochastic regularization[2] (SR) which keeps the white noise of the original formalism, and the latter the non-Markovian[3,4] SR which makes use of the colored noise.

According to Bern, Chan, and Halpern,[5] the Markovian SR can be carried out in gauge theories coupled to fermions. On the other hand, it has been shown by some authors[6] that the non-Markovian SR does not work consistently in non-Abelian gauge theories with the stochastic gauge fixing.[7] Hence, any serious comparison between the two SR schemes in the Dirac theory should be made only if the gauge field is regarded as external. In fact, both schemes for external-field problems are equivalent[5,8] to the regularization at the action level, and the authors in Ref. 5 noted that the regularized equilibrium actions take similar forms. As one (K.M.) of us has recently pointed out,[9] they become identical if the two regularization functions employed are related to each other via the Laplace transform.

The purpose of this paper, which is an expanded version of Ref. 9, is threefold. First, we will show that the equilibrium limit of the stochastically regularized Dirac theory in the external field exists in both SR schemes for arbitrary kernel of the fermion Langevin equation[10–12] under the same condition as in the unregularized case. This generalizes the previous result[5,8] on the equivalence of both SR schemes to the action regularization, and proves the equilibrium condition implicitly assumed in Ref. 9. Second, we will take up the much-discussed subject of the chiral anomalies[13] in the background gauge field. As is well known, they depend on the regularization methods in the ordinary perturbation theory or on the fermion measure in Fujikawa's path-integral method.[14] In the stochastic quantization method they should not depend on the SR schemes since the latter are equivalent to each other as long as the gauge field is regarded as external. Moreover, the stochastic quantization method dispenses with the definition of the fermion measure. We, therefore, claim that the various types[15] of anomalies originate from different kernels. This was pointed out in Ref. 16 using non-Markovian SR, and extended[9] to Markovian SR, too. We are now able to describe the kernel dependence of anomalies by the generating functional method. Third, we will apply Fujikawa's path-integral method[14] to the regularized equilibrium fermion action, where the fermion measure is expanded in terms of the complete set of plane waves. The connection of the two methods was briefly studied in Ref. 5 without mentioning the measure. We will make it more precise in view of a possibility of different kernels in perturbation theory.

This paper is organized as follows. In the next section we derive the regularized generating functional for an external-field fermion Langevin equation in both SR schemes and show the existence of the equilibrium limit. The generating functional method will be applied in Sec. III to show that the types of anomalies in the background gauge field are determined at the Langevin level. The conventional approach[17,16] to anomalies in the Parisi-Wu method will be reviewed in Sec. IV where, in addition to the known kernels leading to the consistent and covariant anomalies, a new kernel is given, which yields the consistent form in $D=2$ and a new form in $D=4$. Section V gives an application[5] of Fujikawa's path-integral method[14] to the regularized equilibrium fermion action obtained via the SR schemes. The final section is devoted to a summary and discussions.

II. REGULARIZED GENERATING FUNCTIONAL FROM STOCHASTIC REGULARIZATIONS

Let us consider Euclidean Dirac theory with the action

$$S=\int d^D x\, \bar{\psi}(x)\Gamma\psi(x)\ , \tag{1}$$

where D is the dimensionality of the Euclidean space-time, and the Dirac operator Γ is assumed to depend on the external field(s) only. To quantize the theory one may use either the canonical or path-integral or stochastic formulations. They are all equivalent in the unregularized theory, but a question remains open whether or not the equilibrium limit in the last formulation exists also in the regularized version except the cases treated in Refs. 5 and 8. To answer the question in the more general cases, we employ in this paper the Parisi-Wu stochastic quantization method.[1] Suppose that the D-dimensional, quantum Euclidean Dirac theory is regarded as the equilibrium state of the $(D+1)$-dimensional stochastic field theory with the random, Grassmann spinor fields $\psi(x,\tau)$ and $\bar{\psi}(x,\tau)$, where τ denotes the extra time variable. The approach to the equilibrium state at $\tau\to\infty$ is simulated by the fermion Langevin equations[3,10–12]

$$\dot{\psi}(x,\tau)=-K\Gamma\psi(x,\tau)+\eta(x,\tau)\ , \tag{2a}$$

$$\dot{\bar{\psi}}(x,\tau)=-\bar{\psi}(x,\tau)\bar{\Gamma}\bar{K}+\bar{\eta}(x,\tau)\ , \tag{2b}$$

where the overdot stands for the derivative with respect to τ, K is an arbitrary kernel not existing at the Lagrangian level, $\bar{Q}=Q|_{\partial_\mu\to-\overleftarrow{\partial}_\mu}$ for any operator Q (here $Q=K,\Gamma$), and η and $\bar{\eta}$ are the Gaussian, Grassmann noise fields with $\langle\eta\rangle=\langle\bar{\eta}\rangle=\langle\eta\eta\rangle=\langle\bar{\eta}\bar{\eta}\rangle=0$, and[18]

$$\langle\eta(x,\tau)\bar{\eta}(x',\tau')\rangle=2K\delta^D(x-x')\delta(\tau-\tau')\ . \tag{2c}$$

