Poincaré-invariant Lee model

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A relativistic generalization of the Lee model is developed within the framework of light-front dynamics. The model describes three particles V, N, and θ interacting through the virtual process $V \rightleftarrows N+\theta$, with no antiparticles present. The model consists of ten generators which satisfy the commutation relations of the Poincaré group. The Lorentz invariance of the S-matrix elements that arise is demonstrated. The dressed V-particle state is given, and the integral equations that arise in the θV-$\theta\theta N$ sector are derived. The interacting spin operator for the model is analyzed in some detail. The integral equations for the θV-$\theta\theta N$ sector are transformed to a new picture and a partial-wave analysis of the transformed equations is carried out. It is shown that the θV elastic scattering amplitudes can be obtained from the solution of uncoupled, one-dimensional integral equations. The model has many of the features of few-particle systems involving pions and nucleons and moreover provides insight into dealing with the interacting angular momentum operators of light-front dynamics.

I. INTRODUCTION

In its original formulation,[1] the Lee model consists of two fermions V and N and a boson θ, interacting according to the vertex $V \rightleftarrows N+\theta$. The θ particle is treated relativistically, while the V and N particles are assumed to be static.

Since there are no antiparticles in the model, the Fock space breaks up into a direct sum of invariant subspaces which are spanned by restricted sets of basis vectors. For example, one sector is spanned by bare V and θN states, while another is spanned by bare θV and $\theta\theta N$ states. The physical V-particle state and the θN scattering amplitude were obtained in Lee's original paper.[1] The θV elastic scattering amplitude and the amplitude for the production process $V+\theta \rightarrow N+2\theta$ were first derived by Amado[2] using the methods of dispersion theory.

Originally[1] the Lee model was used to illustrate mass and coupling-constant renormalization in a quantum field theory. Subsequently, it was used[3] to study the problem of ghost states and the accompanying indefinite metric. The description of unstable particles[4] has also been studied with the help of the Lee model.

One of the more interesting applications of the Lee model and its variations is the construction of solvable models for few-particle systems. This type of application was first developed by Amado[5] in the context of a simple model for nucleon-deuteron scattering as well as a model for deuteron stripping. The practicality of this approach to few-particle systems was subsequently established by Amado and his co-workers.[6]

The V-θN and θV-$\theta\theta N$ sectors of the Lee model incorporate many of the features of the low- and intermediate-energy pion-nucleon system. This fact was exploited in a three-body calculation of pion-nucleon scattering by Aaron.[7] The relationship between various versions of the Lee model and the πN system has also been studied by the author,[8,9] and the insights gained thereby have been used to develop a set of integral equations for the coupled $N\pi$-$N\pi\pi$ system.[10] It is interesting to note that the cloudy bag model of baryon structure leads rather naturally to a model[11] of N's and Δ's interacting with pions, which is a combination of the Lee model and the Chew-Low model.

The VN-θNN and VV-θNV-$\theta\theta NN$ sectors of the Lee model have many of the features of the system consisting of two nucleons coupled explicitly to pions. This aspect of the Lee model has been studied by a number of workers.[12]

One of the serious shortcomings of the original Lee model[1] is that it does not take into account the requirements of special relativity. In particular it does not specify a complete set of ten generators for the Poincaré group, and moreover it does not lead to scattering amplitudes that behave properly under Lorentz transformations.

One of the problems in constructing Poincaré-invariant theories is that in order to satisfy the commutation relations for the generators more than one generator must contain interactions. As Dirac[13] pointed out some time ago, there are various ways to separate the ten generators into a subset which contains interactions, i.e., is dynamical, and a subset which is kinematical. Since the commutator of two kinematical generators is a linear combination of kinematical generators, this subset of generators must be associated with some subgroup of the Poincaré group. Such subgroups are called stability groups[14] or kinematic subgroups.[15] Each such subgroup is associated with a three-dimensional hypersurface which is invariant under a subgroup of the Poincaré transformations $x'=ax+b$ and intersects every world line once. The choice of the invariant hypersurface determines the form of the dynamics. Dirac[13] considered the instant, point, and front forms for which the hypersurfaces can be taken to be $t=0$, $t^2-\mathbf{x}^2=a^2>0$ with $t>0$, and $t+z=0$, respectively. The instant and point forms have four dynamical generators, while the front form has only three. It is known[14] that there are two other possible hypersurfaces, but these both require six dynamical generators, which makes them rather unattractive.

An instant-form version of the V-θN sector of the Lee model has been treated by de Dormale.[16] As he points out it does not appear to be straightforward to extend the instant-form approach to the other sectors. There is a brief discussion of the relativistic Lee model in Ref. 15; however, no explicit model is given and no analysis is carried out. Relativistic models, involving the N, the Δ isobar, and the π, which ignore the NN interaction and only consider the $\Delta \rightleftarrows \pi + N$ vertex are essentially relativistic Lee models.[17]

Here we will construct and analyze a Poincaré-invariant Lee model within the framework of the front form of relativistic dynamics. As in the original Lee model[1] there are three particles V, N, and θ interacting through the vertex $V \rightleftarrows N+\theta$, and no antiparticles. We will take all three particles to be bosons. Of course, the Fock space of this model also breaks up into a direct sum of invariant subspaces.

The statement of the model involves the specification of ten Poincaré generators, three of which are dynamical. The generator associated with translations in the light-front variable $x^0=(t+z)/\sqrt{2}$ will be taken to have the same structure as the Hamiltonian in the original Lee model. We will see that it is not difficult to guess the other two dynamical generators and verify that all ten generators satisfy the necessary commutation relations.

One of the main concerns in relativistic theories is the transformation properties of the various scattering amplitudes. We will see that the model constructed here leads to S-matrix elements that are Lorentz-invariant functions of the initial and final four-momenta involved. This is a very encouraging result since it shows that it is possible to construct tractable models that change particle numbers and still lead to amplitudes that satisfy the requirements of special relativity.

The analysis of the V-θN sector of the Poincaré-invariant Lee model goes through very much as in the treatment of the version of the Lee model analyzed by Schweber,[18] although the kinematics are different. In Schweber's version, the total three-momentum is conserved at each vertex, whereas here it is three *light-front* components of the four-momentum that are conserved.

Here we will analyze the θV-$\theta\theta N$ sector in some detail since it has some of the features of the intermediate-energy πN system, and moreover serves as a prototype for a relativistic three-particle model. Fortunately the techniques[8] used to derive the integral equations for this sector of the original Lee model work here as well. The integral equation that has to be solved to obtain all of the quantities of interest in this sector turns out to be three dimensional.

In relativistics dynamics it is often convenient to focus on the operators which describe the internal structure of the system, i.e., the mass operator M and the internal angular momentum or spin operator $\mathcal{J}$ (Refs. 14 and 15). It is possible to construct light-front models in which $\mathcal{J}$ does not contain interactions;[19–21] however, it turns out that for the Poincaré-invariant Lee model two components of $\mathcal{J}$ contain interactions. Part of the reason for constructing this model was to gain insight into this peculiar feature of light-front dynamics. The model does not disappoint in this regard.

Here we will study in some detail the action of the spin operator $\mathcal{J}$ in the θV-$\theta\theta N$ sector. It turns out that this study pretty much dictates a natural relative momentum variable for this sector.

In general, carrying out a partial-wave analysis when an interacting spin operator $\mathcal{J}$ is in effect is nontrivial. In order to deal with this, the author has developed[22] a unitary transformation which when applied to the state vectors and operators of a light-front model leads to a "new picture" in which the construction of angular momentum eigenstates appears to be kinematical. This unitary transformation depends on two parameters of a lightlike vector ξ, and has accordingly been referred to as the "ξ picture." If in a rest frame we write the conventional components of ξ in the form $(\xi^\mu)=\xi^0(1,-\mathbf{n})$ where $\mathbf{n}$ is a unit vector, the two parameters in the unitary transformation can be identified with the two parameters needed to specify the direction of $\mathbf{n}$. We will see that in the ξ picture it is fairly straightforward to carry out a partial-wave analysis of the integral equations that arise in the θV-$\theta\theta N$ sector of the Poincaré-invariant Lee model. In particular we will find that it is only necessary to solve one-dimensional integral equations in order to determine all of the quantities of interest in this sector.

It is important to note that these integral equations do not suffer from the spurious singularities[23] that occur in earlier integral equations developed for relativistic three-particle systems. It has been shown that such singularities can lead to spurious bound-state solutions.[24]

The outline of the paper is as follows. In Sec. II the ten generators of the Poincaré-invariant Lee model are given and it is verified that they satisfy the commutation relations of the Poincaré algebra. The Lorentz invariance of the S-matrix elements is established in Sec. III. An analysis of some of the states and amplitudes of the model is presented in Sec. IV. In particular, the dressed V-particle state is given, and the integral equations that arise in the θV-$\theta\theta N$ sector are derived. The interacting spin operator for the model is treated in Sec. V, and it is shown that its structure in the θV-$\theta\theta N$ sector suggests a natural relative momentum variable for this sector. The integral equations for the θV-$\theta\theta N$ sector are transformed to the new picture in Sec. VI, and a partial-wave analysis of the transformed equations is carried out. In this section it is shown that the θV elastic scattering amplitudes can be obtained from the solution of uncoupled, one-dimensional integral equations. The Appendix summarizes some useful, general relations for the light-front description of a system of particles.

II. POINCARÉ INVARIANCE

The generators of the Poincaré group satisfy the well-known commutation relations[20,25]

$$[P_\mu,P_\nu]=0\ , \tag{2.1}$$

$$[J_{\mu\nu},P_\rho]=i(g_{\nu\rho}P_\mu-g_{\mu\rho}P_\nu)\ , \tag{2.2}$$

$$[J_{\mu\nu},J_{\rho\lambda}]=i(g_{\mu\lambda}J_{\nu\rho}+g_{\nu\rho}J_{\mu\lambda}-g_{\mu\rho}J_{\nu\lambda}-g_{\nu\lambda}J_{\mu\rho})\ . \tag{2.3}$$

The P_μ generate spacetime translations, while the $J_{\mu\nu}$ generate spacetime rotations. Since our model will be constructed within the framework of light-front dynamics, we will use the metric

$$(g_{\mu\nu})=(g^{\mu\nu})=\begin{bmatrix}0&0&0&1\\0&-1&0&0\\0&0&-1&0\\1&0&0&0\end{bmatrix}\ . \tag{2.4}$$

Dot products and the raising and lowering of indices are carried out in the usual way, e.g.,

$$x\cdot y=x^\mu g_{\mu\nu}y^\nu=x^\mu y_\mu\ . \tag{2.5}$$

It is convenient to distinguish the components of the generators by introducing the notation

$$P^\mu=(P^0,\mathbf{P}_\perp,H)=(\bar{P},H)\ , \tag{2.6a}$$

$$(J_{\mu\nu})=\begin{bmatrix}0&-S_1&-S_2&K_3\\S_1&0&J_3&B_1\\S_2&-J_3&0&B_2\\-K_3&-B_1&-B_2&0\end{bmatrix}\ . \tag{2.6b}$$

We will also let

$$\mathbf{B}=(B_1,B_2),\quad \mathbf{S}=(S_1,S_2)\ . \tag{2.7}$$

Here H is the light-front Hamiltonian and generates translations in x^0, which plays the role of time in light-front dynamics. We will refer to $\bar{P}$ as the "trimomentum." If we distinguish the conventional components of the four-momentum by a caret, then the light-front components are related to them by

$$P^0=(\hat{P}^0+\hat{P}^3)/\sqrt{2}\ , \tag{2.8a}$$

$$P^r=\hat{P}^r,\quad r=1,2\ , \tag{2.8b}$$

$$P^3=H=(\hat{P}^0-\hat{P}^3)/\sqrt{2}\ . \tag{2.8c}$$

The components of $J_{\mu\nu}$ are related to the components of the angular momentum $\mathbf{J}$ and boost operator $\mathbf{K}$ by

$$\mathbf{J}=(J_1,J_2,J_3)\ , \tag{2.9}$$

$$\mathbf{K}=(K_1,K_2,K_3)\ , \tag{2.10}$$

$$J_r=\epsilon_{rs}(S_s-B_s)/\sqrt{2}\ , \tag{2.11}$$

$$K_r=(S_r+B_r)/\sqrt{2}\quad (r,s=1,2)\ , \tag{2.12}$$

where ϵ_{rs} is the two-dimensional Levi-Civita symbol.

