Derivative expansions for affinely quantized field theories

Carl M. Bender
Department of Physics, Washington University, St. Louis, Missouri 63130

R. J. Rivers and C. C. Wong
Physics Department, Imperial College, London SW7 2BZ, United Kingdom
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We examine the existence of an affinely (noncanonically) quantized free scalar theory via an expansion in powers of derivatives of the field. We show that with the simplest choice of measure affine quantization does not provide a useful basis for a nontrivial, well-defined theory.

I. INTRODUCTION

It is now many years since Klauder[1] argued that gravity be quantized noncanonically so as to preserve the signature of the three-metric from quantum fluctuations. The idea was to replace the *canonical* relations (on a given three-surface)

$$[g_{ij}(\mathbf{x}),\pi^{kl}(\mathbf{y})]=\frac{i}{2}(\delta_i^k\delta_j^l+\delta_i^l\delta_j^k)\delta(\mathbf{x}-\mathbf{y}) \tag{1.1}$$

that formally permit unitarily implementable translations of the metric $g_{ij}(\mathbf{x})$ by the *affine* relations

$$[g_{ij}(\mathbf{x}),\pi_k^l(\mathbf{y})]=\frac{i}{2}(\delta_i^l g_{jk}+\delta_j^l g_{ik})\delta(\mathbf{x}-\mathbf{y}) \tag{1.2}$$

that permit unitary implementation of scaling of the metric. [$\pi_k^l=(g^{li}\pi_{ik}+\pi_{km}g^{ml})/2$ in terms of the inverse metric g^{ij}.]

Such an approach, with its avoidance of wormholes, is conceptually much simpler than canonical quantization (e.g., the inverse metric g^{ij} is guaranteed to be defined) but it remains equally insolvable. This is despite much work from Isham and collaborators[2] and, via strong gravity, Pilati and co-authors[3] (although see de Alfaro, Fubini, and Furlan[4] for a parallel approach). In fact, gravity is so complicated that the features of noncanonical behavior cannot be isolated easily. It is not surprising that, despite the role of gravity in motivating a noncanonical approach, such understanding as we have of noncanonical quantization comes from much simpler theories, and it is to them that we shall turn.

The increased complexity of noncanonical quantization can be seen most easily in the case of a single real scalar field $\phi(x)$, for which the affine relations that are the counterpart of (1.2) take the form

$$[\phi(\mathbf{x},t),K(\mathbf{y},t)]=i\phi(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{y}) , \tag{1.3}$$

where $K=(\phi\pi+\pi\phi)/2$ is the generator of scaling transformations of ϕ. The relations (1.3) are *not* necessarily identical[5] to the canonical relations because K involves an operator product, and is defined through its renormalization.

Klauder has done much work to develop a program for the noncanonical quantization of a scalar field.[6] In particular, he has observed that the noncanonical *Euclidean* scalar field, in the presence of a source $j(x)$, has a generating functional that can be written formally as

$$Z[j]=\int \bar{D}\phi \exp\left[-S[\phi]+\int j\phi\right] . \tag{1.4}$$

For simplicity, the Euclidean action $S[\phi]$ in d dimensions is taken to be

$$S[\phi]=\int d^dx\left[\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}m_0^2\phi^2+\frac{\lambda_0}{4}\phi^4\right] . \tag{1.5}$$

The formal insertion of (1.3) in the analog of the Dyson-Schwinger equations for $Z[j]$ shows that the measure $\bar{D}\phi$ does *not* display the translation invariance $\bar{D}(\phi+\Lambda)=\bar{D}\phi$ appropriate to a canonically quantized theory but, rather, the *scale covariance*[7]

$$\bar{D}(\Lambda\phi)=F(\Lambda)\bar{D}\phi,\quad \Lambda(x)>0\quad \forall\ x \tag{1.6}$$

relevant for the unitary implementation of scaling.

This permits a formal realization of $\bar{D}\phi$ in terms of the canonical translation-invariant "measure" $D\phi$ [$D(\phi+\Lambda)=D\phi$] as[7]

$$\bar{D}\phi=\frac{D\phi}{\prod_x|\phi(x)|^B} \tag{1.7}$$

for some $B\le 1$. Note that we have chosen units so that $\hbar=1$. Had we not, the effect of changing the "measure" from $D\phi$ to $\bar{D}\phi$ is, nominally, to introduce a term in the action:

$$S[\phi]\to S[\phi]+B\hbar\delta(0)\int dx\,\ln|\phi(x)| . \tag{1.8}$$

Classically ($\hbar\to 0$) S is unchanged and B plays the role of a "hidden" quantum degree of freedom.

In many-body theory we have learned to treat unrenormalized path integrals such as (1.4) with a certain amount of scepticism. We know that, despite superficial differences, many measures give rise to the same theory after renormalization (universality). [Conversely, measures of the form (1.7) are not the only way to implement the affine relations.] Thus, changing B will not necessarily give rise to a different quantum theory. However, the possibility exists that, if B is fined-tuned correctly, we may find ourselves in a different universality class from the canonical theory ($B=0$). A concrete example was given by one of us (R.R.) in an O(N) generalization of (1.4) (Ref. 8). In the large-N limit it happens that, for $B=\frac{5}{9}$ in $d=5$ dimensions, the theory is *different* from the canonical theory. [A more complicated choice of B gives a noncanonical theory in $d=4$ dimensions also (Ref. 8).]

Apart from special cases such as this, noncanonical theories do not permit an analytic solution. This is unfortunate, since we know so little about them that we have a very poor understanding of their utility, or even their existence. For this reason alone there is merit in almost any work that takes our knowledge further since, *a priori*, noncanonical theories have much promise, gravitation apart. In particular, noncanonical quantization has been seen as an alternative that enables us to give sense to theories that are perturbatively nonrenormalizable when quantized canonically.[9] For the $\lambda\phi^4$ theory of (1.5) this corresponds to taking the space-time dimension $d>4$ (and even $d=4$). As a result we might expect to evade the triviality that such theories are known to possess.