It is then possible to show that the generating functional for equal-τ correlation functions

$$W[\zeta,\bar{\zeta},\tau]\equiv\left\langle\exp\left[\int d^D x[\bar{\zeta}(x)\psi(x,\tau)+\bar{\psi}(x,\tau)\zeta(x)]\right]\right\rangle_\eta\ , \tag{3}$$

where the η average $\langle\ \rangle_\eta$ is defined by Eqs. (2a)–(2c) and ζ and $\bar{\zeta}$ are external Grassmann sources, satisfies the functional differential equation

$$\dot{W}[\zeta,\bar{\zeta},\tau]=\left[\int\Lambda\right]W[\zeta,\bar{\zeta},\tau]\ , \tag{4a}$$

where $\int\Lambda\equiv\int d^D x\,\Lambda(\zeta(x),\bar{\zeta}(x))$ with

$$\Lambda(\zeta,\bar{\zeta})\equiv\bar{\zeta}\left[-L\frac{\delta}{\delta\bar{\zeta}}+K\zeta\right]+\left[\frac{\delta'}{\delta\zeta}\tilde{L}+\bar{\zeta}K\right]\zeta\ . \tag{4b}$$

Here $L\equiv K\Gamma$, $\tilde{L}=\Gamma K$, and $\delta'/\delta\zeta$ denotes the Grassmann derivative $\delta/\delta\zeta$ not acting on ζ standing just to the right of it. Equations (4a) and (4b) are obtained by applying Fukuda and Higurashi's method[19] to the present model. By assumption L and K do not depend on τ, and Eqs. (4a) and (4b) are integrated to obtain

$$W[\zeta,\bar{\zeta},\tau]\equiv\exp\left[\int d^D x\,\bar{\zeta}(x)\Gamma^{-1}(1-e^{-2\tilde{L}\tau})\zeta(x)\right]\ , \tag{5}$$

where we have chosen the initial conditions $\psi(x,0)=\bar{\psi}(x,0)=0$. The equilibrium limit

$$\lim_{\tau\to\infty}W[\zeta,\bar{\zeta},\tau]\equiv W[\zeta,\bar{\zeta}]$$

$$=\exp\left[\int d^D x\,\bar{\zeta}(x)\Gamma^{-1}\zeta(x)\right]\ , \tag{6}$$

exists if $e^{-2\tilde{L}\tau}$ goes to zero as $\tau\to\infty$. This condition is always met in the perturbative sense if $\tilde{L}$ (or, equivalently, L) are decomposed into a free, positive-definite Hermitian operator plus perturbations. In particular, for $K=\Gamma^\dagger$, the dagger denoting Hermitian conjugation, L and $\tilde{L}$ themselves are positive-definite Hermitian operators so that the limit (6) exists nonperturbatively as proved in the Fokker-Planck formalism. If the free operator is not Hermitian, we must require that its eigenvalues have positive real parts as in Ref. 10 for massive fermions. Equation (6) establishes the equivalence[20] of the Parisi-Wu stochastic quantization to the conventional ones.

The equivalence problem has also been studied[5,8] when the SR schemes are introduced at the Langevin level. The authors in Ref. 8 proved the existence of the equilibrium limit for the non-Markovian SR in the Fokker-Planck formalism, but failed to generalize the proof for gauge-invariant fermion Langevin equation[12] to arbitrary K. On the other hand, the authors in Ref. 5 assumed the gauge-invariant fermion Langevin equation[12] in the Markovian SR to show that the latter in the background field becomes in the equilibrium limit equivalent to the action regularization as verified in Ref. 8 for non-Markovian SR. In what follows we present a simple proof of the equivalence for arbitrary K using the functional method of Ref. 19.

The Markovian[2,5] SR replaces K by $R_\Lambda K$ in Eq. (2c), where the regularization function R_Λ is assumed not to depend on the fields and goes to unity as $\Lambda\to\infty$, while the non-Markovian[3,4] SR generalizes the delta function $\delta(\tau-\tau')$ in Eq. (2c) to the stochastic regularization function $\alpha_\Lambda(\tau-\tau')$ with the properties $\alpha_\Lambda(\tau'-\tau)=\alpha_\Lambda(\tau-\tau')$, $\lim_{\Lambda\to\infty}\alpha_\Lambda(\tau-\tau')=\delta(\tau-\tau')$, and $\int_{-\infty}^{\infty}d\tau'\alpha_\Lambda(\tau-\tau')=1$. Accordingly we introduce the regularized generating functional $W_{\rm reg}[\zeta,\bar{\zeta},\tau]$ through Eq. (3) with the η average $\langle\ \rangle_\eta$ defined as above. It also satisfies Eq. (4a) with $\Lambda\to\Lambda^{\rm reg}$, which amounts to replacing K in Eq. (4b) by $K_{\rm reg}$, where

$$K_{\rm reg}=\begin{cases} R_\Lambda K \quad \text{for Markovian SR,} & \text{(7a)}\\ f_\tau(L/\Lambda^2)K \quad \text{for non-Markovian SR,} & \text{(7b)}\end{cases}$$

with

$$f_\tau(x/\Lambda^2)=2\int_0^\tau d\tau' \alpha_\Lambda(\tau')e^{-\tau' x} . \tag{7c}$$