Throughout we will work with only on-mass-shell particles. Accordingly (2.4) and (2.5) imply that for a particle of mass m, we have

$$(p^\mu)=\left[\bar{p},\frac{\mathbf{p}_\perp^2+m^2}{2p^0}\right]\ . \tag{2.13}$$

We will use the notation $L_{\mu\nu}(p)$ for the single-particle generators of spacetime rotations, which are given by

$$L_{r3}(p)=B_r(p)=-ip^0\frac{\partial}{\partial p^r}\ , \tag{2.14a}$$

$$L_{03}(p)=K_3(p)=-ip^0\frac{\partial}{\partial p^0}\ , \tag{2.14b}$$

$$L_{12}(p)=J_3(p)=i\left[\frac{\partial}{\partial p^1}p^2-\frac{\partial}{\partial p^2}p^1\right]\ , \tag{2.14c}$$

$$L_{r0}(p)=S_r(p)=-i\left[p^r\frac{\partial}{\partial p^0}+p^3\frac{\partial}{\partial p^r}\right]\ . \tag{2.14d}$$

In order to construct the Lee model, we introduce operators $a_\alpha(p_\alpha)$ and $a_\alpha^\dagger(p_\alpha)$, which annihilate and create particles of momentum p_α. We assume the commutation relations

$$[a_\alpha(p_\alpha),a_\beta^\dagger(k_\beta)]=\delta_{\alpha\beta}(2\pi)^3 2p_\alpha^0\delta^3(\bar{p}_\alpha-\bar{k}_\alpha)\ , \tag{2.15}$$

with all other commutators zero.

We will denote noninteracting operators by a subscript or superscript 0. For a noninteracting second-quantized theory, we can take, for the generators,

$$P_0^\mu=\sum_\alpha\int dp_\alpha a_\alpha^\dagger(p_\alpha)p_\alpha^\mu a_\alpha(p_\alpha)\ , \tag{2.16}$$

$$J_0^{\mu\nu}=\sum_\alpha\int dp_\alpha a_\alpha^\dagger(p_\alpha)L^{\mu\nu}(p_\alpha)a_\alpha(p_\alpha)\ , \tag{2.17}$$

where

$$dp=\frac{d\bar{p}\,\theta(p^0)}{(2\pi)^3 2p^0}\ . \tag{2.18}$$

It is straightforward to verify that since the single-particle generators satisfy (2.1)–(2.3), so do the generators given by (2.16) and (2.17).

In light-front dynamics the seven generators $\bar{P}$, $\mathbf{B}$, K_3, and J_3 are taken to be noninteracting or kinematical, and the interactions are put into the remaining three; i.e., we assume

$$\bar{P}=\bar{P}_0,\quad \mathbf{B}=\mathbf{B}_0,\quad K_3=K_3^0,\quad J_3=J_3^0\ , \tag{2.19}$$

$$H=H_0+H_i,\quad \mathbf{S}=\mathbf{S}_0+\mathbf{S}_i\ . \tag{2.20}$$

The generators given by (2.19) generate a subgroup of the Poincaré group called the stability group of the null plane or the kinematic subgroup. This subgroup is the set of transformations which maps the null-plane hypersurface

$$x^0=0\ \text{(null plane)} \tag{2.21}$$

into itself. The commutator of any two members of the set (2.19) is a linear combination of members of the set, so it is consistent to take them all to be noninteracting. According to (2.2) and (2.6),

$$[S_r,P^s]=-i\delta_{rs}H\ , \tag{2.22}$$

so if H contains an interaction so must $\mathbf{S}$.

In the Lee model[1] it is assumed that there are three particles V, N, and θ and no antiparticles. These particles interact according to the vertex

$$V \rightleftarrows N + \theta \ . \tag{2.23}$$

Accordingly we take, for the interaction in H,

$$\begin{aligned} H_i = (2\pi)^3 g_0 \int dp_\theta dp_N dp_V \delta^3(\bar{p}_V - \bar{p}_N - \bar{p}_\theta) \\ \times [a_V^\dagger(p_V) a_N(p_N) a_\theta(p_\theta) \\ + a_\theta^\dagger(p_\theta) a_N^\dagger(p_N) a_V(p_V)] \\ - \int dp_V a_V^\dagger(p_V) \frac{m_V^2 - m_{V0}^2}{2p_V^0} a_V(p_V) \ , \end{aligned} \tag{2.24}$$

where we have included a mass counterterm for the V particle. It turns out that the masses of the θ and N particles, m_θ and m_N, are not renormalized. It is straightforward to verify that a choice for $\mathbf{S}_i$ that agrees with (2.22) is given by

$$\begin{aligned} S_r^i = -(2\pi)^3 g_0 \int dp_\theta dp_N dp_V i \frac{\partial \delta^3(\bar{p}_V - \bar{p}_N - \bar{p}_\theta)}{\partial p_V^r} \\ \times [a_V^\dagger(p_V) a_N(p_N) a_\theta(p_\theta) \\ - a_\theta^\dagger(p_\theta) a_N^\dagger(p_N) a_V(p_V)] \\ + \int dp_V a_V^\dagger(p_V) \frac{m_V^2 - m_{V0}^2}{2p_V^0} i \frac{\partial}{\partial p_V^r} a_V(p_V) \ . \end{aligned} \tag{2.25}$$

We must now check that all of the commutation relations that involve H and $\mathbf{S}$ are satisfied. With the help of the relations

$$[P_\mu^0, a_\alpha(p_\alpha)] = -p_{\mu\alpha} a_\alpha(p_\alpha) \ , \tag{2.26}$$

$$[J_{\mu\nu}^0, a_\alpha(p_\alpha)] = -L_{\mu\nu}(p_\alpha) a_\alpha(p_\alpha) \ , \tag{2.27}$$

and the fact that the single-particle generators $L_{\mu\nu}(p)$ are Hermitian with respect to the metric dp, it is straightforward to verify that the commutators of H and $\mathbf{S}$ with the kinematic generators (2.19) come out correctly.

The only commutation relations that remain to be checked are

$$[H, S_r] = [S_1, S_2] = 0 \ . \tag{2.28}$$

From (2.22) it follows that

$$[H, S_r] = i\{[S_s, S_r] P^s - P^s [S_s, S_r]\} \qquad (r \neq s;\ r \text{ and } s = 1,2) \ , \tag{2.29}$$

so if S_1 and S_2 commute so do H and $\mathbf{S}$. Using (2.27), it is straightforward to verify that

$$[\hat{S}_1^i, \hat{S}_2^0] + [\hat{S}_1^0, \hat{S}_2^i] = 0 \ . \tag{2.30}$$

Here $\hat{S}_r^0$ is given by (2.17) and (2.14d) with bare masses, and $\hat{S}_r^i$ is given by (2.25) without the mass counterterm. With a little bit of manipulation it is possible to show that

$$\begin{aligned} [\hat{S}_1^i, \hat{S}_2^i] = (2\pi)^6 g_0^2 \int dp_\theta dp_N dp_V dp'_\theta dp'_N dp'_V \delta^3(\bar{p}_V - \bar{p}_N - \bar{p}_\theta) \delta^3(\bar{p}'_V - \bar{p}'_N - \bar{p}'_\theta) \\ \times \sum_{\alpha = \theta, N, V} \tfrac{1}{2} \left[\frac{\partial^2}{\partial p_\alpha^1 \partial p_\alpha^2} + \frac{\partial^2}{\partial p_\alpha'^1 \partial p_\alpha'^2} \right] (2\pi)^3 2p_\alpha^0 \delta^3(\bar{p}_\alpha - \bar{p}'_\alpha) [O_\alpha(p_\beta, p_\gamma; p'_\beta, p'_\gamma) - O_\alpha(p'_\beta, p'_\gamma; p_\beta, p_\gamma)] \\ (\alpha \neq \beta, \gamma;\ \beta \neq \gamma) \ , \end{aligned} \tag{2.31}$$

where the O_α's are products of creation and annihilation operators. Since the integrand is antisymmetric in p's and p''s, the integral vanishes and

$$[\hat{S}_1^i, \hat{S}_2^i] = 0 \ . \tag{2.32}$$

This completes the proof that the Lee-model generators given by (2.16), (2.17), (2.19), (2.20), (2.24), and (2.25) satisfy the commutation relations (2.1)–(2.3).

III. INVARIANCE OF THE S MATRIX

In this section we will demonstrate that the S-matrix elements for the various processes that the Lee model gives rise to are Lorentz-invariant functions of the initial and final four-momenta. The technique we will use is based on the well-known Lehmann-Symanzik-Zimmermann (LSZ) formalism, which has been used to[5,26] investigate the original Lee model. Justification for the use of this approach can be found in Ref. 27.

We begin by defining in $(+)$ and out $(-)$ operators by

$$Z_\alpha^{1/2} a_\alpha^{(\pm)}(p_\alpha) = \lim_{\tau \to \mp\infty} A_\alpha(\tau, p_\alpha) \ , \tag{3.1}$$

where Z_α is a wave-function renormalization constant and

$$\begin{aligned} A_\alpha(\tau, p_\alpha) &= e^{ip_\alpha^3 \tau} e^{iH\tau} a_\alpha(p_\alpha) e^{-iH\tau} \ , \\ &= e^{iH\tau} e^{-iH_0\tau} a_\alpha(p_\alpha) e^{iH_0\tau} e^{-iH\tau} \ . \end{aligned} \tag{3.2}$$

The equivalence of the two expressions for A_α can be demonstrated by using (2.26). From (3.2) we find the equation

$$i \frac{\partial}{\partial \tau} A_\alpha(\tau, p_\alpha) = [A_\alpha(\tau, p_\alpha), H] - p_\alpha^3 A_\alpha(\tau, p_\alpha) \ . \tag{3.3}$$

Since (3.1) implies that $A_\alpha(\tau, p_\alpha)$ becomes independent of τ for large τ, we see that (3.3) implies that

$$[H, a_\alpha^{(\pm)}(p_\alpha)] = -p_\alpha^3 a_\alpha^{(\pm)}(p_\alpha) \ . \tag{3.4}$$

According to (2.1) and (2.19) $\bar{P}$ commutes with H and H_0, so (3.2) and (2.26) lead to

$$[\bar{P}, A_\alpha(\tau, p_\alpha)] = -\bar{p}_\alpha A_\alpha(\tau, p_\alpha) \ . \tag{3.5}$$

Combining this with (3.1) and (3.4), we obtain

$$[P^\mu, a_\alpha^{(\pm)}(p_\alpha)] = -p_\alpha^\mu a_\alpha^{(\pm)}(p_\alpha) \; . \tag{3.6}$$

The in and out operators can be used to create the in and out states for the interacting theory. For example, the in and out θV scattering states are given by

$$\begin{aligned} |p_\theta p_V\rangle^{(\pm)} &= a_\theta^{(\pm)\dagger}(p_\theta) a_V^{(\pm)\dagger}(p_V)|0\rangle \; , \\ &= a_\theta^{(\pm)\dagger}(p_\theta)|p_V\rangle^{(\pm)} \; , \end{aligned} \tag{3.7}$$

where $|0\rangle$ is the vacuum state. It follows from (3.6) that

$$P^\mu |p_\theta p_V\rangle^{(\pm)} = (p_\theta^\mu + p_V^\mu)|p_\theta p_V\rangle^{(\pm)} \; . \tag{3.8}$$

The S-matrix elements for θV elastic scattering are given by

$$S(p_\theta, p_V; k_\theta, k_V) = {}^{(-)}\langle p_\theta p_V | k_\theta k_V\rangle^{(+)} \; . \tag{3.9}$$

We wish to prove the Lorentz invariance of matrix elements such as (3.9). For the Lorentz transformation of the spacetime points x and the state vectors $|\Psi\rangle$, we use the notation

$$x' = \Lambda x \; , \tag{3.10}$$

$$|\Psi'\rangle = U(\Lambda)|\Psi\rangle \; , \tag{3.11}$$

where Λ is the underlying Lorentz transformation and $U(\Lambda)$ is a unitary operator that corresponds to it. For an infinitesimal transformation we have

$$\Lambda_{\mu\nu} = g_{\mu\nu} + \epsilon_{\mu\nu} \quad (\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}) \; , \tag{3.12}$$

and

$$U(\Lambda) = 1 - \frac{i}{2}\epsilon_{\mu\nu} J^{\mu\nu} \; . \tag{3.13}$$

For the noninteracting theory, (3.13) and (2.27) imply

$$U_0(\Lambda) a_\alpha(p_\alpha) U_0(\Lambda)^{-1} = a_\alpha(p'_\alpha) \; , \tag{3.14}$$

with

$$p'_\alpha = \Lambda p_\alpha \; . \tag{3.15}$$

We will now show that (3.14) holds with U_0 and a_α replaced by U and $a_\alpha^{(\pm)}$, respectively.

We first consider the Lorentz transformations generated by $\mathbf{B}$, K_3, and J_3. These transformations belong to the stability group of the null plane, and we will denote them by Λ_s. In general, for a Poincaré-invariant theory the components of the four-momentum operator transform according to[28]

$$U(\Lambda) P^\mu U(\Lambda)^{-1} = P^\nu \Lambda_\nu{}^\mu \; . \tag{3.16}$$

As a result of (2.19), we have

$$U(\Lambda_s) = U_0(\Lambda_s) \; . \tag{3.17}$$

Using these relations, the fact that $\bar{P} = \bar{P}_0$, and (3.14), it is straightforward to show that

$$U(\Lambda_s) A_\alpha(\tau, p_\alpha) U(\Lambda_s)^{-1} = A_\alpha(\Lambda_{s0}^0 \tau, \Lambda_s p_\alpha) \; . \tag{3.18}$$

For a proper Lorentz transformation Λ_{s0}^0 is positive, so we find from (3.1) that

$$U(\Lambda_s) a_\alpha^{(\pm)}(p_\alpha) U(\Lambda_s)^{-1} = a_\alpha^{(\pm)}(\Lambda_s p_\alpha) \; . \tag{3.19}$$

Using this, (3.12), and (3.13), we find that

$$[J_{\mu\nu}^s, a_\alpha^{(\pm)}(p_\alpha)] = -L_{\mu\nu}(p_\alpha) a_\alpha^{(\pm)}(p_\alpha) \; . \tag{3.20}$$

By letting Λ_s in (3.18) be an infinitesimal transformation and then letting $\tau \to \mp\infty$, we find that in order to have consistency with (3.20), we must have

$$\lim_{\tau \to \mp\infty} \tau \frac{\partial A_\alpha(\tau, p_\alpha)}{\partial \tau} = 0 \; . \tag{3.21}$$

We will soon make use of this identity.