Because of the non-Gaussian starting point provided by the free theory with measure (1.6) it has been necessary to fall back upon numerical methods. Given the interest in the triviality of $\lambda(\phi^4)_4$ theory essentially all calculations hitherto have been performed in $d=4$ dimensions. At the most prosaic we can always attempt Monte Carlo calculations for lattice versions of the theory, but the inclusion of the additional parameter B makes this very time consuming,[10] given the difficulty of obtaining reliable results for $B=0$ (Ref. 11). As an alternative to pure Monte Carlo calculations it has been found helpful[12] to combine them with high-temperature series for the analogue continuous-spin ferromagnet (for which all diagrams have been calculated to eleventh order in $d\le 4$ dimensions.[13]) However, even then, large values of B are poorly understood.

A case of particular interest is given by $B\to 1^-$. Not only does this make the (now *scale invariant*) $\bar{D}\phi$ of (1.6) maximally singular[5] but it is the choice imposed by the important independent-valued model[6,14] (IVM), in which all field derivatives are omitted in $S[\phi]$ in the first instance. Normally such a starting point would be a free theory (in the continuum limit) but, for $B\to 1^-$, this is not necessarily the case.[6] The significance of $B\to 1^-$ is that the IVM permits a description of the operator-valued field $\phi(x)$ as a bilinear in a complex extended field $B(x,\lambda)$ (real c number λ), defined on a "translated" pseudo-Fock space, as

$$\phi(x)=\int d\lambda\, B^\dagger(x,\lambda)\lambda B(x,\lambda)\ . \tag{1.9}$$

This is almost a prototype for a similar-seeming bilinear construction for an ultralocal model[15] (ULM), in which time derivatives are restored. Although there are fundamental differences between the IVM and ULM they share a common problem: namely, that the classical derivative-deficient theory in which there is no variation in the fields (spatially, at least) is replaced by a quantum theory in which the fields at adjacent points are completely uncorrelated. It would not be surprising if this antithetical picture forced the field theory into triviality or nonexistence. A failure to be able to use ϕ of (1.9) as a base for a full noncanonical theory reenforces our unease for strong gravity,[3] the most sophisticated ULM to date.

We shall return to case $B\to 1^-$ in a subsequent publication. In the meantime we shall use this most pathological case to motivate our tactics. In particular, as $B\to 1^-$ the nontriviality of the IVM requires[6,16] that, on a hypercubic lattice of size a in d dimensions,

$$B=1-2ba^d m^d \tag{1.10}$$

(m,b arbitrary fixed parameters) as $a\to 0$ in the continuum limit. Unfortunately, the high-temperature series is of little use for such behavior. The resulting situation, in which the single-site distribution depends explicitly on a through (1.8) (and hence on the correlation length that it defines) is far too complicated for conventional many-body tactics to handle. It makes more sense to attempt to evaluate the $B\to 1^-$ limit case by the expansion in powers of derivatives (corresponding to a strong-coupling expansion for a canonical theory) pioneered by one of us (C.M.B.), in a series of papers some years back.[17–19] Not only does this enable us to take the $a\to 0$ limit directly, but it provides a natural way to incorporate the derivatives omitted in the IVM.

Anticipating the usefulness of derivative expansions for $B\to 1^-$, in this work we shall use the same tactics to look at the more general question of the role of noncanonical quantization. In particular, we are interested in (perturbatively nonrenormalizable) scalar field theories in large space-time dimensions for which noncanonical quantization is favored. For simplicity, we work at *fixed* $B\neq 0$, and contrast our results with the canonical case ($B=0$). Our results, which are very dimension specific, complement the $d=4$ work of Refs. 10 and 12.

II. THE PSEUDOFREE THEORY

The expansion of the noncanonical theory in powers of field derivatives corresponds to rewriting $Z[j]$ of (1.4),

$$Z[j]=\int \bar{D}\phi \exp\left[-\int d^dx\left[\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}m_0^2\phi^2+\frac{\lambda_0}{4}\phi^4 -j\phi\right]\right], \tag{2.1}$$

as a power series in c:

$$Z[j]=\exp\left[\frac{c}{2}\int\frac{\delta}{\delta j(x)}D^{-1}(x,y)\frac{\delta}{\delta j(y)}\right] \times\int\bar{D}\phi\exp\left[-\int\left[\tfrac{1}{2}m_0^2\phi^2+\frac{\lambda_0}{4}\phi^4-j\phi\right]\right] \tag{2.2}$$

for $c=1$. In (2.2) $D^{-1}(x,y)=\nabla^2\delta(x-y)$ is the *inverse* scalar field propagator. The coefficient of c^p consists of all p-line diagrams in this inverse propagator.

Because of the extremely singular nature of an inverse propagator expansion attempts to impose analytic renormalization have not been successful.[20] We find it convenient to evaluate Z on a (hypercubic) lattice in d dimensions of size a and to recover the continuum limit by taking $a \to 0$. The lattice inverse propagator terminates at vertices given by the derivatives of[17,18]

$$F(y)=\int_0^\infty \frac{dx}{x^B}\exp[-a^d(m_0^2x^2/2+\lambda_0x^4/4-yx)] , \quad (2.3)$$

where we have used the form (1.6) for $\overline{D}\phi$. If

$$A_{2n}=\int_0^\infty \frac{dx}{x^B}(xa^d)^{2n}\exp[-a^d(m_0^2x^2/2+\lambda_0x^4/4)] \quad (2.4)$$

the $2n$-leg vertices V_{2n} are given by

$$\sum_{n=0}^{\infty} \frac{V_{2n}}{(2n)!}y^{2n}=\ln\left[\sum_{n=0}^{\infty}\frac{A_{2n}}{(2n)!}y^{2n}\right] . \quad (2.5)$$

Observe that V_{2n} depends on the three parameters m_0,λ_0,B in addition to a. It is only practical to calculate diagrams with $p \le 6$ interval lines, and such short series make it very difficult to extract useful information for general values of the parameters. To simplify the situation we restrict ourselves to the case $\lambda_0 \to 0$. That is, form (2.1) we are quantizing a *free* field in a noncanonical way. The resulting *pseudofree* theory is *not* free, formally, although we have our usual caveats concerning universality.

The role of pseudofree theories has been discussed in great detail by Klauder.[9] In particular, the pseudofree scalar theory can be interpreted as the correct starting point for a perturbation expansion in λ for a perturbatively nonrenormalizable theory (i.e., $d>4$ and even $d=4$ dimensions). Using singular potentials as a simile for nonrenormalizability,[14] it is arguable that canonical nonrenormalizability is a signal that the incorrect $\lambda \to 0$ theory is being used.