At $\tau\to\infty$, f_τ becomes Fujikawa's regularization function $f\equiv f_\infty$ with $f(0)=1$ and $f'(\infty)=f''(\infty)=\cdots=0$. Integration gives

$$W_{\rm reg}[\zeta,\bar\zeta,\tau]=\exp\left[\int d^Dx\,\bar\zeta(x)\Gamma^{-1}X_\tau\zeta(x)\right] , \tag{8a}$$

where

$$X_\tau=\begin{cases}(1-e^{-2\tilde L\tau})\tilde R_\Lambda \quad \text{for Markovian SR,} & \text{(8b)}\\ 2\int_0^\tau d\tau'\alpha_\Lambda(\tau')(e^{-\tilde L\tau'}-e^{-\tilde L(2\tau-\tau')}) & \\ \qquad \text{for non-Markovian SR .} & \text{(8c)}\end{cases}$$

In Eq. (8b) we have assumed $\tilde R_\Lambda\equiv K^{-1}R_\Lambda K=\Gamma R_\Lambda\Gamma^{-1}$, whence $R_\Lambda=R_\Lambda(L)$ and $\tilde R_\Lambda=R_\Lambda(\tilde L)$. It is clear that the $\tau\to\infty$ limit of Eqs. (8b) and (8c) exists under the same condition as in the unregularized theory, obtaining

$$\lim_{\tau\to\infty} W_{\rm reg}[\zeta,\bar\zeta,\tau]\equiv W_{\rm reg}[\zeta,\bar\zeta] =\exp\left[\int d^Dx\,\bar\zeta(x)\Gamma^{-1}X_\infty\zeta(x)\right] , \tag{9}$$

where $X_\infty=\tilde R_\Lambda(f(\tilde L/\Lambda^2))$ for Markovian (non-Markovian) SR. Therefore, both Markovian and non-Markovian SR schemes in the background field become at $\tau\to\infty$ equivalent[5,8] to the action regularization depending on the kernel. If we further assume[9] that

$$\begin{aligned} &R_\Lambda=f(L/\Lambda^2) ,\\ &\tilde R_\Lambda=K^{-1}R_\Lambda K=K^{-1}f(L/\Lambda^2)K=f(\tilde L/\Lambda^2) ,\end{aligned} \tag{10}$$

both SR schemes in the equilibrium limit are identical to each other.

III. ANOMALIES IN THE BACKGROUND GAUGE FIELD

Using the regularized generating functional (9) for Green's functions in the theory, we next discuss the chiral anomalies in the background gauge field for even $D=2n$ to clarify the role of the kernel.

Put

$$\Gamma=\gamma\cdot(\partial+A)+m , \tag{11}$$

where we choose Dirac matrices to be Hermitian and satisfy $\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}$, $\mu,\nu=1,\ldots,2n$, $A_\mu=-iA_\mu^aT^a$ denotes the prescribed external non-Abelian gauge field with T^a being the Hermitian generators of the gauge group taken to be in the fermion representation, and m is the fermion mass. In this case consider the variation of the action (1) under the infinitesimal gauge and chiral transformations:

$$\delta\psi(x)=i\begin{bmatrix}\alpha(x)\\ \gamma_{2n+1}\beta(x)\end{bmatrix}\psi(x) , \tag{12a}$$

$$\delta\bar\psi(x)=i\bar\psi(x)\begin{bmatrix}-\alpha(x)\\ \gamma_{2n+1}\beta(x)\end{bmatrix} , \tag{12b}$$

where $\alpha=\alpha_AT^A$, $\beta=\beta_AT^A$, $A=0,a$ with $T^0=1$, $\gamma_{2n+1}=(-i)^n\gamma_1\gamma_2\cdots\gamma_{2n}$, and $\alpha_A(x)$ and $\beta_A(x)$ are assumed to vanish on the boundary of the integration domain of Eq. (1). We write the variation as

$$\begin{aligned}\frac{\delta S}{\delta\alpha_A(x)}&=i(-\bar\psi T^A\Gamma\psi+\bar\psi\bar\Gamma T^A\psi)(x)\\ &=-i(D_\mu J_\mu)^A(x) ,\end{aligned} \tag{13a}$$

and

$$\begin{aligned}\frac{\delta S}{\delta\beta_A(x)}&=i(\bar\psi T^A\gamma_{2n+1}\Gamma\psi+\bar\psi\bar\Gamma T^A\gamma_{2n+1}\psi)(x)\\ &=-i[(D_\mu J_{5\mu})^A-2mJ_{2n+1}^A](x) ,\end{aligned} \tag{13b}$$