We now consider Lorentz transformations generated by $\mathbf{S}$. From (2.26), (2.27), (2.24), and (2.25) it follows that

$$\begin{aligned} [S_r, a_\alpha(p_\alpha)] = {} & -S_r(p_\alpha) a_\alpha(p_\alpha) \\ & - i \frac{\partial}{\partial p_\alpha^r} \{ [H, a_\alpha(p_\alpha)] + p_\alpha^3 a_\alpha(p_\alpha) \} \; . \end{aligned} \tag{3.22}$$

Using this in conjunction with (2.28), (2.14d), and (3.2), we find

$$\begin{aligned} [S_r, A_\alpha(\tau, p_\alpha)] = {} & -S_r(p_\alpha) A_\alpha(\tau, p_\alpha) \\ & - e^{ip_\alpha^3 \tau} i \frac{\partial}{\partial p_\alpha^r} e^{-ip_\alpha^3 \tau} \{ [H, A_\alpha(\tau, p_\alpha)] \\ & + p_\alpha^3 A_\alpha(\tau, p_\alpha) \} \; , \end{aligned} \tag{3.23}$$

which in turn when combined with (3.1), (3.4), (3.3), and (3.21) leads to

$$[S_r, a_\alpha^{(\pm)}(p_\alpha)] = -S_r(p_\alpha) a_\alpha^{(\pm)}(p_\alpha) \; . \tag{3.24}$$

Putting together (3.19) and (3.24), we conclude that

$$U(\Lambda) a_\alpha^{(\pm)}(p_\alpha) U(\Lambda)^{-1} = a_\alpha^{(\pm)}(p'_\alpha) = a_\alpha^{(\pm)}(\Lambda p_\alpha) \; , \tag{3.25}$$

which when applied, for example, to (3.7) and (3.9) leads to

$$U(\Lambda)|p_\theta p_V\rangle^{(\pm)} = |p'_\theta p'_V\rangle^{(\pm)} \; , \tag{3.26}$$

and

$$S(p_\theta, p_V; k_\theta, k_V) = S(p'_\theta, p'_V; k'_\theta, k'_V) \; . \tag{3.27}$$

This completes the proof of the invariance of the S-matrix elements.

IV. ANALYSIS OF STATES AND AMPLITUDES

In analyzing the states of the Lee model it is convenient to introduce operators $J_\alpha(p_\alpha)$ defined by

$$[H, a_\alpha^\dagger(p_\alpha)] = p_\alpha^3 a_\alpha^\dagger(p_\alpha) + J_\alpha(p_\alpha) \; . \tag{4.1}$$

Using this definition, as well as (3.1)–(3.3), it is trivial to show that

$$a_\alpha(p_\alpha) = Z_\alpha^{1/2} a_\alpha^{(\pm)}(p_\alpha) - i \int_{\mp\infty}^0 d\tau \, J_\alpha^\dagger(\tau, p_\alpha) e^{ip_\alpha^3 \tau} \; , \tag{4.2}$$

where

$$J_\alpha(\tau, p_\alpha) = e^{iH\tau} J_\alpha(p_\alpha) e^{-iH\tau} \; . \tag{4.3}$$

We let $|k^3 q\rangle$ be an eigenstate of $\bar{H}$, i.e.,

$$H|k^3 q\rangle = k^3|k^3 q\rangle \ , \tag{4.4}$$

where q stands for any additional quantum numbers that are needed to specify the state. If we let (4.2) or its adjoint act on such a state, and put a convergence factor $e^{\pm\epsilon\tau}$ in the integral, we obtain

$$a_\alpha(p_\alpha)|k^3 q\rangle = Z_\alpha^{1/2} a_\alpha^{(\pm)}(p_\alpha)|k^3 q\rangle + \frac{1}{k^3 - p_\alpha^3 \pm i\epsilon - H} J_\alpha^\dagger(p_\alpha)|k^3 q\rangle \ , \tag{4.5}$$

and

$$Z_\alpha^{1/2} a_\alpha^{(\pm)\dagger}(p_\alpha)|k^3 q\rangle = \left[a_\alpha^\dagger(p_\alpha) + \frac{1}{k^3 + p_\alpha^3 \pm i\epsilon - H} J_\alpha(p_\alpha) \right] |k^3 q\rangle \ . \tag{4.6}$$

Putting (4.1) in these relations we find

$$Z_\alpha^{1/2} a_\alpha^{(\pm)}(p_\alpha)|k^3 q\rangle = \pm \frac{i\epsilon}{k^3 - p_\alpha^3 \pm i\epsilon - H} a_\alpha(p_\alpha)|k^3 q\rangle \ , \tag{4.7}$$

and

$$Z_\alpha^{1/2} a_\alpha^{(\pm)\dagger}(p_\alpha)|k^3 q\rangle = \pm \frac{i\epsilon}{k^3 + p_\alpha^3 \pm i\epsilon - H} a_\alpha^\dagger(p_\alpha)|k^3 q\rangle \ . \tag{4.8}$$

We now consider various states. The bare vacuum state $|0\rangle$ is defined by

$$a_\alpha(p_\alpha)|0\rangle = 0, \quad \alpha = \theta, N, V \ . \tag{4.9}$$

Since the interaction terms in H and $\mathbf{S}$ give 0 when acting on $|0\rangle$, we have

$$P_\mu|0\rangle = 0 \ , \tag{4.10}$$

$$J_{\mu\nu}|0\rangle = 0 \ ; \tag{4.11}$$

i.e., the bare vacuum is the physical vacuum and moreover is invariant under spacetime translations and rotations. By choosing $|k^3 q\rangle = |0\rangle$ in (4.7), it is easy to check that the in and out annihilation operators annihilate the vacuum, i.e.,

$$a_\alpha^{(\pm)}(p_\alpha)|0\rangle = 0, \quad \alpha = \theta, N, V \ . \tag{4.12}$$

The bare single-particle states are defined by

$$|p_\alpha\rangle = a_\alpha^\dagger(p_\alpha)|0\rangle, \quad \alpha = \theta, N, V \ . \tag{4.13}$$

According to (2.16), (2.19), (2.20), and (2.24), we have

$$P^\mu|p_\alpha\rangle = p_\alpha^\mu|p_\alpha\rangle, \quad \alpha = \theta, N \ ; \tag{4.14}$$

so we see that the θ particle and N particle are not dressed by the interaction. There is however a distinction between the bare and dressed V particle. The dressed single-particle states are defined by

$$|p_\alpha\rangle^{(\pm)} = a_\alpha^{(\pm)\dagger}(p_\alpha)|0\rangle \tag{4.15}$$

and according to (3.6) and (4.10) satisfy

$$P^\mu|p_\alpha\rangle^{(\pm)} = p_\alpha^\mu|p_\alpha\rangle^{(\pm)} \ . \tag{4.16}$$

We choose the Z_α in (3.1) so that the dressed single-particle states have the same normalization as the bare ones, i.e.,

$$^{(\pm)}\langle p_\alpha|k_\alpha\rangle^{(\pm)} = (2\pi)^3 2k_\alpha^0 \delta^3(\bar{p}_\alpha - \bar{k}_\alpha) \ . \tag{4.17}$$

If in (4.7) we choose $|k^3 q\rangle = |k_\alpha\rangle^{(\pm)}$ and contract with $\langle 0|$, we find

$$\langle p_\alpha|k_\alpha\rangle^{(\pm)} = Z_\alpha^{1/2} (2\pi)^3 2k_\alpha^0 \delta^3(\bar{p}_\alpha - \bar{k}_\alpha) \ . \tag{4.18}$$

We will assume the Z_α's are real.

If we choose $|k^3 q\rangle = |0\rangle$ in (4.8), and use (4.13), (4.15), and (4.14), we obtain

$$Z_\theta = Z_N = 1 \ , \tag{4.19}$$

$$|p_\alpha\rangle^{(\pm)} = |p_\alpha\rangle, \quad \alpha = \theta, N \ . \tag{4.20}$$

The dressed V-particle state can be found in precisely the same way as in Schweber's version of the Lee model.[18] The result is

$$|k_V\rangle^{(\pm)} = Z_V^{1/2} \left[|k_V\rangle + \int |p_\theta p_N\rangle dp_\theta dp_N \times \frac{(2\pi)^3 2k_V^0 \delta^3(\bar{p}_\theta + \bar{p}_N - \bar{k}_V)}{m_V^2 - (p_\theta + p_N)^2} g_0 \right] , \tag{4.21}$$

where the states on the right are bare states. The eigenvalue equation for the physical mass is

$$m_V^2 - m_{V0}^2 = \Sigma(m_V^2) \ , \tag{4.22}$$

while the renormalization constant is given by

$$Z_V^{-1} = 1 - \Sigma'(m_V^2) \ . \tag{4.23}$$

The function Σ is defined by

$$\Sigma(s) = g_0^2 \int_0^1 d\eta \int \frac{d\rho}{(2\pi)^3 2\eta(1-\eta)} \frac{1}{s - W^2(\eta,\rho)} \ , \tag{4.24}$$

where the variables η and ρ are described in the Appendix and $W^2 = (p_\theta + p_N)^2$ is given by (A8). We will assume that

$$m_V < m_\theta + m_N \ , \tag{4.25}$$

as a result of which there is no singularity arising from the denominator in (4.21) and the in and out V-particle states are the same.

It is straightforward to verify that

$$^{(\pm)}\langle p_\alpha|k_\beta\rangle^{(\pm)} = \delta_{\alpha\beta}(2\pi)^3 2k_\beta^0 \delta^3(\bar{p}_\beta - \bar{k}_\beta) \ , \tag{4.26}$$

which according to (4.12) and (4.15) is consistent with

$$[a_\alpha^{(\pm)}(p_\alpha), a_\beta^{(\pm)\dagger}(k_\beta)] = \delta_{\alpha\beta}(2\pi)^3 2k_\beta^0 \delta^3(\bar{p}_\beta - \bar{k}_\beta) \ . \tag{4.27}$$

Using the techniques of Ref. 8 it is not difficult to convince oneself that this relation is valid in general.

We now consider the θV scattering states and S-matrix elements defined by (3.7) and (3.9), respectively. We designate the initial and final four-momenta by (k_θ,k_V) and (p_θ,p_V), respectively, and the corresponding total four-momenta by

$$k=k_\theta+k_V\ , \tag{4.28}$$

$$p=p_\theta+p_V\ . \tag{4.29}$$

From (4.6) and (4.19) it follows that

$$|k_\theta k_V\rangle^{(\pm)}=\left[a_\theta^\dagger(k_\theta)+\frac{1}{k^3\pm i\epsilon-H}J_\theta(k_\theta)\right]|k_V\rangle^{(\pm)}\ . \tag{4.30}$$

Using this, as well as (4.4), it is straightforward to show that

$$\begin{aligned}S(p_\theta,p_V;k_\theta,k_V)=&\langle p_\theta p_V|k_\theta k_V\rangle\\&-2\pi i\delta^3(p^3-k^3)\\&\times{}^{(+)}\langle p_V|J_\theta^\dagger(p_\theta)|k_\theta k_V\rangle^{(+)}\ ,\end{aligned} \tag{4.31}$$

where $\langle p_\theta p_V|k_\theta k_V\rangle$ is given by (A9). From (4.1) and (4.4),

$${}^{(+)}\langle p_V|J_\theta^\dagger(p_\theta)|k_\theta k_V\rangle^{(+)}=(k^3-p^3)\,{}^{(+)}\langle p_V|a_\theta(p_\theta)|k_\theta k_V\rangle^{(+)}\ . \tag{4.32}$$

Using the results outlined in the Appendix, as well as (2.27), (3.20), and (4.11), it can be shown that ${}^{(+)}\langle p_V|a_\theta(p_\theta)$ and $|k_\theta k_V\rangle^{(+)}$ satisfy (A12) and (A13) and therefore the matrix element on the right-hand side of (4.32) can be expressed in the form (A14). From (A15) and (A8), we find that we can write

$${}^{(+)}\langle p_V|J_\theta^\dagger(p_\theta)|k_\theta k_V\rangle^{(+)}=(2\pi)^3\delta^3(\bar p-\bar k)T(\eta,\boldsymbol{\rho};\not\eta,\not\rho)\ , \tag{4.33}$$

with

$$\eta=p_\theta^0/p^0,\quad \boldsymbol{\rho}=\mathbf{p}_{\theta\perp}-\eta\mathbf{p}_\perp\ , \tag{4.34}$$

$$\not\eta=k_\theta^0/k^0,\quad \not\rho=\mathbf{k}_{\theta\perp}-\not\eta\mathbf{k}_\perp\ . \tag{4.35}$$