With $\lambda_0=0$ the A_{2n}'s of (2.4) become Γ functions, and the V_{2n}'s finite series in powers of B. With the definition (1.8) in mind we replace B by f, where

$$B=1-2f, \quad 0<f\le\infty . \quad (2.6)$$

The first several V_{2n} are then compared to be

$$\begin{aligned} &V_2=2fm_0^{-2} ,\\ &V_4=4f(1-2f)a^dm_0^{-4} ,\\ &V_6=16f(1-2f)(1-4f)a^{2d}m_0^{-6} , \qquad (2.7)\\ &V_8=32f(1-2f)(3-34f+68f^2)a^{3d}m_0^{-8} ,\\ &V_{10}=256f(1-2f)(1-4f)(124f^2-62f+3)a^{4d}m_0^{-10} . \end{aligned}$$

We observe that V_{2n} $(n>1)$ is necessarily zero for the canonical choice $f=\frac{1}{2}$, but it is otherwise nonzero, in general. That is, prior to renormalization, the change in measure generates self-interactions, as we anticipated. To determine whether these persist in the renormalized continuum limit we need to construct the diagram of the c expansion (2.2), and extract their continuum limit.

III. ALGEBRAIC RULES FOR CALCULATION OF d-DIMENSIONAL DIAGRAMS

Although the computational rules for a "strong-coupling" lattice expansion have been given by one of us (C.M.B.) elsewhere,[17,18] it is useful to recapitulate the basic elements of the constructional techniques.

On the lattice $D^{-1}(x,y)$ can be represented as a generalized "matrix." For example, on a one-dimensional lattice with spacing a,

$$D_{ij}^{-1}=a^{-3}(\delta_{i,j+1}-2\delta_{i,j}+\delta_{i+1,j}) , \quad (3.1)$$

which may be represented as the row matrix

$$D^{-1}=a^{-3}(1 \quad -2 \quad 1) . \quad (3.2)$$

On a two-dimensional lattice D^{-1} takes the form

$$D^{-1}=a^{-4}\begin{vmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{vmatrix} . \quad (3.3)$$

For $d>2$ dimensions we need a more general notation since such arrays are difficult to visualize. We adopt the following vector notation: (**0**) represents the position at the center of the d-dimensional matrix; (**1**) represents the positions one unit away from the center of the matrix in all directions (there are $2d$ such positions); (**n**) represents the $2d$ positions n units away from the matrix center along the major axes; (**11**) represents the $2d(d-1)$ positions, one unit from each of any two different axes; (**12**) represents the $4d(d-1)$ positions, the unit from any one axis and two units from a different axis; (**111**) represents the $4d(d-1)(d-2)/3$ positions one unit away from each of any three distinct axes. Further extensions of the notation are transparent.

As an example, in two dimensions the "matrix"

$$\begin{vmatrix} & & & 3 & & & \\ & & 5 & 2 & 5 & & \\ & 5 & 4 & 1 & 4 & 5 & \\ 3 & 2 & 1 & 9 & 1 & 2 & 3 \\ & 5 & 4 & 1 & 4 & 5 & \\ & & 5 & 2 & 5 & & \\ & & & 3 & & & \end{vmatrix} \quad (3.4)$$

is represented by $9(\mathbf{0})+1(\mathbf{1})+2(\mathbf{2})+3(\mathbf{3})+4(\mathbf{11})+5(\mathbf{12})$. In d dimensions, the vector representation of D^{-1} is

$$D^{-1}=\frac{1}{a^{d+2}}[(\mathbf{1})-2d(\mathbf{0})] . \quad (3.5)$$

The graphs that comprise the c expansion of (2.2) contain combinations of inverse propagators in two different ways that we term "series" or "parallel." The simplest parallel diagram is the loop of Fig. 1(a), represented by

$$D^{-1}\cdot D^{-1}=a^{-d-2}[(\mathbf{1})-2d(\mathbf{0})]\cdot a^{-d-2}[(\mathbf{1})-2d(\mathbf{0})] . \quad (3.6)$$

The vectors $(\mathbf{0}),(\mathbf{1}),\ldots$ satisfy the simple projective dot product relations

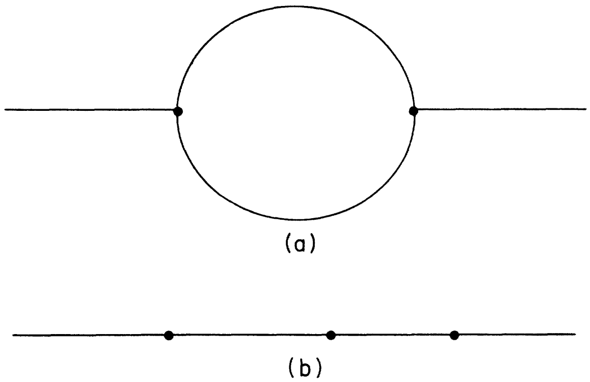

FIG. 1. (a) Propagator graphs with repeated connections between two points. (b) Propagator graphs with series structures.

$$(0)\cdot(0)=(0)\,,\quad (1)\cdot(1)=(1)\,,\quad (2)\cdot(2)=(2)\,,\quad (11)\cdot(11)=(11)\,,\quad (0)\cdot(1)=(1)\cdot(11)=(2)\cdot(12)=0\,, \tag{3.7}$$

and so on, to give

$$D^{-1}\cdot D^{-1}=a^{-2d-4}[(1)+4d^2(0)]\,. \tag{3.8}$$

The simplest series diagram, given by Fig. 1(b), represents the convolution

$$D^{-1}\times D^{-1}=a^{-2d-4}[(1)-2d(0)]\times[(1)-2d(0)]\,. \tag{3.9}$$

The rules for evaluating "convolutions" are

$$\begin{aligned}&(0)\times(\text{any})=a^2(\text{any})\,,\\&(1)\times(1)=a^d[2d(0)+(2)+2(11)]\,,\\&(1)\times(2)=a^d[(1)+(3)+(12)]\,,\\&(2)\times(2)=a^d[2d(0)+2(22)+(4)]\,,\\&(1)\times(11)=a^d[2(d-1)(1)+(12)+3(111)]\,,\end{aligned} \tag{3.10}$$