where $D_\mu J_\mu=\partial_\mu J_\mu+[A_\mu,J_\mu]$ and similarly for $J_{5\mu}$, with $J_\mu^A=\bar\psi\gamma_\mu T^A\psi$, $J_{5\mu}^A=\bar\psi\gamma_\mu\gamma_{2n+1}T^A\psi$, and $J_{2n+1}^A=\bar\psi\gamma_{2n+1}T^A\psi$. For the $A=0$ component the non-Abelian parts $[A_\mu,J_\mu]$ and $[A_\mu,J_{5\mu}]$ are absent. By the action principle both $\delta S/\delta\alpha_A$ and $\delta S/\delta\beta_A$ vanish if the classical field equations, $\Gamma\psi=\bar\psi\bar\Gamma=0$, are satisfied, leading to classical "conservation" equations of the vector and axial-vector currents, J_μ and $J_{5\mu}$, respectively. It is well known, however, that, upon quantization, the (regularized) vacuum expectation values of them,

$$H^A(x)=\left\langle i\frac{\delta S}{\delta\alpha_A(x)}\right\rangle_{\rm reg} \tag{14a}$$

$$=\lim_{\Lambda\to\infty}\frac{\delta}{\delta\zeta(x)}(\bar\Gamma T^A-T^A\Gamma)\frac{\delta}{\delta\bar\zeta(x)}W_{\rm reg}[\zeta,\bar\zeta]\bigg|_{\zeta=\bar\zeta=0} , \tag{14b}$$

$$G^A(x)=\left\langle i\frac{\delta S}{\delta\beta_A(x)}\right\rangle_{\rm reg} \tag{15a}$$

$$=\lim_{\Lambda\to\infty}\frac{\delta}{\delta\zeta(x)}(\bar\Gamma T^A\gamma_{2n+1}+T^A\gamma_{2n+1}\Gamma)\times\frac{\delta}{\delta\bar\zeta(x)}W_{\rm reg}[\zeta,\bar\zeta]\bigg|_{\zeta=\bar\zeta=0} , \tag{15b}$$

do not, in general, vanish due to the possible presence of anomalies.

Given Eq. (9) the functional derivatives in Eqs. (14b) and (15b) can be evaluated by first splitting the point x and then taking the same-x limit in the end. The result for the Markovian SR turns out to be

$$H^A(x)=\lim_{\Lambda\to\infty}{\rm tr}T^A(R_\Lambda-\tilde R_\Lambda)(x,x) , \tag{16a}$$

$$G^A(x)=\lim_{\Lambda\to\infty}{\rm tr}\gamma_{2n+1}T^A(R_\Lambda+\tilde R_\Lambda)(x,x) , \tag{16b}$$

where we have used the relation $\tilde R_\Lambda\Gamma=\Gamma R_\Lambda$ and tr means the trace over Dirac and internal symmetry matrices, while the same result holds also in the non-Markovian case by the substitution (10). Hence, we conclude[9] that both Markovian and non-Markovian SR schemes give the same answer as far as the anomalies in the background gauge field are concerned, and that the types of anomalies depend on the kernel K in the Langevin equations (2a) and (2b) through the operators $L=K\Gamma$ and $\tilde{L}=\Gamma K$. (Remember that the anomalies do not depend on the detailed form of the regularization function as verified by Fujikawa.[14])

IV. STOCHASTIC PERTURBATION THEORY OF ANOMALIES

The stochastic perturbation theory designed to evaluate anomalies in the background gauge field for given K is based on either Eqs. (16a) and (16b) or an equivalent formula[16] valid in the non-Markovian SR:

$$H^A(x)=2\lim_{\Lambda\to\infty}\int_0^\infty d\tau\,\alpha_\Lambda(\tau)\mathrm{tr}T^A[G(x,x;\tau,0)-\bar{G}(x,x;0,\tau)]\ , \tag{17a}$$

$$G^A(x)=2\lim_{\Lambda\to\infty}\int_0^\infty d\tau\,\alpha_\Lambda(\tau)\mathrm{tr}\gamma_{2n+1}T^A[G(x,x;\tau,0)+\bar{G}(x,x;0,\tau)]\ , \tag{17b}$$

where $G=G(x,x';\tau,\tau')$ and $\bar{G}=\bar{G}(x',x;\tau',\tau)$ are Green's functions of Eqs. (2a) and (2b), respectively, satisfying $\dot{G}=LG+1$ and $\dot{\bar{G}}=-\bar{G}\tilde{L}+1$ with 1 representing the product of delta functions in x and τ. Equations (17a) and (17b) are obtained from the stochastic version of Eqs. (14a) and (15a) using the equilibrium condition $\dot{W}_{\mathrm{reg}}\to 0$ at $\tau\to\infty$. The method of Ref. 16 makes use of Eqs. (17a) and (17b) assuming the kernel K to be either constant or linear in the derivative so that $K=K_0+K_1$, $K_0=-\gamma\cdot\partial+m$, and K_1 does not contain the derivative. In the first case[10] we put $K=l^{-1}$, where l is a constant with a dimension of length. Then $H^A=0$, whereas we find that[9]

$$G^A=\frac{i}{2\pi}\epsilon_{\mu\nu}\mathrm{tr}T^AF_{\mu\nu}\quad (D=2)\ , \tag{18}$$

where $\epsilon_{\mu\nu}=-\epsilon_{\nu\mu}$ with $\epsilon_{12}=+1$ and $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]$, and[16]

$$G^A=-\frac{1}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}T^AF_{\mu\nu}F_{\rho\sigma}\quad (D=4)\ , \tag{19}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the alternating symbol with $\epsilon_{1234}=+1$ (Ref. 21). In Eqs. (18) and (19) tr refers only to the internal symmetry matrices.[22] The same remark also applies to the following equations with tr [ ] when the object [ ] involves no γ matrices.