It is important to note that because of the δ function in (4.33), we can set

$$\bar p=\bar k \tag{4.36}$$

in (4.34) and (4.35). It should also be noted that as a result of the δ functions in (4.31) and (4.33), the only T-matrix elements that occur in the S matrix are those for which

$$p^2=k^2 \tag{4.37}$$

with

$$p^2=W_p^2=W^2(\eta,\rho)=\frac{\rho^2+m_\theta^2}{\eta}+\frac{\rho^2+m_V^2}{1-\eta}\ , \tag{4.38}$$

$$k^2=W_k^2=W^2(\not\eta,\not\rho)\ . \tag{4.39}$$

In order to derive an equation for the T matrix, we will first derive an equation for the Fock-space components $\langle p_\theta p_V|k_\theta k_V\rangle^{(\pm)}$. According to (A14) these can be factored in the form

$$\langle p_\theta p_V|k_\theta k_V\rangle^{(\pm)}=(2\pi)^3 2k^0\delta^3(\bar p-\bar k)\psi^{(\pm)}(\eta,\rho;\not\eta,\not\rho)Z_V^{1/2}\ , \tag{4.40}$$

so our goal is to find an equation for the internal wave function $\psi^{(\pm)}$. In (4.5) we let $|k^3q\rangle=|k_\theta k_V\rangle^{(\pm)}$ and $a_\alpha(p_\alpha)=a_\theta(p_\theta)$, and then use (4.19) and (4.27). After contracting with $\langle p_V|$, we use (4.18) to obtain

$$\begin{aligned}\langle p_\theta p_V|k_\theta k_V\rangle^{(\pm)}=&Z_V^{1/2}\langle p_\theta p_V|k_\theta k_V\rangle\\&+\left\langle p_V\left|\frac{1}{k^3\pm i\epsilon-p_\theta^3-H}J_\theta^\dagger(p_\theta)\right|k_\theta k_V\right\rangle^{(\pm)}\ .\end{aligned} \tag{4.41}$$

We will now insert a complete set of states to the left of $J_\theta^\dagger$. As a result of the nature of the interaction (2.23), only $|V\rangle$ and $|\theta N\rangle$ states will contribute. According to (4.1) and (2.24), $J_\theta^\dagger$ involves the products $a_N^\dagger a_V$; therefore, if bare states are used only $|\theta N\rangle$ states survive. Thus we have

$$\begin{aligned}&\left\langle p_V\left|\frac{1}{k^3\pm i\epsilon-p_\theta^3-H}J_\theta^\dagger(p_\theta)\right|k_\theta k_V\right\rangle^{(\pm)}\\&=\int\left\langle p_V\left|\frac{1}{k^3\pm i\epsilon-p_\theta^3-H}\right|p'_\theta p_N\right\rangle dp'_\theta dp_N\\&\qquad\times\langle p'_\theta p_N|J_\theta^\dagger(p_\theta)|k_\theta k_V\rangle^{(\pm)}\ .\end{aligned} \tag{4.42}$$

From (4.1) and (4.40) it follows that

$$\begin{aligned}\langle p'_\theta p_N|J_\theta^\dagger(p_\theta)|k_\theta k_V\rangle^{(\pm)}=&(2\pi)^3\delta^3(\bar p'-\bar k)g\frac{\theta(1-\eta')}{1-\eta'}\\&\times\psi^{(\pm)}(\eta',\rho';\not\eta,\not\rho)\ ,\end{aligned} \tag{4.43}$$

where g is the renormalized coupling constant defined by

$$g=Z_V^{1/2}g_0\ , \tag{4.44}$$

and

$$p'=p_\theta+p'_\theta+p_N\ . \tag{4.45}$$

As a result of the δ functions in (4.43) and (4.40),

$$\bar p=\bar p'=\bar k\ . \tag{4.46}$$

By using the identity[8]

$$\frac{1}{z-H}a_\alpha^\dagger(p_\alpha)=\left[a_\alpha^\dagger(p_\alpha)+\frac{1}{z-H}J_\alpha(p_\alpha)\right]\frac{1}{z-p_\alpha^3-H}\ , \tag{4.47}$$

as well as (4.14), we can show that

$$\begin{aligned}&\left\langle p_V\left|\frac{1}{k^3\pm i\epsilon-p_\theta^3-H}\right|p'_\theta p_N\right\rangle\\&=\left\langle p_V\left|\frac{1}{k^3\pm i\epsilon-p_\theta^3-H}\right|p'_V\right\rangle\frac{g_0\theta(p_V'^0)}{2p_V'^0(k^3\pm i\epsilon-p'^3)}\\&\qquad(\bar p'_V=\bar p'_\theta+\bar p_N)\ .\end{aligned} \tag{4.48}$$

The matrix element on the right-hand side of (4.48) is proportional to the V-particle propagator which in turn is proportional to the θN scattering amplitude. Following Schweber,[18] it can be shown that

$$\left\langle p_V \left| \frac{1}{k^3 \pm i\epsilon - p_\theta^3 - H} \right| p'_\theta p_N \right\rangle \delta^3(\bar{p}' - \bar{k})$$
$$= Z_V^{1/2}(2\pi)^3 2k^0 \delta^3(\bar{p} - \bar{k}) \times \frac{(1-\eta)\theta(1-\eta)}{D[(k-p_\theta)^2 \pm i\epsilon]} \frac{g\delta^3(\bar{p}' - \bar{k})}{k^3 \pm i\epsilon - p'^3}, \quad (4.49)$$

with

$$D(s) = Z_V[s - m_{V0}^2 - \Sigma(s)] . \quad (4.50)$$

As a result of (4.22)–(4.24),

$$D(s) \underset{s \to m_V^2}{\sim} s - m_V^2 . \quad (4.51)$$

If we use the δ function in (4.43) to eliminate the integral over p_N in (4.42), replace $d\bar{p}'_\theta$ with $d\eta' d\rho'$, and use (A8), (A9), and (A15), we find

$$\frac{\psi^{(\pm)}(\eta,\rho;\acute{\eta},\acute{\rho})}{1-\eta} = (2\pi)^3 2\eta \delta(\eta - \acute{\eta}) \delta^2(\rho - \acute{\rho}) + \frac{1}{D[(k-p_\theta)^2 \pm i\epsilon]} \int \frac{d\eta' d\rho' \theta(\eta')}{(2\pi)^3 2\eta'} B(\eta,\rho;\eta',\rho';k^2 \pm i\epsilon) \frac{\psi^{(\pm)}(\eta',\rho';\acute{\eta},\acute{\rho})}{1-\eta'} , \quad (4.52)$$

with

$$B(\eta,\rho;\eta',\rho';k^2 \pm i\epsilon) = \frac{\theta(1-\eta-\eta')g^2}{(1-\eta-\eta')(k^2 \pm i\epsilon - p'^2)} , \quad (4.53)$$

and

$$p'^2 = W^2(\eta,\rho;\eta',\rho') = \frac{\rho^2 + m_\theta^2}{\eta} + \frac{(\rho+\rho')^2 + m_N^2}{1-\eta-\eta'} + \frac{\rho'^2 + m_\theta^2}{\eta'} . \quad (4.54)$$

If in (4.41) we insert a complete set of physical states to the left of $J_\theta^\dagger$, and use (4.16), (4.18), (4.33), and (A15), we find

$$\lim_{p^2 \to k^2} (k^2 - p^2) \left\langle p_V \left| \frac{1}{k^3 \pm i\epsilon - p_\theta^3 - H} J_\theta^\dagger(p_\theta) \right| k_\theta k_V \right\rangle^{(\pm)} = Z_V^{1/2}(2\pi)^3 2k^0 \delta^3(\bar{p} - \bar{k}) T(\eta,\rho;\acute{\eta},\acute{\rho}) , \quad (4.55)$$

where the arguments of T are constrained by (4.37)–(4.39). According to (4.51), this pole is carried by the D in (4.52). Using (4.29), (4.36), and (4.34), we have

$$k^2 - p^2 = \frac{(k-p_\theta)^2 - m_V^2}{1-\eta} , \quad (4.56)$$

so we obtain

$$T(\eta,\rho;\acute{\eta},\acute{\rho}) = X(\eta,\rho;\acute{\eta},\acute{\rho};k^2 + i\epsilon) \quad (p^2 = k^2) , \quad (4.57)$$

where we are introducing a half-off-shell amplitude by

$$X(\eta,\rho;\acute{\eta},\acute{\rho};k^2 \pm i\epsilon) = \int \frac{d\eta' d\rho' \theta(\eta')}{(2\pi)^3 2\eta'} \times B(\eta,\rho;\eta',\rho';k^2 \pm i\epsilon) \times \frac{\psi^{(\pm)}(\eta',\rho';\acute{\eta},\acute{\rho})}{1-\eta'} . \quad (4.58)$$

Combining this with (4.52), we see that a fully off-shell amplitude can be defined as the solution of

$$X(\eta,\rho;\eta',\rho';k^2 \pm i\epsilon) = B(\eta,\rho;\eta',\rho';k^2 \pm i\epsilon) + \int \frac{d\eta'' d\rho'' \theta(\eta'')}{(2\pi)^3 2\eta''} B(\eta,\rho;\eta'',\rho'';k^2 \pm i\epsilon) \times \frac{X(\eta'',\rho'';\eta',\rho';k^2 \pm i\epsilon)}{D[(k-p''_\theta)^2 \pm i\epsilon]} . \quad (4.59)$$

The on-shell matrix elements $[W^2(\eta,\rho) = W^2(\eta',\rho') = k^2]$ of X coincide with the on-shell elements of T, which in turn are Lorentz-invariant functions of the initial and final four-momenta of the θV scattering process.

Comparing (4.58) and (4.52), we see that the internal wave function $\psi^{(\pm)}$ can be expressed in terms of the half-off-shell X. If in (4.4) we choose $|k^3 q\rangle = |k_\theta k_V\rangle^{(\pm)}$, contract with $\langle p_\theta p'_\theta p_N|$, and use (2.24), (4.40), (4.44), and (A15), we find

$$\langle p_\theta p'_\theta p_N | k_\theta k_V \rangle^{(\pm)} = (2\pi)^3 2k^0 \delta^3(\bar{p}' - \bar{k}) \frac{g\theta(1-\eta-\eta')}{k^2 \pm i\epsilon - p'^2} \times \left[\frac{\psi^{(\pm)}(\eta',\rho';\acute{\eta},\acute{\rho})}{1-\eta'} + \frac{\psi^{(\pm)}(\eta,\rho;\acute{\eta},\acute{\rho})}{1-\eta} \right] . \quad (4.60)$$

We see that all of the Fock-space components of $|k_\theta k_V\rangle^{(\pm)}$ can be expressed in terms of the half-off-shell X.

V. SPIN OPERATORS AND NEW VARIABLES

Leutwyler and Stern[14] have shown that the internal angular momentum or spin operator $\mathcal{J}$ for a system is given by

$$M\mathcal{J}_r=\epsilon_{rs}(S_sP^0-K_3P^s-B_sH)-\mathcal{J}_3P^r \tag{5.1a}$$

$$=M_0\mathcal{J}_r^0+\epsilon_{rs}(S_s^iP^0-B_sH_i)\ , \tag{5.1b}$$

$$\mathcal{J}_3=J_3+\frac{B_1}{P^0}P^2-\frac{B_2}{P^0}P^1=\mathcal{J}_3^0\ , \tag{5.2}$$

where M and M_0 are the mass operators for the interacting and noninteracting systems, respectively; and $\mathcal{J}_0$ is the noninteracting spin operator. The equivalence of (5.1a) and (5.1b) follows from (2.19) and (2.20). The components of $\mathcal{J}$ are Hermitian operators which satisfy angular momentum commutation relations. Moreover each component of $\mathcal{J}$ commutes with M as well as $\bar{P}$, $\mathbf{B}$, and K_3, while $\mathcal{J}^2$ is a Casimir operator; i.e., it commutes with all of the generators of the Poincaré group.