and so on. The multiplication symbol implies that an integration is being performed, so it is associated with a factor of a^d. Using (3.10) gives

$$D^{-1}\times D^{-1}=a^{-d-4}[(2d+4d^2)(0)-4d(1)+(2)+2(11)]\,. \tag{3.11}$$

In practice we need the momentum-space Fourier transform of the lattice diagrams, just as in continuum theory. The Fourier transforms of $(0),(1),\ldots$ are

$$\begin{aligned}&F(0)=a^d\,,\\&F(1)=2da^d-p_E^2a^{2+d}\,,\\&F(2)=2da^d-4p_E^2a^{2+d}+p_E^4a^{4+d}\,,\\&F(11)=(2d^2-2d)a^d-p_E^2(2d-2)a^{2+d}\,,\end{aligned} \tag{3.12}$$

and so on, where p_E^2 is the Euclidean momentum squared. The Fourier transform of $D^{-1}=\nabla^2\delta(x-y)$ is simply $-p_E^2$.

IV. THE EFFECTIVE POTENTIAL

The effective potential is constructed in the usual way. The generating functional for the connected Green's functions W_{2n} is $W[j]=-\ln Z[j]$. The effective action $\Gamma[\phi]$ that generates the one-particle-irreducible (1PI) Green's functions Γ_{2n} is the Laplace transform of W with respect to the semiclassical field $\phi=-\delta W/\delta j$:

$$\Gamma[\phi]=W[j]-\int\phi(x)j(x)d^dx\,. \tag{4.1}$$

In turn, the effective potential $V(\phi)$ (the energy density of the vacuum when $\langle\hat{\phi}\rangle$ is held to the value ϕ) is defined from $\Gamma[\phi]$ for *constant* ϕ by

$$\Gamma[\phi]=(2\pi)^d\delta(0)V(\phi)\,. \tag{4.2}$$

If Λ_{2n} is the sum of all 1PI diagrams with $2n$ external legs, the relationship of Λ_{2n} to Γ_{2n} is obtained in a straightforward but tedious way as

$$\begin{aligned}&\Gamma_2=-D^{-1}+\Lambda_2^{-1}\,,\\&\Gamma_4=-\Gamma_4(\Lambda_2^{-1})^4\,,\\&\Gamma_6=-\Gamma_6(\Lambda_2^{-1})^6+10(\Lambda_2^{-1})^6\Lambda_4\Lambda_2^{-1}\Lambda_4\,,\\&\Gamma_8=-\Gamma_8(\Lambda_2^{-1})^8+56(\Lambda_2^{-1})^8\Lambda_6\Lambda_2^{-1}\Lambda_4\\&\qquad-280(\Lambda_2^{-1})^8\Lambda_4\Lambda_2^{-1}\Lambda_4\Lambda_2^{-1}\Lambda_4\,,\end{aligned} \tag{4.3}$$

and so on, where integration of the (implicit) repeated argument has been performed. Details are given in Ref. 18.

The final step is to observe that $V(\phi)$ has the Taylor expansion

$$V(\phi)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}\tilde{\Gamma}_{2n}(0,0,0,\ldots,0)\phi^{2n}\,, \tag{4.4}$$

where $\tilde{\Gamma}_{2n}(0,0,\ldots,0)$ is the $2n$-leg 1PI Green's function for zero external momenta. The relationship of the $\tilde{\Gamma}_{2n}(0,0,\ldots,0)$ to the Fourier transforms $\tilde{\Lambda}_{2n}$ follows directly from (4.3). To evaluate a graph having $2n$ legs at zero momentum corresponds to treating the graph as a vacuum graph, with no external legs.

As an example, we consider the reduction of propagator graphs to vacuum graphs. for each translation-invariant propagator graph $p(x,y)$ in coordinate space, the corresponding legless vacuum graph gives a contribution $\int dx\,p(x,y)$. The equivalent operation on the lattice is obtained by summing over all the lattice points using the formulas

$$\begin{aligned}&a^d\sum(0)=a^d\,,\\&a^d\sum(\mathbf{n})=2da^d\quad(n\geq 1)\,,\\&a^d\sum(11)=2d(d-1)a^d\,,\end{aligned} \tag{4.5}$$