In the second case we consider two choices $K=K_0$ and $K=\Gamma^\dagger$, separately. For $K=\Gamma^\dagger$ we obtain[23] ($D=2n$)

$$G^A=\frac{(i)^n}{2^{2n-1}\pi^n n!}\epsilon_{\mu_1\nu_1\cdots\mu_n\nu_n}\mathrm{tr}T^A(F_{\mu_1\nu_1}\cdots F_{\mu_n\nu_n})\ , \tag{20}$$

where the alternating symbol is fixed by $\epsilon_{12\cdots 2n}=+1$. Equation (20) has repeatedly been derived in the literature,[24] and is reduced to Eqs. (18) and (19) for $n=1$ and 2, respectively. Accordingly, for $K=l^{-1}$ and $K=\Gamma^\dagger$ the covariant form[15,14] of anomalies is reproduced in $D=2$ and 4 dimensions.[25] We shall see in the next section that the case $K=l^{-1}$ also reproduces Eq. (20) in any higher even dimensions. Another choice $K=K_0$[3,11] leads[16] to the chiral consistent anomaly[15] for vector plus axial-vector gauge couplings. For vector coupling under consideration it gives, for $D=2$ (Ref. 9),

$$H^A=\frac{1}{2\pi}\mathrm{tr}T^A\partial\cdot A\ , \tag{21a}$$

$$G^A=\frac{i}{2\pi}\epsilon_{\mu\nu}\mathrm{tr}T^A\partial_\mu A_\nu\ , \tag{21b}$$

while, for $D=4$ (Ref. 16),

$$H^A=\frac{1}{8\pi^2}\mathrm{tr}T^A(\tfrac{1}{6}\{\partial_\mu A_\nu+\partial_\nu A_\mu, A_\mu A_\nu\}-\tfrac{1}{12}\{\partial\cdot A, A^2\}+\tfrac{1}{2}A_\mu\partial\cdot A A_\mu+\tfrac{1}{3}D^2\partial\cdot A-\tfrac{1}{3}[A_\mu, D_\nu F_{\nu\mu}]-2m^2\partial\cdot A+\kappa\Lambda^2\partial\cdot A)\ , \tag{22a}$$

$$G^A=-\frac{1}{16\pi^2}\frac{4}{3}\partial_\mu\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}T^A(A_\nu\partial_\rho A_\sigma+\tfrac{1}{2}A_\nu A_\rho A_\sigma)\ , \tag{22b}$$

where $D_\mu=\partial_\mu+[A_\mu,\ ]$, $\{A,B\}\equiv AB+BA$ (of course, $[A,B]=AB-BA$), and κ is a divergent constant defined by Eq. (44) of Ref. 16. It is not difficult to verify that both Eqs. (21a) and (22a) take the form

$$H^A=2i\ \mathrm{tr}T^AD_\mu j'_\mu\ , \tag{23}$$

for appropriately chosen vector current j'_μ. Therefore, the subtracted current $J_\mu-j'_\mu$ is anomaly-free. Equations (21b) and (22b) are called consistent[15] form of anomalies.[26]

A new choice

$$K=(-\partial^2+m^2)\Gamma^{-1}\ , \tag{24}$$

for which $L=K\Gamma=(-\partial^2+m^2)$ and $\tilde{L}=\Gamma K=\Gamma(-\partial^2+m^2)\Gamma^{-1}$, requires a perturbative treatment based on Eqs. (16a) and (16b) rather than Eqs. (17a) and (17b). For the Pauli-Villars-type regularization function $\alpha_\Lambda(\tau)=(\Lambda^2/2)e^{-\Lambda^2|\tau|}$ which has been assumed to obtain the previous Eqs. (20)–(22b), $f(x)=(1+x)^{-1}$ so that Eq. (10) gives

$$R_\Lambda=[1+(-\partial^2+m^2)/\Lambda^2]^{-1}\ , \tag{25a}$$

$$\tilde{R}_\Lambda=\Gamma[1+(-\partial^2+m^2)/\Lambda^2]^{-1}\Gamma^{-1}\ . \tag{25b}$$

Neglecting m^2 compared with Λ^2 which would eventually be made infinite, we get, from Eqs. (16a) and (16b),

$$H^A(x)=-\lim_{\Lambda\to\infty}\mathrm{tr}T^A\Gamma[(1-\partial^2/\Lambda^2)^{-1},\Gamma^{-1}](x,x)\ , \tag{26a}$$

$$G^A(x)=\lim_{\Lambda\to\infty}\mathrm{tr}\gamma_{2n+1}T^A\Gamma[(1-\partial^2/\Lambda^2)^{-1},\Gamma^{-1}](x,x)\ . \tag{26b}$$