It is not difficult to determine the action of $\mathcal{J}$ on the eigenvectors of the Lee model. According to (3.20) and (3.24),

$$[J_{\mu\nu},a_\alpha^{(\pm)\dagger}(p_\alpha)]=-L_{\mu\nu}(p_\alpha)a_\alpha^{(\pm)\dagger}(p_\alpha)\ , \tag{5.3}$$

where the $L_{\mu\nu}$'s are given by (2.14). Using this relation, as well as (4.15), (4.11), and (4.16), it can be shown that

$$\mathcal{J}|p_\alpha\rangle^{(\pm)}=0,\quad \alpha=\theta,N,V\ , \tag{5.4}$$

which simply states that the particles in our model are spinless. Similarly it can be shown that

$$\mathcal{J}|p_\theta p_V\rangle^{(\pm)}=-\mathcal{J}_0(\eta,\rho)|p_\theta p_V\rangle^{(\pm)}\ , \tag{5.5}$$

where

$$W(\eta,\rho)\mathcal{J}_r^0(\eta,\rho)=\epsilon_{rs}\left[-\rho^s i\frac{\partial}{\partial\eta}+W(\eta,\rho)\left[\frac{2\eta-1}{2}W(\eta,\rho)-\frac{m_\theta^2-m_V^2}{2W(\eta,\rho)}\right]i\frac{\partial}{\partial\rho^s}\right]\ , \tag{5.6a}$$

$$\mathcal{J}_3(\eta,\rho)=\mathcal{J}_3^0(\eta,\rho)=i\frac{\partial}{\partial\rho^1}\rho^2-i\frac{\partial}{\partial\rho^2}\rho^1\ . \tag{5.6b}$$

If we define[20]

$$\mathbf{q}=(\boldsymbol{\rho},q_z)\ , \tag{5.7a}$$

where

$$q_z=\frac{2\eta-1}{2}W(\eta,\rho)-\frac{m_\theta^2-m_V^2}{2W(\eta,\rho)}\ , \tag{5.7b}$$

then $\mathcal{J}_0$ can be rewritten in the more familiar form

$$\mathcal{J}_0(\eta,\rho)=\mathcal{J}_0(\mathbf{q})=i\nabla_q\times\mathbf{q}\ . \tag{5.8}$$

It can be shown that $\mathbf{q}=\mathbf{p}_\theta$ in a frame in which $\mathbf{p}_\theta+\mathbf{p}_V=0$.

Eigenstates of angular momentum can be constructed from the states given by (3.7). If we define

$$|pj\lambda\rangle^{(\pm)}=\int|p_\theta p_V\rangle^{(\pm)}d\Omega_q Y_j^\lambda(\hat{q})\ , \tag{5.9}$$

with p given by (4.29), then we find from (5.5), (5.8), and (3.8) that

$$\mathcal{J}^2|pj\lambda\rangle^{(\pm)}=j(j+1)|pj\lambda\rangle^{(\pm)}\ , \tag{5.10a}$$

$$\mathcal{J}_3|pj\lambda\rangle^{(\pm)}=\lambda|pj\lambda\rangle^{(\pm)}\ , \tag{5.10b}$$

$$P^\mu|pj\lambda\rangle^{(\pm)}=p^\mu|pj\lambda\rangle^{(\pm)}\ . \tag{5.10c}$$

In order to carry out partial-wave analyses of (4.52) and (4.59), we need to know the Fock-space representatives of $\mathcal{J}$. It follows from (5.2), (A12), (2.27), (2.14), (4.29), and (4.34) that

$$\langle p_\theta p_V|\mathcal{J}_3=\mathcal{J}_3(\eta,\rho)\langle p_\theta p_V|\ . \tag{5.11}$$

Using (5.1b), the fact that $\mathbf{B}$ commutes with H_i, (A12), (4.10), and (4.11), we can show that

$$\langle p_\theta p_V|M\mathcal{J}_r=W(\eta,\rho)\mathcal{J}_r^0(\eta,\rho)\langle p_\theta p_V|+\epsilon_{rs}\langle 0|\{p^0[a_\theta(p_\theta)a_V(p_V),S_s^i]-B_r(p)[a_\theta(p_\theta)a_V(p_V),H_i]\}\ , \tag{5.12}$$

with p given by (4.29) and $W\mathcal{J}_r^0$ by (5.6a). With the help of (3.22), (2.14), (4.10), (4.14), (4.29), and (4.34), this can be rewritten in the form

$$\langle p_\theta p_V|M\mathcal{J}_r=W(\eta,\rho)\mathcal{J}_r^0(\eta,\rho)\langle p_\theta p_V|+\epsilon_{rs}i\frac{\eta}{2}\frac{\partial}{\partial\rho^s}\langle p_\theta p_V|(M^2-p^2) \tag{5.13a}$$

with

$$p^2=W^2(\eta,\rho)=W_p^2\ . \tag{5.13b}$$

The mass operator M is defined by

$$M^2=2P^0P^3-\mathbf{P}_\perp^2=M_0^2+2P^0H_i\ . \tag{5.14}$$

If we contract (5.11) and (5.13a) with the θV scattering state $|k_\theta k_V\rangle^{(\pm)}$, which is an eigenstate of P^μ with eigenvalue $k^\mu=k_\theta^\mu+k_V^\mu$, we find with the help of (4.40), (5.6), and (4.38) that

$$\begin{aligned}\langle p_\theta p_V|\mathcal{J}|k_\theta k_V\rangle^{(\pm)}&=\mathcal{J}(\eta,\rho;W_k)\langle p_\theta p_V|k_\theta k_V\rangle^{(\pm)}\\&=(2\pi)^3 2k^0\delta^3(\bar{p}-\bar{k})\mathcal{J}(\eta,\rho;W_k)\psi^{(\pm)}(\eta,\rho;\eta',\rho')Z_V^{1/2}\ ,\end{aligned} \tag{5.15}$$

where

$$\mathcal{J}_r(\eta,\rho;W_k)=(1-\eta)\epsilon_{rs}\left[-\frac{\rho^s}{W_k}i\frac{\partial}{\partial\eta}+u_z i\frac{\partial}{\partial\rho^s}\right]\frac{1}{1-\eta}\ , \tag{5.16a}$$

$$\mathcal{J}_3(\eta,\rho;W_k)=\mathcal{J}_3(\eta,\rho)\ , \tag{5.16b}$$

$$k^2=W^2(\eta',\rho')=W_k^2\ . \tag{5.16c}$$

Here

$$u_z = \frac{1}{2W_k}\left[\eta W_k^2 - \frac{\rho^2+m_\theta^2}{\eta}\right], \tag{5.17}$$

which can be combined with ρ to define the three-momentum

$$\mathbf{u}=(\rho, u_z) . \tag{5.18}$$

Reexpressing $\mathcal{J}$ in terms of $\mathbf{u}$, we find

$$\mathcal{J}(\eta,\rho;W_k)=(1-\eta)\mathbf{L}(\mathbf{u})\frac{1}{1-\eta}, \tag{5.19}$$

with

$$\mathbf{L}(\mathbf{u})=i\nabla_u \times \mathbf{u} . \tag{5.20}$$

In order to determine the meaning of $\mathbf{u}$, we consider a Lorentz transformation from the x frame to a rest frame of k, which we call the x_k frame. We have

$$x_k=\Lambda(k)x , \tag{5.21}$$

where the elements of Λ are given by

$$x_k^0=\frac{W_k}{\sqrt{2}k^0}x^0 , \tag{5.22a}$$

$$\mathbf{x}_{\perp k}=\mathbf{x}_\perp - \frac{\mathbf{k}_\perp}{k^0}x^0 , \tag{5.22b}$$

$$x_k^3=\frac{\sqrt{2}}{W_k}k\cdot x - \frac{W_k}{\sqrt{2}k^0}x^0 . \tag{5.22c}$$

This transformation is an element of the stability group of the null plane, since it maps $x^0=0$ to $x_k^0=0$. Obviously

$$(k_k^\mu)=\left[\frac{W_k}{\sqrt{2}},0,0,\frac{W_k}{\sqrt{2}}\right] . \tag{5.23}$$

According to (4.40) we have $\bar{p}=\bar{k}$, and so from (5.22) $\bar{p}_k=\bar{k}_k$, and therefore

$$(p_k^\mu)=\left[\frac{W_k}{\sqrt{2}},0,0,\frac{W_p^2}{\sqrt{2}W_k}\right] . \tag{5.24}$$

Using (5.22)–(5.24) in conjunction with (4.34), we find

$$(p_{\theta k}^\mu)=\left[\frac{W_k\eta}{\sqrt{2}},\rho,\frac{\rho^2+m_\theta^2}{\sqrt{2}W_k\eta}\right] . \tag{5.25}$$

The relation between conventional components $\hat{x}^\mu$ and light-front components x^μ is

$$\hat{x}^0=(x^0+x^3)/\sqrt{2} , \tag{5.26a}$$

$$\hat{x}^r=x^r,\quad r=1,2 , \tag{5.26b}$$

$$\hat{x}^3=(x^0-x^3)/\sqrt{2} , \tag{5.26c}$$

so (5.17), (5.18), and (5.25) imply

$$\mathbf{u}=\mathbf{p}_{\theta k} . \tag{5.27}$$

From (5.24), (5.25), and (5.26), we find

$$\eta=\frac{p_{\theta k}^0}{p_k^0}=\frac{\omega_u+u_z}{W_k} , \tag{5.28}$$

$$\omega_u=(\mathbf{u}^2+m_\theta^2)^{1/2} . \tag{5.29}$$

So we see that $\mathbf{u}$ is the three-momentum that arises from transforming the four-momentum of the θ particle associated with the Fock-space basis state $\langle p_\theta p_V|$ to the rest frame associated with the system eigenstate $|k_\theta k_V\rangle^{(\pm)}$, which has total four-momentum k. Moreover (5.18) and (5.28) show that it is a simple matter to replace the variables (η,ρ) with $\mathbf{u}$.

Using the techniques that led to (5.19)–(5.20), it is also possible to show that

$$\langle p_\theta p_\theta' p_N|\mathcal{J}|k_\theta k_V\rangle^{(\pm)} = \mathcal{J}(\eta,\rho;\eta',\rho';W_k)\langle p_\theta p_\theta' p_N|k_\theta k_V\rangle^{(\pm)} , \tag{5.30}$$

where

$$\mathcal{J}(\eta,\rho;\eta',\rho';W_k)=(1-\eta-\eta')[\mathbf{L}(\mathbf{u})+\mathbf{L}(\mathbf{u}')]\frac{1}{1-\eta-\eta'} . \tag{5.31}$$

Thus the Fock-space representatives of $\mathcal{J}$ for the θV sector come out rather simple when expressed in terms of the variable $\mathbf{u}$. This suggests that we use $\mathbf{u}$'s rather than η's and ρ's in (4.52). If we evaluate the argument of D in the x_k frame, and use (5.23) and (5.27), we find

$$(k-p_\theta)^2=(W_k-\omega_u)^2-\mathbf{u}^2 . \tag{5.32}$$

With the help of (4.45), (4.46), (A15), and (4.34), the denominator of B in (4.53) can be reexpressed as

$$\begin{aligned}(1-\eta-\eta')(k^2-p'^2)&=(k-p_\theta-p_\theta')^2-m_N^2\\ &=(W_k-\omega_u-\omega_{u'})^2-(\mathbf{u}+\mathbf{u}')^2-m_N^2 .\end{aligned} \tag{5.33}$$

Using (5.28) we find

$$d\eta\, d\rho/\eta=d\mathbf{u}/\omega_u . \tag{5.34}$$

We have to be a little careful about the range of the variable u_z. From the inverse of (5.26) it is clear that $p_\theta^0>0$ and $p_V^0>0$; therefore, (4.29) and (4.34) imply

$$0<\eta<1 , \tag{5.35}$$

so from (5.17) we find

$$-\infty<u_z<\frac{W_k^2-\rho^2-m_\theta^2}{2W_k}\le\frac{W_k^2-m_\theta^2}{2W_k} . \tag{5.36}$$

The lower and upper limits correspond to $\eta\to 0$ and $\eta\to 1$, respectively. In (4.52) the $\int d\eta'=\int_0^1 d\eta'$, but because of $\theta(1-\eta-\eta')$ in (4.53), we can let $\int_0^1 d\eta'=\int_0^\infty d\eta'$, which allows us to formally integrate over all values of the components of $\mathbf{u}'$.

The δ function in (4.52) can be transformed by using (5.18) and (5.28). Putting everything together, we arrive at

$$\chi^{(\pm)}(\mathbf{u},\not{\mathbf{u}})=(2\pi)^3 2\omega_u \delta^3(\mathbf{u}-\not{\mathbf{u}})+\frac{1}{D[(k-p_\theta)^2\pm i\epsilon]}\times\int\frac{d\mathbf{u}'}{(2\pi)^3 2\omega_{u'}}B^{(\pm)}(\mathbf{u},\mathbf{u}')\chi^{(\pm)}(\mathbf{u}',\not{\mathbf{u}}) , \tag{5.37}$$

where

$$\chi^{(\pm)}(\mathbf{u},\not{\mathbf{u}})=\frac{\psi^{(\pm)}(\eta,\boldsymbol{\rho};\not\eta,\not{\boldsymbol{\rho}})}{1-\eta} , \tag{5.38}$$

and

$$B^{(\pm)}(\mathbf{u},\mathbf{u}')=B(\eta,\boldsymbol{\rho};\eta',\boldsymbol{\rho}';k^2\pm i\epsilon)=\frac{\theta(1-\eta-\eta')g^2}{(W_k-\omega_u-\omega_{u'})^2-(\mathbf{u}+\mathbf{u}')^2\pm i\epsilon-m_N^2} . \tag{5.39}$$

The argument of the D is given by (5.32), and the η and η' in (5.39) can be expressed in terms of $\mathbf{u}$ and $\mathbf{u}'$ by using (5.28). The three-vector $\not{\mathbf{u}}$ is given by (5.17) and (5.18) with $(\eta,\boldsymbol{\rho})$ replaced by $(\not\eta,\not{\boldsymbol{\rho}})$, or more simply by

$$\not{\mathbf{u}}=\mathbf{k}_{\theta k} . \tag{5.40}$$

Since $\mathbf{k}_{\theta k}$ can be chosen arbitrarily $\not{u}_z$ does not have its range restricted as does u_z in (5.36). More formally this follows from the observation that W_k in (5.17) is given by (4.38) and (4.39) so that $0<\not\eta<1$ is mapped onto $-\infty<\not{u}_z<\infty$.