and so on. For example, the vacuum graph corresponding to Fig. 1(a) is

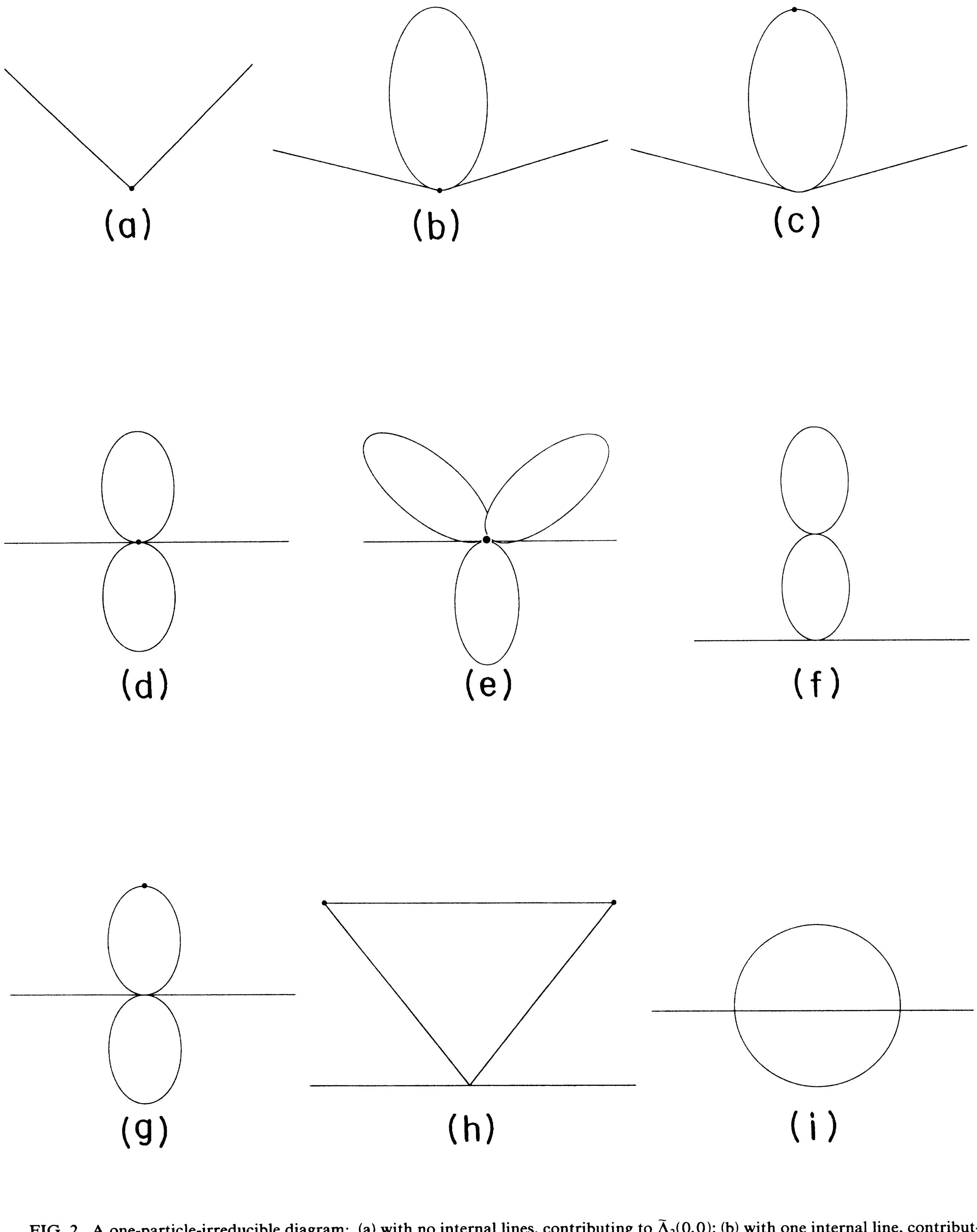

FIG. 2. A one-particle-irreducible diagram: (a) with no internal lines, contributing to $\tilde{\Lambda}_2(0,0)$; (b) with one internal line, contributing to $\tilde{\Lambda}_2(0,0)$; (c) with two internal lines, contributing to $\tilde{\Lambda}_2(0,0)$; (d) with two internal lines, contributing to $\tilde{\Lambda}_2(0,0)$; (e) with three internal lines, contributing to $\tilde{\Lambda}_2(0,0)$; (f) with three internal lines, contributing to $\tilde{\Lambda}_2(0,0)$; (g) with three internal lines, contributing to $\tilde{\Lambda}_2(0,0)$; (h) with three internal lines, contributing to $\tilde{\Lambda}_2(0,0)$; (i) with three internal lines, contributing to $\tilde{\Lambda}_2(0,0)$.

$$a^d \sum (D^{-1} \cdot D^{-1}) = a^{-d-4} \sum [(1) + 4d^2(0)]$$
$$= a^{-d-4}(4d^2 + 2d) \ . \tag{4.6}$$

Finally, we observe that the vacuum graphs which are 1-vertex reducible can be written as the product of 1-vertex-irreducible subgraphs (OVIS's). All vacuum OVIS's having from one to six lines are shown in Fig. 7 of Ref. 18.

As an example of this procedure we list all the 1PI graphs contributing to Λ_2 up to three internal lines in Figs. 2(a)–2(i). We see from (4.3) that Λ_2 is an essential ingredient in the construction of $\tilde{\Gamma}_{2n}$. Consider the particular graph in Fig. 2(g). The graph decomposes into its OVIS components

 and 

evaluated as $(2d + 4d^2)a^{-d-4}$ and $-2da^{-d-2}$, respectively. The vertices involved are $V_6 V_2$. The contribution of this graph to Λ_2 is then

$$2 \times \tfrac{1}{2} V_6 V_2 (2d + 4d^2) a^{-d-4}(-2da^{-d-2}) \ , \tag{4.7}$$

where 2 and $\frac{1}{2}$ are the external symmetry number (permutation of external lines) and the internal symmetry number (permutation of internal lines), respectively.

Similarly, for the graph in Fig. 2(i) the inverse propagators give the factor

$$(D^{-1} \cdot D^{-1} \cdot D^{-1}) = a^{-3d-6}[(1) - (2d^3)(0)] \ , \tag{4.8}$$

whose Fourier transform

$$F(D^{-1} \cdot D^{-1} \cdot D^{-1}) = a^{-3d-6}(2da^d - p_E^2 a^{2+d} - 8d^3 a^d) \ . \tag{4.9}$$

With vertices V_4^2 and overall symmetry number $1/3! = \frac{1}{6}$, the graph's contribution to Λ_2 ($p_E^2 = 0$) is

$$V_4^2(2d - 8d^3)a^{-2d-6}/6 \ . \tag{4.10}$$

As it stands, the effective potential (4.4) needs field renormalization, which we implement at $p = 0$. The field renormalization constant Z is defined by

$$Z^{-1} = \left. \frac{\partial W_2^{-1}}{\partial (p_M^2)} \right|_{p_M^2 = 0}$$
$$= 1 - \left. \frac{\partial}{\partial (p_M^2)} \tilde{\Lambda}_2^{-1}(p_M^2) \right|_{p_M^2 = 0} \ , \tag{4.11}$$

where p_M^2 is the square of the Minkowski momentum, $p_M^2 = -p_E^2$. Thus, for example, the contribution to Z^{-1} obtained from the graphs with up to three lines discussed above is

$$Z^{-1} = 1 + \frac{1}{\tilde{\Lambda}_2^2(p_M^2)} \frac{\partial}{\partial (p_M^2)} (V_4^2 p_M^2 a^{-2d-4}/6)$$
$$= 1 + \frac{V_4^2 a^{-2d-4}}{6\Lambda_2^2} \ . \tag{4.12}$$

The renormalized 1PI Green's functions are, as a consequence,

$$\tilde{\Gamma}_{2n}^R(0,0,\ldots,0) = Z^n \Gamma_{2n}(0,0,\ldots,0) \ . \tag{4.13}$$