The commutator is evaluated by expanding Γ^{-1} into a perturbation series

$$\Gamma^{-1}=(\Gamma_0+\Gamma_1)^{-1}$$
$$=\Gamma_0^{-1}-\Gamma_0^{-1}\Gamma_1\Gamma_0^{-1}+\Gamma_0^{-1}\Gamma_1\Gamma_0^{-1}\Gamma_1\Gamma_0^{-1}+\cdots$$
$$(\Gamma_0\equiv\gamma\cdot\partial+m)$$

and using the formula

$$[(1-\partial^2/\Lambda^2)^{-1},\phi](x,x)=\{\Lambda^{-2}[2(\partial_\mu\phi)\partial_\mu+\partial^2\phi](1-\partial^2/\Lambda^2)^{-2}+4\Lambda^{-4}[(\partial_\lambda\partial_\mu\phi)\partial_\lambda+\partial_\mu\partial^2\phi]\partial_\mu(1-\partial^2/\Lambda^2)^{-3}$$
$$+8\Lambda^{-6}(\partial_\mu\partial_\nu\partial_\lambda\phi)\partial_\mu\partial_\nu\partial_\lambda(1-\partial^2/\Lambda^2)^{-4}+\cdots\}(x,x)\ , \tag{27}$$

where the ellipsis in the curly brackets stands for terms with more than third derivatives of ϕ. Then we carry out the Dirac algebra to obtain that, for $D=2$, Eqs. (26a) and (26b) reproduce the consistent anomalies (21a) and (21b), while for $D=4$, they lead to new types of anomalies:

$$H^A=\frac{1}{16\pi^2}\mathrm{tr}T^A\{\kappa\Lambda^2\partial\cdot A-\tfrac{2}{3}[-\partial^2\partial\cdot A+\partial_\mu A_\nu(\partial_\mu A_\nu+\partial_\nu A_\mu)+(\partial\cdot A)^2+\partial^2 A_\mu A_\mu$$
$$+2\partial_\mu\partial\cdot A A_\mu-\partial\cdot A A^2-\partial_\mu A_\nu(A_\mu A_\nu+A_\nu A_\mu)]\}\ , \tag{28a}$$

and

$$G^A=-\frac{1}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}T^A\partial_\mu A_\nu F_{\rho\sigma}\ . \tag{28b}$$

Equation (28a) can be brought into the form (23), implying that the vector current is anomaly-free after subtraction. (Unlike the previous case the current j'_μ cannot be given by a variation of the counterterm.) The anomaly (28b) is related to the covariant one (19) via the gauge-Bianchi identity. It is similar to a relation[15] between covariant and consistent anomalies.

V. RELATION TO FUJIKAWA'S PATH-INTEGRAL METHOD

The stochastic quantization method introduces non-Hermitian operators L and $\tilde{L}$ but dispenses with the definition of the fermion measure, while Fujikawa's path-integral method[14] regards the measure as a source for anomalies using Hermitian operators. In this section we investigate a connection between the two methods by generalizing the regularization procedure of Ref. 5.

For Hermitian L and $\tilde{L}$ which are supposed to be different,[27] there exist the complete sets of eigenfunctions

$$L\phi_n(x)=\lambda_n^2\phi_n(x)\ ,$$
$$\int d^{2n}x\ \phi_n^\dagger(x)\phi_m(x)=\delta_{nm}\ ,$$
$$\sum_n\phi_n(x)\phi_n^\dagger(x')=\delta^{2n}(x-x')\ , \tag{29a}$$

and

$$\tilde{L}\varphi_n(x)=\mu_n^2\varphi(x)\ ,$$
$$\int d^{2n}x\ \varphi_n^\dagger(x)\varphi_m(x)=\delta_{nm}\ ,$$
$$\sum_n\varphi_n(x)\varphi_n^\dagger(x')=\delta^{2n}(x-x')\ , \tag{29b}$$

where λ_n^2 and μ_n^2 denote the eigenvalues. The Green's functions G and $\bar{G}$ are then given by

$$G(x,x';\tau,\tau')=\theta(\tau-\tau')\sum_n\phi_n(x)\phi_n^\dagger(x')e^{-\lambda_n^2(\tau-\tau')}\ , \tag{30a}$$

$$\bar{G}(x',x;\tau',\tau)=\theta(\tau-\tau')\sum_n\varphi_n(x')\varphi_n^\dagger(x)e^{-\mu_n^2(\tau-\tau')}\ , \tag{30b}$$

which are substituted into Eqs. (17a) and (17b), giving $(\gamma_{D+1}\equiv\gamma_{2n+1})$

$$H^A(x)=2\lim_{\Lambda\to\infty}\int_0^\infty d\tau\,\alpha_\Lambda(\tau)\sum_n[\phi_n^\dagger(x)T^A e^{-\lambda_n^2\tau}\phi_n(x)-\varphi_n^\dagger(x)T^A e^{-\mu_n^2\tau}\varphi_n(x)]\ , \tag{31a}$$

$$G^A(x)=2\lim_{\Lambda\to\infty}\int_0^\infty d\tau\,\alpha_\Lambda(\tau)\sum_n[\phi_n^\dagger(x)\gamma_{D+1}T^A e^{-\lambda_n^2\tau}\phi_n(x)+\varphi_n^\dagger(x)\gamma_{D+1}T^A e^{-\mu_n^2\tau}\varphi_n(x)]\ . \tag{31b}$$