From (4.58), (5.38), and (5.39), we see that we can write the half-off-shell amplitude for θV-scattering in the form

$$A^{(\pm)}(\mathbf{u},\not{\mathbf{u}})=X(\eta,\boldsymbol{\rho};\not\eta,\not{\boldsymbol{\rho}};k^2\pm i\epsilon)=\int\frac{d\mathbf{u}'}{(2\pi)^3 2\omega_{u'}}B^{(\pm)}(\mathbf{u},\mathbf{u}')\chi^{(\pm)}(\mathbf{u}',\not{\mathbf{u}}) . \tag{5.41}$$

Even though (5.37) is rather simple in appearance, a partial-wave analysis of it is nontrivial. Recalling the θ function in (5.39) and the expression for η in terms of $\mathbf{u}$, i.e., (5.28), it is straightforward to verify that $\chi^{(\pm)}(\mathbf{u},\not{\mathbf{u}})$ must vanish for $\eta>1$. Since $\eta\rightarrow 1$ gives the upper limit in (5.36), we see that $\chi^{(\pm)}(\mathbf{u},\not{\mathbf{u}})$ must vanish for

$$u_z>\frac{W_k-m_\theta^2-u_x^2-u_y^2}{2W_k} . \tag{5.42}$$

Clearly this does not define a spherically symmetric region in $\mathbf{u}$ space. The next section shows how to deal with this.

VI. THE NEW PICTURE

In this section we will see how to adapt the general formalism of Ref. 22 to the problem of carrying out partial-wave analyses of the integral equations that arise in the θV sector of the Poincaré-invariant Lee model. We begin by considering three-rotations R in the rest frame defined by (5.21) and (5.22).

We have

$$U(R)|\mathbf{k}_{\theta k}\mathbf{k}_{Vk}\rangle^{(\pm)}=|R\mathbf{k}_{\theta k}R\mathbf{k}_{Vk}\rangle^{(\pm)} , \tag{6.1a}$$

$$U_0(R)|\mathbf{p}_{\theta k}\mathbf{p}_{Vk}\rangle=|R\mathbf{p}_{\theta k}R\mathbf{p}_{Vk}\rangle , \tag{6.1b}$$

where U and U_0 are the unitary operators corresponding to R for the interacting and noninteracting theories, respectively. These operators can be expressed in terms of the angular momentum operators $\mathbf{J}$ and $\mathbf{J}_0$, and since in light-front dynamics $\mathbf{J}\neq\mathbf{J}_0$, we have $U(R)\neq U_0(R)$ and therefore $\langle\mathbf{p}_{\theta k}\mathbf{p}_{Vk}|\mathbf{k}_{\theta k}\mathbf{k}_{Vk}\rangle^{(\pm)}$ is not a rotationally invariant function of the three-vectors that appear.

In order to put the basis vectors $|p_{\theta k}p_{Vk}\rangle=|\mathbf{p}_{\theta k}\mathbf{p}_{Vk}\rangle$ and the state vectors $|k_{\theta k}k_{Vk}\rangle^{(\pm)}=|\mathbf{k}_{\theta k}\mathbf{k}_{Vk}\rangle^{(\pm)}$ on an equal footing with regard to three rotations we will introduce a unitary transformation $C(\mathbf{n})$ which depends on the two parameters of a unit vector $\mathbf{n}$. In this connection we also introduce a special three-rotation $R(\mathbf{n})$ which has the property

$$R(\mathbf{n})\mathbf{n}=\boldsymbol{\epsilon}_3=(0,0,1) . \tag{6.2}$$

In an active interpretation $R(\mathbf{n})$ rotates the unit vector $\mathbf{n}$ into a vector $\boldsymbol{\epsilon}_3$ in the positive z direction. This rotation is not unique since any such rotation can always be followed by a rotation about $\boldsymbol{\epsilon}_3$. In order to make it unique we specify $R(\mathbf{n})$ to be a right-handed rotation of $\mathbf{n}$ into $\boldsymbol{\epsilon}_3$ about $\mathbf{n}\times\boldsymbol{\epsilon}_3$.

If R is an arbitrary rotation, we can write

$$\mathbf{n}'=R\mathbf{n} , \tag{6.3}$$

and

$$\boldsymbol{\epsilon}_3=R(\mathbf{n}')\mathbf{n}' . \tag{6.4}$$

From (6.2)–(6.4), it seems reasonable that

$$R(\mathbf{n}')R=R_3R(\mathbf{n}) , \tag{6.5}$$

where R_3 is a rotation about $\boldsymbol{\epsilon}_3$. It is not difficult to prove this conjecture.

We now define the unitary operator $C(\mathbf{n})$ by

$$C(\mathbf{n})=U_0[R(\mathbf{n})]^{-1}U[R(\mathbf{n})] . \tag{6.6}$$

By using (6.5) as well as the identities

$$U(R)U(R')=U(RR') , \tag{6.7a}$$

$$U(R^{-1})=U(R)^{-1} , \tag{6.7b}$$

$$U_0(R_3)=U(R_3) , \tag{6.7c}$$

it is straightforward to show that

$$U_0(R)C(\mathbf{n})=C(R\mathbf{n})U(R) . \tag{6.8}$$

We now subject our Fock-space basis vectors to the unitary transformation $C^{-1}(\mathbf{n})$. We define

$$|\mathbf{p}_{\theta k}\mathbf{p}_{Vk}\mathbf{n}\rangle=C(\mathbf{n})^{-1}|\mathbf{p}_{\theta k}\mathbf{p}_{Vk}\rangle \tag{6.9}$$

and observe that (6.8) and (6.1b) imply that

$$U(R)|\mathbf{p}_{\theta k}\mathbf{p}_{Vk}\mathbf{n}\rangle=|\mathbf{p}'_{\theta k}\mathbf{p}'_{Vk}\mathbf{n}'\rangle , \tag{6.10a}$$

$$\mathbf{x}'=R\mathbf{x} . \tag{6.10b}$$

As a consequence of (6.10) and (6.1a), $\langle\mathbf{p}_{\theta k}\mathbf{p}_{Vk}\mathbf{n}|\mathbf{k}_{\theta k}\mathbf{k}_{Vk}\rangle^{(\pm)}$ is a rotationally invariant function of the three-vectors that appear. We see that by introducing the unit vector $\mathbf{n}$ it is possible to obtain a description of rotations in light-front dynamics in terms of the rotation of three-vectors.

From (6.9), (6.6), and (6.1) it follows that

$$\langle \mathbf{p}_{\theta k}\mathbf{p}_{Vk}\mathbf{n}|\mathbf{k}_{\theta k}\mathbf{k}_{Vk}\rangle^{(\pm)}=\langle \mathbf{p}^n_{\theta k}\mathbf{p}^n_{Vk}|\mathbf{k}^n_{\theta k}\mathbf{k}^n_{Vk}\rangle^{(\pm)}$$
$$=\langle p^n_{\theta k}p^n_{Vk}|k^n_{\theta k}k^n_{Vk}\rangle^{(\pm)}\ , \tag{6.11}$$

with

$$\mathbf{x}^n_k=R(\mathbf{n})\mathbf{x}_k\ . \tag{6.12}$$

The right-hand side of (6.11) is a light-front wave function of the type treated in Secs. IV and V.

The rest-frame transformation (5.21) and (5.22) is generated by $\mathbf{B}$ and K_3, so as a result of (2.19) $U_0[\Lambda(k)]=U[\Lambda(k)]$. Accordingly, we have

$$\langle p^n_{\theta k}p^n_{Vk}|k^n_{\theta k}k^n_{Vk}\rangle^{(\pm)}=\langle p^n_{\theta}p^n_{V}|k^n_{\theta}k^n_{V}\rangle^{(\pm)}\ , \tag{6.13}$$

with

$$x^n_k=\Lambda(k^n)x^n\ , \tag{6.14a}$$

$$k^n=k^n_\theta+k^n_V\ . \tag{6.14b}$$

We can identify the right-hand side of (6.13) with (4.40), and simply add the label n to the momentum variables.

It follows from (5.22), (5.23), (5.26), (6.2), and (6.12) that

$$k^{0n}\delta^3(\bar p^{\,n}-\bar k^{\,n})=k^{0n}_k\delta^3(\bar p^{\,n}_k-\bar k^{\,n}_k)$$
$$=W_k\delta^3[\mathbf{p}_k+(\hat p^{\,0}_k-W_k)\mathbf{n}]\ . \tag{6.15}$$

Using this, (6.11), (6.13), (4.40), (5.38), (5.27), (5.40), (5.28), (6.12), and (6.2), we find

$$\langle \mathbf{p}_{\theta k}\mathbf{p}_{Vk}\mathbf{n}|\mathbf{k}_{\theta k}\mathbf{k}_{Vk}\rangle^{(\pm)}$$
$$=(2\pi)^3 2W_k\delta^3[\mathbf{p}_k+(\hat p^{\,0}_k-W_k)\mathbf{n}]$$
$$\times(1-\eta)\phi^{(\pm)}(\mathbf{u},\mathbf{n},\not{\mathbf{k}})Z_V^{1/2}\ , \tag{6.16}$$

where

$$\phi^{(\pm)}(\mathbf{u},\mathbf{n},\not{\mathbf{k}})=\chi^{(\pm)}(\mathbf{u}^n,\not{\mathbf{k}}^n)\ , \tag{6.17}$$

with

$$\mathbf{u}^n=R(\mathbf{n})\mathbf{u},\quad \not{\mathbf{k}}^n=R(\mathbf{n})\not{\mathbf{k}}\ , \tag{6.18}$$

$$\mathbf{u}=\mathbf{p}_{\theta k},\quad \not{\mathbf{k}}=\mathbf{k}_{\theta k}\ , \tag{6.19}$$

and

$$\eta=\frac{\omega_u+\mathbf{n}\cdot\mathbf{u}}{W_k}\ . \tag{6.20}$$

Thus we see that the representatives of the state vector $|k_{\theta k}k_{Vk}\rangle^{(\pm)}$ in the basis defined by (6.9) can be obtained by simply making the substitutions (6.18) in $\chi^{(\pm)}$.

In order to determine the representative of the angular momentum operator $\mathbf{J}$ in the new basis we can use the relation

$$\langle \mathbf{p}_{\theta k}\mathbf{p}_{Vk}\mathbf{n}|U(R)|\mathbf{k}_{\theta k}\mathbf{k}_{Vk}\rangle^{(\pm)}$$
$$=(2\pi)^3 2W_k\delta^3[\mathbf{p}_k+(\hat p^{\,0}_k-W_k)\mathbf{n}]$$
$$\times(1-\eta)\phi^{(\pm)}(R^{-1}\mathbf{u},R^{-1}\mathbf{n},\not{\mathbf{k}})Z_V^{1/2}\ , \tag{6.21}$$

which is a consequence of (6.10), (6.16), and (6.20). By taking the arbitrary rotation R to be an infinitesimal rotation, we find

$$\langle \mathbf{p}_{\theta k}\mathbf{p}_{Vk}\mathbf{n}|\mathbf{J}|\mathbf{k}_{\theta k}\mathbf{k}_{Vk}\rangle^{(\pm)}$$
$$=(2\pi)^3 2W_k\delta^3[\mathbf{p}_k+(\hat p^{\,0}_k-W_k)\mathbf{n}]$$
$$\times(1-\eta)\mathbf{S}(\mathbf{u},\mathbf{n})\phi^{(\pm)}(\mathbf{u},\mathbf{n},\not{\mathbf{k}})Z_V^{1/2}\ , \tag{6.22}$$

where

$$\mathbf{S}(\mathbf{u},\mathbf{n})=\mathbf{L}(\mathbf{u})+\mathbf{L}(\mathbf{n})\ . \tag{6.23}$$

The $\mathbf{L}$'s are defined as in (5.20). It should be noted that η, as given by (6.20), and $\mathbf{S}$ commute.