In this intermediate renormalization scheme, the renormalized mass M and the renormalized coupling G are given in terms of the coefficients of V:

$$M^2 = \tilde{\Gamma}_2^R(0,0) \tag{4.14}$$

and

$$G = 24\tilde{\Gamma}_4^R(0,0,0,0) \ . \tag{4.15}$$

If we now use the techniques that we have sketched earlier we find that M^{-2}, G, Γ_6^R all permit power-series expansions in $(A^2 m_0^2)^{-1}$, with coefficients that are polynomials in $F = \frac{1}{2}(1 - B)$. For example, at the level of the first three terms we have

$$\frac{1}{a^2 M^2} = \frac{1}{a^2 m_0^2} \left[2f + (8df^2 - 4df) \frac{1}{a^2 m_0^2} + [(96d^2 - 48d + 16)f^3 + (-96d^2 + 24d - 16)f^2 + (24d^2 + 4)f] \frac{1}{(a^2 m_0^2)^2} + \cdots \right] , \tag{4.16}$$

$$G = 24M^8 a^{d+4} \frac{1}{(a^2 m_o^2)^2} \left[(8f^2 - 4f) + (128df^3 - 96df^2 + 16df) \frac{1}{a^2 m_0^2} - [(3840d^2 - 1344d + 128)f^4 + (-4224d^2 + 1152d - 192)f^3 + (1440d^2 - 240d + 96)f^2 + (-144d^2 - 16)f] \frac{1}{(a^2 m_0^2)^2} + \cdots \right] , \tag{4.17}$$

$$\Gamma_6^R = M^{12} a^{2d+6} \frac{1}{(a^2 m_0^2)^3} \left[(192f^3 - 244f^2 + 64f) + (4608df^4 - 6528df^3 + 2880df^2 - 384df) \frac{1}{a^2 m_0^2} + [(64\,512d^2 - 9216d + 1536)f^5 + (-112\,128d^2 + 16\,896d - 3328)f^4 + (69\,120d^2 - 9600d + 2688)f^3 + (-17\,664d^2 + 1728d - 960)f^2 + (1536d^2 - 128)f] \frac{1}{(a^2 m_0^2)^2} + \cdots \right] . \tag{4.18}$$

V. REARRANGING THE SERIES

Given the expansions (4.16)–(4.18) for M^2, G, Γ_6^R, going to the continuum limit requires the ability to choose $m_0^2(a^2)$ as a function of a^2 so that $M^2(a^2,m_0^2,f)$ remains fixed as $a\to 0$. The behavior of G,Γ_6^R, etc., then determine the continuum theory.

Before tackling the full problem, it is worthwhile examining truncated versions of the model. The simplest is just to consider the theory having no inverse propagators, in which we set $c=0$ (or, equivalently, D^{-1} to zero) in (2.2), rather than $c=1$. This gives

$$\frac{1}{a^2M^2}=\frac{2f}{a^2m_0^2}\ , \tag{5.1}$$

$$G=-24M^4 4f(1-2f)\frac{a^d}{m_0^4}\ , \tag{5.2}$$

$$\Gamma_6^{(R)}=16M^{12}\left[-\frac{f(1-2f)(1-4f)}{m_0^6}+\frac{10M^2f^2(1-2f)^2}{m_0^8}\right]a^{2d}\ , \tag{5.3}$$

and so on. Using (5.1) to solve for $m_0^2=2fM^2$, it follows that $G=O(a^d)$, $\Gamma_6^{(R)}=O(a^{2d})$ vanish as $a\to 0$, as do all higher Green's functions. [It can be proved[16] that this "triviality" is inevitable for all fixed f. Only for the choice (1.8) do we get nontrivial G,Γ_6^R.]

Second, consider the case in which we neglect all cross terms (nondiagonal terms) in D^{-1} of (3.5). That is, we take

$$D^{-1}\to -2d(\mathbf{0})/a^{d+2}\ . \tag{5.4}$$

This is the starting point for the high-temperature expansion of the analogue continuous-spin ferromagnet realization of the theory.[13] The series expansions for M^{-2}, G now become

$$\frac{1}{a^2M^2}=\frac{2f}{a^2m_0^2}\left[1-\frac{2d}{a^2m_0^2}+\frac{4d^2}{(a^2m_0^2)^2}-\frac{8d^3}{(a^2m_0^2)^2}+\cdots\right], \tag{5.5}$$

$$G=-24M^8\frac{(4f)(1-2f)}{(a^2m_0^2)^2}a^{d+4}\left[1-\frac{4d}{a^2m_0^2}+\frac{12d^2}{(a^2m_0^2)^2}-\frac{32d^3}{(a^2m_0^2)^3}+\cdots\right]. \tag{5.6}$$

We know that, since the sum can be performed to give

$$\frac{1}{a^2M^2}=\frac{2f}{a^2m_0^2+2d}\ , \tag{5.7}$$

$$G=\frac{-24M^8(4f)(1-2f)a^{d+4}}{(a^2m_0^2+2d)^2}\ , \tag{5.8}$$

we can again substitute m_0^2 for M^2 in G to give a trivial theory as $a\to 0$. However, suppose we had not observed this. If x,y are defined as

$$y=\frac{1}{a^2M^2},\quad x=\frac{1}{a^2m_0^2}\ , \tag{5.9}$$

then Eq. (5.5) becomes

$$y=2xf(1-2dx+4d^2x^2-8d^3x^3+\cdots) \tag{5.10}$$

$$=x\sum_{i=0}^{\infty}a_i^0(f,d)x^i\ , \tag{5.11}$$

where the a_i^0 can be read directly from (5.10). Similarly, (5.6) takes the form

$$G=-96\frac{M^{4-d}}{y^{d/2+2}}f(1-2f)x^2(1-4dx+12d^2x^2-32d^3x^3+\cdots) \tag{5.12}$$

$$=-\frac{M^{4-d}}{y^{d/2+2}}x^2\sum_{i=0}^{\infty}c_i^0(f,d)x^i\ . \tag{5.13}$$

We note that the Eqs. (4.16) and (4.17) for the full theory have the same generic form as (5.11) and (5.13).