If L and $\tilde{L}$ are not Hermitian, no general rule is available to imitate the above derivation. The case $K=l^{-1}$ is, however, exceptional. Then we choose $\phi_n=\varphi_n\equiv u_n$ to be the eigenfunction of the massless Dirac operator $i\gamma\cdot(\partial+A)\equiv i\not{D}$:

$$i\not{D}u_n(x)=\epsilon_n u_n(x),\quad \epsilon_n\ \text{real},\quad \int d^{2n}x\, u_n^\dagger(x)u_m(x)=\delta_{nm},\quad \sum_n u_n(x)u_n^\dagger(x')=\delta^{2n}(x-x')\ . \tag{32}$$

This gives $H^A=0$ as should be the case for $K=l^{-1}$, while

$$G^A(x)=4\lim_{\Lambda\to\infty}\int_0^\infty d\tau\,\alpha_\Lambda(\tau)\sum_n u_n^\dagger(x)\gamma_{D+1}T^A e^{-l^{-1}(-i\epsilon_n+m)\tau}u_n(x)$$

$$=2\lim_{\Lambda\to\infty}\sum_n u_n^\dagger(x)\gamma_{D+1}T^A f\left[\frac{-i\epsilon_n+m}{\Lambda}\right]u_n(x)=2\lim_{\Lambda\to\infty}\mathrm{tr}\gamma_{D+1}T^A f\left[-\frac{\not{D}^2}{\Lambda^2}\right](x,x)\ . \tag{33}$$

In obtaining Eq. (33) we have neglected the mass compared with Λ, used the anticommutativity of $\not{D}$ and γ_{D+1} and assumed $f(x)+f(-x)=2f(-x^2)$ as is the case for $f(x)=(1+x)^{-1}$ (Ref. 28). According to Matsuki[24] Eq. (33) precisely reproduces the covariant anomaly (20).

On the other hand, Fujikawa's Jacobian factor for the infinitesimal change δ (12a) and (12b) of the integration variables in the fermion path integral for the partition function Z determines anomalies through

$$i\delta J=\langle i\delta S\rangle\ . \tag{34}$$

Fujikawa expands ψ and $\bar{\psi}$ as

$$\psi=\sum_n a_n\phi_n\ , \tag{35a}$$

$$\bar{\psi}=\sum_n\varphi_n^\dagger\bar{b}_n\ , \tag{35b}$$

where a_n and $\bar{b}_n$ are independent Grassmann variables, and ϕ_n and φ_n are the orthonormalized complete sets (29a) and (29b) (the case $K=l^{-1}$ should also be included.) The fermion measure is then defined by

$$D\psi D\bar{\psi}=\prod_n da_n d\bar{b}_n\ , \tag{36}$$

yielding

$$i\delta J=\int d^{2n}x\times\begin{cases}\alpha_A(x)H^A_{\mathrm{unreg}}(x), & (37a)\\ \beta_A(x)G^A_{\mathrm{unreg}}(x), & (37b)\end{cases}$$

where[14]

$$H^A_{\mathrm{unreg}}(x)=\sum_n[\phi_n^\dagger(x)T^A\phi_n(x)-\varphi_n^\dagger(x)T^A\varphi_n(x)]\ , \tag{38a}$$

$$G^A_{\mathrm{unreg}}(x)=\sum_n[\phi_n^\dagger(x)\gamma_{D+1}T^A\phi_n(x)+\varphi_n^\dagger(x)\gamma_{D+1}T^A\varphi_n(x)]\ . \tag{38b}$$

These equations correspond to the white noise $\alpha_\Lambda(\tau)=\delta(\tau)$ in Eqs. (31a) and (31b).

It should be emphasized that the case $K=K_0$ or Eq. (24) are not included in the above derivation. This is because for such K, L and $\tilde{L}$ are not both Hermitian. Thus the (chiral) consistent anomaly is not derivable[29] from Fujikawa's equation (34).

With the arbitrary K allowed in perturbation theory, we consider the regularized partition function Z_{reg} with the measure

$$D\psi D\bar{\psi}=\prod_k dc_k d\bar{c}_k\ , \tag{39}$$

where c_k and $\bar{c}_k$ are independent Grassmann variables in place of a_n and $\bar{b}_n$, which are obtained by replacing ϕ_n and $\varphi_n^\dagger$ in Eqs. (35a) and (35b) by the plane-wave solutions ξ_k and $\xi_k^\dagger$, respectively. Considering the regularized infinitesimal transformations[5] δ_Λ such that

$$\delta_\Lambda S_{\mathrm{reg}}=\delta S\ , \tag{40}$$

for finite Λ, where δS is given by Eq. (34), the regularized Fujikawa's equation reads[30]

$$\lim_{\Lambda\to\infty} i\delta_\Lambda J=\langle i\delta S\rangle_{\text{reg}} . \tag{41}$$