We now proceed to derive another expression for the same matrix element. From (6.9), (6.6), (6.1), (6.12) plus the relation

$$U(R)J_iU(R)^{-1}=J_jR_{ji}\ , \tag{6.24}$$

we find

$$\langle \mathbf{p}_{\theta k}\mathbf{p}_{Vk}\mathbf{n}|J_i|\mathbf{k}_{\theta k}\mathbf{k}_{Vk}\rangle^{(\pm)}$$
$$=\langle \mathbf{p}^n_{\theta k}\mathbf{p}^n_{Vk}|J_j|\mathbf{k}^n_{\theta k}\mathbf{k}^n_{Vk}\rangle^{(\pm)}R_{ji}(\mathbf{n})\ . \tag{6.25}$$

By using (5.1), (5.2), (2.11), (6.14), and the fact that $\mathcal{J}$ commutes with $U_0[\Lambda(k)]=U[\Lambda(k)]$, we obtain

$$\langle \mathbf{p}^n_{\theta k}\mathbf{p}^n_{Vk}|\mathbf{J}|\mathbf{k}^n_{\theta k}\mathbf{k}^n_{Vk}\rangle^{(\pm)}=\langle p^n_{\theta}p^n_{V}|\mathcal{J}|k^n_{\theta}k^n_{V}\rangle^{(\pm)}\ . \tag{6.26}$$

Finally from (5.15) with the n's added, (6.15), (5.19), (5.38), (6.18), (6.17), and (6.25) we arrive at

$$\langle \mathbf{p}_{\theta k}\mathbf{p}_{Vk}\mathbf{n}|\mathbf{J}|\mathbf{k}_{\theta k}\mathbf{k}_{Vk}\rangle^{(\pm)}$$
$$=(2\pi)^3 2W_k\delta^3[\mathbf{p}_k+(\hat p^{\,0}_k-W_k)\mathbf{n}]$$
$$\times(1-\eta)\mathbf{L}(\mathbf{u})\phi^{(\pm)}(\mathbf{u},\mathbf{n},\not{\mathbf{k}})Z_V^{1/2}\ . \tag{6.27}$$

By comparing this with (6.22) we conclude that

$$S(\mathbf{u},\mathbf{n})\phi^{(\pm)}(\mathbf{u},\mathbf{n},\not{\mathbf{k}})=\mathbf{L}(\mathbf{u})\phi^{(\pm)}(\mathbf{u},\mathbf{n},\not{\mathbf{k}})\ , \tag{6.28a}$$

when

$$0<\eta<1\ . \tag{6.28b}$$

We recall that (5.36) is a consequence of (5.35), so (6.28b) implies that $\mathbf{n}\cdot\mathbf{u}$ obeys a restriction similar to (5.36).

In order to obtain an integral equation for $\phi^{(\pm)}$ we replace the $\mathbf{u}$'s in (5.37) with $\mathbf{u}^n$'s and then use (6.17) and (6.18). With the help of (5.39) we obtain

$$\phi^{(\pm)}(\mathbf{u},\mathbf{n},\not{\mathbf{k}})=(2\pi)^3 2\omega_u\delta^3(\mathbf{u}-\not{\mathbf{k}})+\frac{1}{D[(k-p_\theta)^2\pm i\epsilon]}$$
$$\times\int\frac{d\mathbf{u}'}{(2\pi)^3 2\omega_{u'}}\theta(1-\eta-\eta')$$
$$\times V_k^{(\pm)}(\mathbf{u},\mathbf{u}')\phi^{(\pm)}(\mathbf{u}',\mathbf{n},\not{\mathbf{k}})\ , \tag{6.29}$$

where

$$V_k^{(\pm)}(\mathbf{u},\mathbf{u}')=g^2[(W_k-\omega_u-\omega_{u'})^2-(\mathbf{u}+\mathbf{u}')^2$$
$$\pm i\epsilon-m_N^2]^{-1}\ . \tag{6.30}$$

In (6.29), η is given by (6.20), as is η' with $\mathbf{u}$ replaced by

$\mathbf{u}'$; and $(k-p_\theta)^2$ is given by (5.32).

The key to making a partial-wave analysis of (6.29) is the observation that (6.28) and (6.23) imply that

$$\mathbf{L}(\mathbf{n})\phi^{(\pm)}(\mathbf{u},\mathbf{n},\not{\mathbf{u}})=0\ , \tag{6.31}$$

when the restriction (6.28b) is satisfied. As a result of this and the presence of $\theta(1-\eta-\eta')$ in the integral term of (6.29), $\phi^{(\pm)}(\mathbf{u}',\mathbf{n},\not{\mathbf{u}})$ is independent of $\mathbf{n}$ and can be replaced by $\xi^{(\pm)}(\mathbf{u}',\not{\mathbf{u}})$ where

$$\xi^{(\pm)}(\mathbf{u},\not{\mathbf{u}})=\int\frac{d\Omega_n}{4\pi}\phi^{(\pm)}(\mathbf{u},\mathbf{n},\not{\mathbf{u}})\ ; \tag{6.32}$$

i.e., we have

$$\begin{aligned}\phi^{(\pm)}(\mathbf{u},\mathbf{n},\not{\mathbf{u}})=(2\pi)^3 2\omega_u\delta^3(\mathbf{u}-\not{\mathbf{u}})+&\frac{1}{D[(k-p_\theta)^2\pm i\epsilon]}\\ &\times\int\frac{d\mathbf{u}'}{(2\pi)^3 2\omega_{u'}}\theta(1-\eta-\eta')V_k^{(\pm)}(\mathbf{u},\mathbf{u}')\\ &\times\xi^{(\pm)}(\mathbf{u}',\not{\mathbf{u}})\ .\end{aligned} \tag{6.33}$$

We now integrate this equation over $d\Omega_n$ to obtain

$$\begin{aligned}\xi^{(\pm)}(\mathbf{u},\not{\mathbf{u}})=(2\pi)^3 2\omega_u\delta^3(\mathbf{u}-\not{\mathbf{u}})+&\frac{1}{D[(k-p_\theta)^2\pm i\epsilon]}\\ &\times\int\frac{d\mathbf{u}'}{(2\pi)^3 2\omega_{u'}}\Sigma_k(\mathbf{u},\mathbf{u}')V_k^{(\pm)}(\mathbf{u},\mathbf{u}')\\ &\times\xi^{(\pm)}(\mathbf{u}',\not{\mathbf{u}})\ ,\end{aligned} \tag{6.34}$$

where

$$\begin{aligned}\Sigma_k(\mathbf{u},\mathbf{u}')&=\int\frac{d\Omega_n}{4\pi}\theta(1-\eta-\eta')\\ &=\tfrac{1}{2}\int_{-1}^{1}dx\ \theta(W_k-\omega_u-\omega_{u'}-|\mathbf{u}+\mathbf{u}'|x)\\ &=\frac{\gamma_+\theta(\gamma_+)-\gamma_-\theta(\gamma_-)}{2|\mathbf{u}+\mathbf{u}'|}\ ,\end{aligned} \tag{6.35a}$$

with

$$\gamma_\pm=W_k-\omega_u-\omega_{u'}\pm|\mathbf{u}+\mathbf{u}'|\ . \tag{6.35b}$$

The last form of Σ_k can be derived by using the argument of θ, rather than x, as the integration variable. Since D, Σ_k, and $V_k^{(\pm)}$ are all rotationally invariant functions of $\mathbf{u}$ and $\mathbf{u}'$, a partial-wave analysis of (6.34) can be carried out in the usual way.

In order to derive an equation for the scattering amplitudes, we replace the $\mathbf{u}$'s in (5.41) with $\mathbf{u}_n$'s and then use (6.17), (6.18), and (6.20) to obtain

$$\begin{aligned}W^{(\pm)}(\mathbf{u},\mathbf{n},\not{\mathbf{u}})&=A^{(\pm)}(\mathbf{u}^n,\not{\mathbf{u}}^n)\\ &=\int\frac{d\mathbf{u}'}{(2\pi)^3 2\omega_{u'}}\theta(1-\eta-\eta')V_k^{(\pm)}(\mathbf{u},\mathbf{u}')\\ &\quad\times\xi^{(\pm)}(\mathbf{u}',\not{\mathbf{u}})\ .\end{aligned} \tag{6.36}$$

We now define

$$\begin{aligned}Z^{(\pm)}(\mathbf{u},\not{\mathbf{u}})&=\int\frac{d\Omega_n}{4\pi}W^{(\pm)}(\mathbf{u},\mathbf{n},\not{\mathbf{u}})\\ &=\int\frac{d\mathbf{u}'}{(2\pi)^3 2\omega_{u'}}\Sigma_k(\mathbf{u},\mathbf{u}')V_k^{(\pm)}(\mathbf{u},\mathbf{u}')\\ &\quad\times\xi^{(\pm)}(\mathbf{u}',\not{\mathbf{u}})\ ,\end{aligned} \tag{6.37}$$

which when combined with (6.34) leads to

$$\begin{aligned}Z^{(\pm)}(\mathbf{u},\not{\mathbf{u}})=&\Sigma_k(\mathbf{u},\not{\mathbf{u}})V_k^{(\pm)}(\mathbf{u},\not{\mathbf{u}})\\ &+\int\frac{d\mathbf{u}'}{(2\pi)^3 2\omega_{u'}}\frac{\Sigma_k(\mathbf{u},\mathbf{u}')V_k^{(\pm)}(\mathbf{u},\mathbf{u}')}{D[(k-p'_\theta)^2\pm i\epsilon]}\\ &\times Z^{(\pm)}(\mathbf{u}',\not{\mathbf{u}})\ .\end{aligned} \tag{6.38}$$

Combining (6.36), (6.34), and (6.37), we obtain

$$\begin{aligned}W^{(\pm)}(\mathbf{u},\mathbf{n},\not{\mathbf{u}})=&\theta(1-\eta-\not\eta)V_k^{(\pm)}(\mathbf{u},\not{\mathbf{u}})\\ &+\int\frac{d\mathbf{u}'}{(2\pi)^3 2\omega_{u'}}\frac{\theta(1-\eta-\eta')V_k^{(\pm)}(\mathbf{u},\mathbf{u}')}{D[(k-p'_\theta)^2\pm i\epsilon]}\\ &\times Z^{(\pm)}(\mathbf{u}',\not{\mathbf{u}})\ ,\end{aligned} \tag{6.39}$$

so the complete amplitude $W^{(\pm)}$ can be obtained from the simpler quantity $Z^{(\pm)}$.

If we substitute

$$Z^{(\pm)}(\mathbf{u},\not{\mathbf{u}})=\sum_{l=0}^{\infty}Z_l^{(\pm)}(u,\not{u})\frac{2l+1}{4\pi}P_l(\hat{\mathbf{u}}\cdot\hat{\not{\mathbf{u}}}) \tag{6.40}$$

in (6.38), we find

$$\begin{aligned}Z_l^{(\pm)}(u,\not{u})=&V_{kl}^{(\pm)}(u,\not{u})\\ &+\int_0^\infty\frac{du'u'^2}{(2\pi)^3 2\omega_{u'}}\frac{V_{kl}^{(\pm)}(u,u')}{D[(k-p'_\theta)^2\pm i\epsilon]}\\ &\times Z_l^{(\pm)}(u',\not{u})\ ,\end{aligned} \tag{6.41}$$

where

$$V_{kl}^{(\pm)}(u,u')=2\pi\int_{-1}^{1}dx\ P_l(x)\Sigma_k(\mathbf{u},\mathbf{u}')V_k^{(\pm)}(\mathbf{u},\mathbf{u}')\ , \tag{6.42a}$$

$$x=\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}'\ . \tag{6.42b}$$

We will now show that the partial-wave amplitudes $Z_l^{(+)}$ determine the partial-wave S-matrix elements. According to (3.27) and (3.9)

$$S(p_\theta,p_V;k_\theta,k_V)={}^{(-)}\langle p_{\theta k}^n p_{Vk}^n|k_{\theta k}^n k_{Vk}^n\rangle^{(+)}\ , \tag{6.43}$$

where $p_{\theta k}^n$, for example, can be obtained by transforming p_θ to the rest frame of $k=k_\theta+k_V$, and then rotating as in (6.12). If we use (4.31) with superscripts n and subscripts k added, as well as (4.33), (4.57), and (5.41), we find

$$\begin{aligned}S(p_\theta,p_V;k_\theta,k_V)=&\langle p_{\theta k}^n p_{Vk}^n|k_{\theta k}^n k_{Vk}^n\rangle\\ &-(2\pi)^4 i\delta^4(p_k^n-k_k^n)A^{(+)}(\mathbf{u}^n,\not{\mathbf{u}}^n)\ .\end{aligned} \tag{6.44}$$

Using the Lorentz invariance of the inner product of the free particle states and the four-dimensional δ function, as well as (6.36), we obtain

$$S(p_\theta,p_V;k_\theta,k_V)=\langle p_\theta p_V|k_\theta k_V\rangle -(2\pi)^4 i\delta^4(p-k)W^{(+)}(\mathbf{u},\mathbf{n},\mathbf{\not{u}}) , \tag{6.45}$$

where $\mathbf{u}$ and $\mathbf{\not{u}}$ are given by (6.19). Since the left-hand side of (6.45) does not depend on $\mathbf{n}$, we can average over the direction of $\mathbf{n}$ and use (6.37) to write

$$S(p_\theta,p_V;k_\theta,k_V)=\langle p_\theta p_V|k_\theta k_V\rangle -(2\pi)^4 i\delta^4(p-k)Z^{(+)}(\mathbf{u},\mathbf{\not{u}}) , \tag{6.46a}$$

with

$$|\mathbf{p}_{\theta k}|=|\mathbf{u}|=|\mathbf{\not{u}}|=|\mathbf{k}_{\theta k}| . \tag{6.46b}$$

Thus we see that the partial-wave amplitudes $Z_l^{(+)}(\not{u},\not{u})$ determine the partial-wave amplitudes of S.