To recover the result that the theory is trivial we can invert (5.10), writing x in therms of y on an order-by-order basis

$$x=\frac{y}{2f}\left[1+\frac{d}{f}y+\frac{d^2}{f^2}y^2+\cdots\right]$$

$$=y\sum_{i=0}^{\infty}A_i^0(f,d)y^i\ . \tag{5.14}$$

Let $x^{(N)}$ be the partial series

$$x^{(N)}=y\sum_{i=0}^{N}A_i^{(0)}(f,d)y^i\ . \tag{5.15}$$

Now let

$$G^{(N)}=\frac{-M^{4-d}}{y^{d/2}}\left[\frac{(x^{(N)})^2}{y^2}\sum_{i=0}^{\infty}c_i^0(f,d)x^{(N)i}\right]_N \tag{5.16}$$

$$=\frac{-M^{4-d}}{y^{d/2}}\sum_{k=0}^{N}h_k^{(0)}(d,f)y^k\ , \tag{5.17}$$

where the subscript N in (5.16) means that we only retain terms up to y^N in the expansion of powers of $x^{(N)}$ in terms of y.

For the case in point $h_k^{(0)}=0$, $k>0$, to give the correct answer as $y\to\infty$,

$$G^{(N)}=\frac{-24M^{4-d}}{y^{d/2}}\frac{1-2f}{f}\to 0,\quad N=1,2,3,\ldots\ . \tag{5.18}$$

We note that the fact that the theory has a continuum limit has nothing to do with the existence or nonexistence of a phase transition (in the language of high-temperature series). For this example, with no linkage from one cell to the next, there can be no long-distance correlations of "continuous" spin.

More generally, the series are not so simple, but the iterative method outlined above is equally applicable. Write (4.16)–(4.18) in terms of dimensionless Green's functions as

$$y=x\{2f+(8df^2-4df)x+[(96d^2-48d+16)f^3+(-96d^2+24d-16)f^2+(24d^2+4)f]x^2+O(x^3)\},$$

$$\tilde{G}=24\gamma_4=M^{d-4}G=24\frac{x^2}{y^{d/2+2}}\{(8f^2-4f)+(128df^3-93df^2+16df)x -[(3840d^2-1344d+128)f^4+(-4224d^2+1152d-192)f^3 +(1440d^2-240d+96)f^2+(-144d^2-16)f]x^2+O(x^3)\}\ , \quad (5.19)$$

$$\gamma_6=\Gamma_6^R(0,0,\ldots,0)M^{2d-6}=\frac{x^3}{y^{d+3}}\{(192f^3-224f^2+64f)+(4608df^4-6528df^3+2880df^2-384df)x +[(64\,512d^2-9612d+1536)f^5+(-112\,128d^2+16\,896d-3328)f^4 +(69\,120d^2-9600d+2688)f^3 +(-17\,664d^2+1728d-960)f^2+(1536d^2-128)f]x^2+O(x^3)\}\ . \quad (5.20)$$

Inverting x iteratively to order y^3 (3 terms) as $x^{(3)}(y)$ gives $\gamma_4^{(3)},\gamma_6^{(3)}$ to the same order as

$$\gamma_4^{(3)}=\frac{-y^{-d/2}}{f^2}\{(2f-1)+4(2df-1)fy+[(8d^2-20d)f^2+(16d-4d^2)f-3d]y^2\}\ , \quad (5.21)$$

$$\gamma_6^{(3)}=\frac{y^{-d}}{f^3}\{(24f^3-28f^2+8f)+(144df^3-168df^2+48df)y +[(288d^2-144d)f^3+(288d-336d^2)f^2+(96d^2+68d)f+30d]y^2\}\ . \quad (5.22)$$

We note that, although factors of $(1-2f)$ have not been displayed explicitly, γ_4 and γ_6 vanish when $2f=1$, as they must for a truly free theory. This is equally true for the higher-leg dimensionless Green's functions

$$\gamma_{2n}=\Gamma_{2n}^R(0,0,\ldots,0)M^{nd-2n-d} \quad (5.23)$$

but we shall have little need of them.

VI. PRELIMINARY SPECULATIONS

From the earlier high-temperature series analysis[12] we would expect the continuum limit either triviality or divergence. With a short series in hand, we feel that it is more reliable to extract some properties of the γ_{2n} which are likely to be finite. One way is to fit the lattice series to

$$\gamma_{2n}\sim\left[\frac{1}{a^2M^2}\right]^{-\nu_{2n}}=y^{-\nu_{2n}} \quad \text{as } y\to\infty \quad (6.1)$$

and then to calculate the critical exponents ν_{2n} order by order in powers of c (the number of internal lines) using the approximants just discussed. (Here we ignore the possibility of confluent singularity[12] and logarithmic behavior[21] for simplicity.) If $\nu_{2n}>0$ $(n=2,3,\ldots)$ the theory is trivial whereas if $\nu_{2n}<0$ the theory does not exist.

Since the lattice series for γ_{2n} in (5.23) can be seen to have a factor $y^{d/2-nd/2}$, it is sufficient to consider the large-y behavior of the remaining power series, which we denote by $A_{2n}(y)$: i.e.,

$$\gamma_{2n}=y^{d/2-nd/2}A_{2n}\ . \quad (6.2)$$

We assume that

$$A_{2n}(y)\sim y^{\rho_{2n}} \quad \text{as } y\to\infty\ , \quad (6.3)$$

whence

$$\nu_{2n}=nd/2-d/2-\rho_{2n}\ . \quad (6.4)$$

The index ρ_{2n} is calculated from the series A_{2n} as

$$\rho_{2n}=\lim_{y\to\infty}y\frac{d}{dy}\ln A_{2n}(y)\ . \quad (6.5)$$

The first few terms in the series for ρ_4, ρ_6, and ρ_8, defined from (5.21), (5.22) (and the equivalent series for γ_8) are

$$\rho_4(d)=\lim_{y\to\infty}(y\{4d-y[(8d^2+20d)f-6d]/f+y^2[(16d^3+120d^2+48d)f^2+(-36d^2-32d)f+4d]/f^2+\cdots\})\ , \quad (6.6)$$

$$\rho_6(d)=\lim_{y\to\infty}(y\{6d-y[(36d^2+3d)f^2+(-24d^2-54d)f+15d]/(3f^2-2f) +y^2[(144d^3+432d^2-360d)f^3+(-96d^3-648d^2+156d)f^2 +(188d^2+38d)f-13d]/(6f^2-4f)+\cdots\})\ , \quad (6.7)$$

$$\rho_8(d)=\lim_{y\to\infty}(y\{8d-y[(2880d^2+2880d)f^3-(4128d^2+6144d)f^2 +(1392d^2+4024d)f-812d]/(180f^3-258f^2+87f) +y^2[(5760d^3+17\,280d^2)f^4+(-8256d^3-36\,864d^2+3456d)f^3 +(2784d^3+24\,144d^2-544d)f^2+(-4872d^2-1184d)f +296d]/(180f^4-258f^3+87f^2)\cdots\})\ . \tag{6.8}$$