For $S_{\text{reg}}=\int d^Dx\, \bar{\psi}(x)\Gamma R_\Lambda^{-1}\psi(x)$ corresponding to (the Markovian case of) Eq. (9), Eq. (40) is satisfied by taking

$$\delta_\Lambda\psi=i\begin{bmatrix} R_\Lambda\alpha \\ R_\Lambda\beta\gamma_{D+1}\end{bmatrix}\psi , \tag{42a}$$

$$\delta_\Lambda\bar{\psi}=i\bar{\psi}\begin{bmatrix} -\alpha\bar{\tilde{R}}_\Lambda \\ \gamma_{D+1}\beta\bar{\tilde{R}}_\Lambda\end{bmatrix} , \tag{42b}$$

which are the generalization of the regularized gauge and chiral transformations introduced by Bern, Chan, and Halpern.[5] The Jacobian factor $i\delta_\Lambda J$ is given by Eqs. (38a) and (38b) with H^A_{unreg} and G^A_{unreg} being replaced by

$$\begin{aligned} H^A_\Lambda(x)&=\sum_k[\xi_k^\dagger(x)\bar{R}_\Lambda T^A\xi_k(x)-\xi_k^\dagger(x)T^A\bar{\tilde{R}}_\Lambda\xi_k(x)] \\ &=\text{tr}T^A(R_\Lambda-\tilde{R}_\Lambda)(x,x) , \end{aligned} \tag{43a}$$

$$\begin{aligned} G^A_\Lambda(x)&=\sum_k[\xi_k^\dagger(x)\gamma_{D+1}\bar{R}_\Lambda T^A\xi_k(x) \\ &\qquad+\xi_k^\dagger(x)T^A\gamma_{D+1}\bar{\tilde{R}}_\Lambda\xi_k(x)] \\ &=\text{tr}\gamma_{D+1}T^A(R_\Lambda+\tilde{R}_\Lambda)(x,x) , \end{aligned} \tag{43b}$$

respectively. These equations go over to Eqs. (16a) and (16b), respectively, in the $\Lambda\to\infty$ limit.[31] No Hermiticity requirement on L and $\tilde{L}$ is needed in the present derivation. Thus, even for such K that L and $\tilde{L}$ are not both Hermitian, the path-integral method is still applicable if we define the measure (39) and consider the regularized transformations (40), now depending on K, in the manner of Bern, Chan, and Halpern.[5] In this sense Fujikawa's claim[14] that the measure (39) is useless for path-integral evaluation of anomalies is not justified. Equation (41) with Eqs. (43a) and (43b) corresponds to the plane-wave regularization of Ref. 14.

VI. SUMMARY AND DISCUSSIONS

We have presented a proof of the existence of the equilibrium limit in Euclidean stochastic quantization of external-field Dirac theory in both SR schemes. The condition for existence is the same as in the unregularized theory. The proof is made possible using the regularized generating functional. Its explicit form derived in Sec. II can also be obtained with the help of Green's functions G and $\bar{G}$.

The generating functional method has been applied to show that anomalies in the background gauge field depend on the kernel in the fermion Langevin equation. For instance, covariant and consistent anomalies come from different kernels.[16,9] A new type of anomaly in $D=4$ dimensions has also been derived from a special choice of the kernel. The reason behind the correspondence remains open.

The relation of the stochastic quantization method to Fujikawa's path-integral one has also been studied on the basis of the plane-wave measure and the regularized transformations of Ref. 5.

ACKNOWLEDGMENTS

The authors are greatly thankful to Professor S. Aramaki and Professor S. Fukuzumi for useful discussions.

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$$\langle\eta_i(x,\tau)\bar{\eta}_j(x',\tau')\rangle=2\delta_{ij}\delta^D(x-x')\delta(\tau-\tau') .$$
The simplified notation (2c) is allowed only if K contains no dynamical fields, which we shall assume in this paper.

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[20]See, for instance, P. M. Damgaard and H. Hüffel, Phys. Rep. **152**, 654 (1987), and references therein.

[21]The sign is opposite to that of Eq. (A7) in Ref. 16 because $\gamma_5=-\gamma_1\gamma_2\gamma_3\gamma_4$ in this paper. This result was also obtained

by Namiki *et al.*, (Ref. 17) in a different way.

[22]In the Abelian gauge theory the index A takes only the value 0 and the trace operation should be deleted accordingly.

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[27]In Bardeen's vector–axial-vector gauge coupling model (Ref. 13) there are two possible choices (Ref. 16) of K to maintain Hermiticity of L and $\tilde{L}$ if the axial-vector field is taken to be either anti-Hermitian or Hermitian; they lead, respectively, to covariant $(L\neq\tilde{L})$ and Bardeen's consistent (Ref. 13) $(L=\tilde{L})$ anomalies as is well known in the path-integral method (Ref. 14).

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[31]Bern, Chan, and Halpern (Ref. 5) derived for $K=\Gamma^\dagger$ the covariant anomalies (20) also based on the regularized transformations (42a) and (42b) in the manner of Fujikawa. It seems that they tacitly assumed the measure (36), although they did not mention it. In fact, this assumption is not really necessary. They could equally assume the measure (39). In other words, no practical difference exists for an important gauge-invariant choice $K=\Gamma^\dagger$.