VII. DISCUSSION

In our analysis of the Poincaré-invariant Lee model we have focused on the θV-$\theta\theta N$ sector. As pointed out in Sec. I, this sector has some of the features of the pion-nucleon system. The VV-θNV-$\theta\theta NN$ sector should be analyzed as well, since it has some of the features of the system consisting of two nucleons coupled explicitly to pions. Such an analysis should give insight into constructing Poincaré-invariant models for pion-deuteron reactions, as well as pion production in nucleon-nucleon collisions. The analysis of this sector may also suggest a way of carrying out an angular momentum or partial-wave analysis of the Weinberg equation.[29] This would be of some importance as this is the basic two-particle equation that is obtained when the ladder diagrams of light-front perturbation theory are summed.

An important feature of the intermediate-energy pion-nucleon system has no analog in the θV sector of the Lee model; i.e., there are no absorption channels analogous to $\pi+N\rightarrow B\rightarrow\pi+N$ where B is an N or Δ resonance. It is not difficult to remedy this shortcoming. Some years ago Bronzan[30] introduced an extension of the Lee model which describes the interaction of four particles V, N, W, and θ through the virtual processes $V\rightleftarrows N+\theta$ and $W\rightleftarrows V+\theta$. In this model θV scattering contains the absorption channel $V+\theta\rightarrow W\rightarrow V+\theta$. The analysis of this extended Lee model is not much more difficult than that of the original Lee model.[9,30,31]

In constructing a relativistic Poincaré-invariant model of the pion-nucleon system along the lines of the Lee model it is, of course, necessary to take the spin of the nucleons into account. Since there exists a version of the Lee model[32] in which the V and N particles are spin-$\frac{1}{2}$ particles, this should be possible. It should be noted that the coupling of intrinsic spin and orbital angular momentum in light-front models entails the use of Melosh transformations.[14,21,33]

Of course with any of the above possible extensions, it will be necessary to verify that they are Poincaré invariant and that the S-matrix elements have the correct behavior under Lorentz transformations. It should be possible to develop such proofs along the lines of those given in Secs. II and III.

The fact that the S-matrix elements of the model presented here have the correct Lorentz transformation properties is of significance for the general program of obtaining few particle equations by summing subsets of light-front perturbation theory diagrams.[34] An example of this is given in Ref. 35 where a set of three-particle equations was obtained by summing diagrams of an invariant version of light-front perturbation theory for a model field theory. This local field theory describes the interaction of a charged scalar particle ψ with a neutral scalar particle ϕ according to the virtual process $\psi\rightleftarrows\psi+\phi$. The three-particle equations were obtained by summing all diagrams with $|\psi\rangle$, $|\phi\psi\rangle$, and $|\phi\phi\psi\rangle$ intermediate states. It turns out that the integral equation that sums the one-ψ irreducible diagrams is identical to (4.59), i.e., the equation for the θV scattering amplitude. This suggests that it is possible to truncate light-front perturbation theory in such a way as to obtain S-matrix elements which behave properly under Lorentz transformations. It would be highly desirable to develop general procedures or guidelines for obtaining such truncations. It is quite possible that projection operator techniques[36] will be of some use in this connection.

As pointed out in Sec. I, and illustrated here in Secs. V and VI, angular momentum and spin operators in light-front models are somewhat peculiar in that they can be interacting. Fortunately, as we have seen in Sec. VI, the author's new picture for light-front dynamics[22] makes it possible to cope with this situation. This is very encouraging as this feature of light-front dynamics has been a stumbling block in many applications of light-front dynamics. It is also encouraging that the new picture made it possible to reduce the problem of finding the θV scattering amplitudes to solving one-variable integral equations.

All of the results obtained here indicate that light-front dynamics, in conjunction with the new picture, provides a practical framework for developing few-particle models that are consistent with the requirements of special relativity, and moreover allow for changes in particle numbers.

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APPENDIX: LIGHT-FRONT KINEMATICS AND WAVE FUNCTIONS

Here we summarize some basic general features of the light-front description of a system of particles. We assume that p_i, the four-momentum of the ith particle, is on the mass shell; i.e.,

$$p_i^2=m_i^2 , \tag{A1}$$

where m_i is the mass of the particle. The light-front components of p_i are given by (2.13) with $p \to p_i$ and $m \to m_i$. The total momentum of the set of particles is given by

$$p=\sum_i p_i \ , \tag{A2}$$

while the generalized Weinberg variables are defined by

$$\eta_i = p_i / p^0 \ , \tag{A3}$$

$$\boldsymbol{\rho}_i = \mathbf{p}_{i\perp} - \eta_i \mathbf{p}_\perp \ , \tag{A4}$$

and satisfy the constraints

$$\sum_i \eta_i = 1 \ , \tag{A5}$$

$$\sum_i \boldsymbol{\rho}_i = 0 \ . \tag{A6}$$

As a result of (2.4), (2.5), (A3), and (A4), we have

$$\boldsymbol{\rho}_i^2 = -(p_i - \eta_i p)^2 \ , \tag{A7}$$

which when combined with (A1), (A2), and (A5) leads to

$$p^2 = W_p^2 = W^2\{\eta_i, \boldsymbol{\rho}_i\} = \sum_i \frac{\boldsymbol{\rho}_i^2 + m_i^2}{\eta_i} \ . \tag{A8}$$

We assume that the inner products of our two-particle states are given by

$$\begin{aligned}\langle p_1 p_2 | k_1 k_2 \rangle &= (2\pi)^3 2p_1^0 \delta^3(\bar{p}_1 - \bar{k}_1)(2\pi)^3 2p_2^0 \\ &\quad \times \delta^3(\bar{p}_2 - \bar{k}_2) \\ &= (2\pi)^3 2p^0 \delta^3(\bar{p} - \bar{k})(2\pi)^3 2\eta(1-\eta) \\ &\quad \times \delta(\eta - \eta\!\!\!/)\delta^2(\boldsymbol{\rho} - \boldsymbol{\rho}\!\!\!/) \ ,\end{aligned} \tag{A9}$$

where $\eta\!\!\!/$ and $\boldsymbol{\rho}\!\!\!/$ are defined as in (A3) and (A4) with p's replaced by k's. Using the definition (2.18), it follows that

$$dp_1 dp_2 = \frac{d\bar{p}\ \theta(p^0)}{(2\pi)^3 2p^0} \frac{d\eta\, d\boldsymbol{\rho}\, \theta(\eta)\theta(1-\eta)}{(2\pi)^3 2\eta(1-\eta)} \ . \tag{A10}$$

We let $|\bar{p}q\rangle$ be an eigenstate of the trimomentum $\bar{P}$, i.e.,

$$\bar{P}|\bar{p}q\rangle = \bar{p}|\bar{p}q\rangle \ , \tag{A11}$$

where q stands for the "internal" quantum numbers of the state. Leutwyler and Stern[14] have shown that it is always possible to construct such a state so that

$$\langle \bar{p}q | B_r = B_r(\bar{p}) \langle \bar{p}q | \ , \tag{A12}$$

$$\langle \bar{p}q | K_3 = K_3(\bar{p}) \langle \bar{p}q | \ , \tag{A13}$$

where $B_r(\bar{p})$ and $K_3(\bar{p})$ are given by (2.14a) and (2.14b). It is straightforward to show that the inner product of any two states that satisfy (A11)–(A13) is given by

$$\langle \bar{p}q | \bar{k}r \rangle = (2\pi)^3 2k^0 \delta^3(\bar{p} - \bar{k}) F(q,r) \ ; \tag{A14}$$

i.e., the inner product can be factored into an external part and an internal part.

We frequently encounter in light-front dynamics a propagator or denominator $(k^3 \pm i\epsilon - p^3)^{-1}$ where $\bar{p} = \bar{k}$. Here $(k^\mu) = (\bar{k}, k^3)$ and $(p^\mu) = (\bar{p}, p^3)$ are the four-momenta of the system and some intermediate state, respectively. It is worth noting that such denominators can be rewritten by using the identity

$$2k^0(k^3 - p^3) = k^2 - p^2 \ . \tag{A15}$$

The k^2 and p^2 can be expressed in terms of the generalized Weinberg variables by using (A8). It is also worth noting that

$$p^3 = \frac{\mathbf{p}_\perp^2 + p^2}{2p^0} \ . \tag{A16}$$

[1]T. D. Lee, Phys. Rev. **95**, 1329 (1954).

[2]R. D. Amado, Phys. Rev. **122**, 696 (1961).

[3]G. Källén and W. Pauli, K. Dan. Vidensk. Selsk. Mat.-Fys. Medd. **30**, 7 (1955); K. W. Ford, Phys. Rev. **105**, 320 (1957); W. Heisenberg, Nucl. Phys. **4**, 532 (1957); B. Ferretti, Nuovo Cimento **12**, 393 (1959).

[4]M. M. Lévy, Nuovo Cimento **13**, 115 (1959); **14**, 612 (1959).

[5]R. D. Amado, Phys. Rev. **132**, 485 (1963).

[6]R. D. Amado, Phys. Rev. **14**, 902 (1966); R. Aaron, R. D. Amado, and Y. Y. Yam, *ibid.* **140**, B1291 (1965); R. D. Amado, Annu. Rev. Nucl. Sci. **19**, 635 (1969).

[7]Ronald Aaron, Phys. Rev. **151**, 1293 (1966).

[8]M. G. Fuda, Phys. Rev. C **25**, 1972 (1982).

[9]M. G. Fuda, Phys. Rev. C **26**, 204 (1982); **29**, 1222 (1984).

[10]M. G. Fuda, Phys. Rev. C **30**, 666 (1984).

[11]G. A. Miller, A. W. Thomas, and S. Théberge, Phys. Lett. **91B**, 192 (1980); S. Théberge, A. W. Thomas, and G. A. Miller, Phys. Rev. D **22**, 2838 (1980).

[12]C. Schmit and J.-P. Maillet, Nucl. Phys. **A312**, 236 (1978); H. R. Petry and M. Sawicki, Z. Naturforsch. **33a**, 1 (1978); M. Sawicki and D. Schütte, *ibid.* **36a**, 1261 (1981); M. Sawicki, Phys. Rev. C **27**, 1415 (1983).

[13]P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).

[14]H. Leutwyler and J. Stern, Ann. Phys. (N.Y.) **112**, 94 (1978).

[15]F. Coester and W. N. Polyzou, Phys. Rev. D **26**, 1348 (1982).

[16]B. de Dormale, J. Math. Phys. **20**, 1229 (1979).

[17]M. Betz and T.-S. H. Lee, Phys. Rev. C **23**, 375 (1981); M. Betz and F. Coester, *ibid.* **21**, 2505 (1980).

[18]S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row, New York, 1961).

[19]M. V. Terent'ev, Yad. Fiz. **24**, 207 (1976) [Sov. J. Nucl. Phys. **24**, 106 (1976)]; V. B. Berestetskii and M. V. Terent'ev, *ibid.* **24**, 1044 (1976) [**24**, 547 (1976)]; **25**, 653 (1977) [**25**, 347 (1977)]; L. A. Kondratyuk and M. V. Terent'ev, **31**, 1087 (1980) [**31**, 561 (1980)]; B. L. G. Bakker, L. A. Kondratyuk, and M. V. Terent'ev, Nucl. Phys. **B158**, 497 (1979).

[20]M. G. Fuda, Phys. Rev. C **36**, 1489 (1987).

[21]P. L. Chung, F. Coester, B. D. Keister, and W. N. Polyzou, Phys. Rev. C **37**, 2000 (1988).

[22]M. G. Fuda, Ann. Phys. (N.Y.) **197**(2), 265 (1990).

[23]J. L. Basevant and R. L. Omnes, Phys. Rev. Lett. **17**, 775 (1966).
[24]H. Garcilazo and L. Mathelitsch, Phys. Rev. C **28**, 1272 (1983); L. Mathelitsch and H. Garcilazo, *ibid.* **32**, 1635 (1985).
[25]Y. S. Kim and Marilyn E. Noz, *Theory and Applications of the Poincaré Group* (Reidel, Boston, 1986).
[26]M. L. Goldberger and S. B. Treiman, Phys. Rev. **113**, 1663 (1959).
[27]M. Stephen Maxon and Richard B. Curtis, Phys. Rev. **137**, B996 (1965); M. Stephen Maxon, *ibid.* **149**, 1273 (1966); Leonard M. Scarfone, J. Math. Phys. **7**, 159 (1966).
[28]S. G. Gasiorowicz, *Elementary Particle Physics* (Wiley, New York, 1966).
[29]Steven Weinberg, Phys. Rev. **150**, 1313 (1966).
[30]J. B. Bronzan, Phys. Rev. **139**, B751 (1965).
[31]F. S. Chen-Cheung, Phys. Rev. **152**, 1408 (1966); J. B. Bronzan, *ibid.* **172**, 1429 (1968).
[32]H. Chew, Phys. Rev. **132**, 2756 (1963).
[33]H. J. Melosh, Phys. Rev. D **9**, 1095 (1974).
[34]J. M. Namyslowski, Prog. Part. Nucl. Phys. **14**, 49 (1984).
[35]Michael F. Fuda, Phys. Rev. C **25**, 226 (1987).
[36]M. G. Fuda, Phys. Rev. C **31**, 1365 (1985); **32**, 2024 (1985).