Taking higher-order terms into account (that we have not included here for lack of space) we observe that, if in the limit of large d we keep the highest power of d multiplying each power of y, the series can be summed to obtain (for fixed $f\neq 1/2$)

$$\rho_4\sim\lim_{y\to\infty}\frac{4yd}{1+2dy}=2\ ,\quad \rho_6\sim\lim_{y\to\infty}\frac{6yd}{1+2dy}=3\ ,\quad \rho_8\sim\lim_{y\to\infty}\frac{8yd}{1+2dy}=4\ . \tag{6.9}$$

Apparently, in large dimension $\rho_{2n}\sim n$. Thus, from (6.4), we have, for large d,

$$\nu_{2n}=\sim nd/2-d/2-n\ . \tag{6.10}$$

This is *exactly* the result that we would expect for a *canonically quantized* ($f=\frac{1}{2}$) *interacting* $\lambda\phi^4$ theory.[18] From this viewpoint, replacing the translationally invariant measure $D\phi$ for a free field by the scale-covariant measure $\bar{D}\phi$ is akin to adding a quartic self-interaction [despite the superficial logarithmic addition (1.7)]. This effect of renormalization was observed in a different context (the large-N limit) in Ref. 8, and is an example of how misleading the unrenormalized path integral can be when trying to determine universality classes.

If $\nu_{2n}>0$, the dimensionless scattering amplitude γ_{2n} vanishes in the continuum limit $a\to 0$. As in the case of the canonical theory,[18] at large dimension the pseudofree theory is trivial and there is nothing to be gained by non-canonical quantization of the type considered here. For small d some caution is required, although if we assume that the result (6.14) is approximately valid in this case (as happens in the canonical theory[18]), then γ_4 vanishes for $d\geq 4$, γ_6 vanishes for $d\geq 3$, and γ_8 for $d\geq\frac{8}{3}$.

VII. EXTRAPOLATION TO THE CONTINUUM

For small d, the series (6.6)–(6.8) are not geometric, and do not permit direct summation. Nominally, they show the seemingly unusual property that as $a\to 0$, with the renormalized mass M kept fixed, each term becomes infinite. However, this is a situation common to all derivative expansions and has been discussed at great length elsewhere[17,18,2] in the context of canonical theories.

The series which we need to evaluate as $y\to\infty$ have the generic form [see (5.21) onwards]

$$f^{(N)}(y)=y^{\alpha}\sum_{k=0}^{N}a_k y^k,\quad a_0\neq 0 \tag{7.1}$$

for which we can only evaluate the first $k\leq 5$ terms conveniently. One method which, despite caveats,[22] has been successful is to convert $\lim_{y\to\infty}f^{(N)}(y)$ into a sequence of extrapolants $f_N(\infty)$ as follows.

First, raising $f^{(N)}(y)$ to the power $1/\alpha$ gives

$$F(y)=f(y)^{1/\alpha}=y\left[\sum_{n=0}^{N}a_n y^n\right]^{1/\alpha}=\frac{y}{\sum_{n=0}^{\infty}\beta_n y^n}\ . \tag{7.2}$$

Define

$$f_n(\infty)=\left[\lim_{y\to\infty}\frac{y^n}{\left[\left[\sum_{i=0}^{\infty}\beta_i y^i\right]^n\right]_{\text{truncated after } n\text{th term}}}\right]^{1/n}\ . \tag{7.3}$$

Knowing $f^{(N)}(y)$ in (7.1) up to N terms gives a series of approximants $f_i(\infty), f_2(\infty),\ldots,f_N(\infty)$ which, in many cases[17,18] converge to, or come very close to $f(\infty)$ when it exists. Variants of this scheme can be constructed[23] which improve the series convergence.

We now give some examples of this method for ρ_4 when $f=\infty$, $f=1$, $f=\frac{3}{8}$, and $f=\frac{1}{4}$ ($B=-\infty,-1,\frac{1}{4},\frac{1}{2}$).

$f=\infty$: The extrapolants for $N=3,4,5$ lines are (all coefficients to nearest integer)

$$(\rho_4)_3(\infty)=\frac{16d}{(64d^2+4d+826)^{1/2}}\ , \tag{7.4}$$

$$(\rho_4)_4(\infty)=\frac{81d}{(66\,430d^3-49\,815d+3\,653\,675)^{1/3}}\ , \tag{7.5}$$

$$(\rho_4)_5(\infty)=\frac{256d}{(268\,435\,456d^4+2\,097\,152d^3+5\,249\,024d^2+4\,717\,039\,624d+55\,022\,766\,610)^{1/4}}\ . \tag{7.6}$$

We plot these estimates for ρ_4 as a function of d in Fig. 3(a).

$f=1$: The extrapolants for $N=3,4,5$ lines are

$$(\rho_4)_3(\infty)=\frac{16d}{(64d^2+4d+435)^{1/2}}, \tag{7.7}$$

$$(\rho_4)_4(\infty)=\frac{81d}{(66\,430d^3-1230d^2+213\,906d-1\,145\,702)^{1/3}}, \tag{7.8}$$

$$(\rho_4)_5(\infty)=\frac{256d}{(268\,435\,456d^4-2\,097\,152d^3+3\,676\,160d^2+11\,001\,542\,660d+4\,827\,593\,699)^{1/4}}. \tag{7.9}$$

We plot these estimates for ρ_4 as a function of d in Fig. 3(b).

$f=\frac{3}{8}$: The extrapolations for ρ_4 at $y=\infty$ for $N=3,4,5$ lines are

$$(\rho_4)_3(\infty)=\frac{16d}{(64d^2+4d+1091)^{1/2}}, \tag{7.10}$$

$$(\rho_4)_4(\infty)=\frac{81d}{(66\,430d^3-1230d^2-182\,060d+417\,105)^{1/3}}, \tag{7.11}$$

$$(\rho_4)_5(\infty)=\frac{256d}{(268\,435\,456d^4+2\,097\,152d^3+1\,054\,720d^2-2\,758\,624\,587d+4\,198\,806\,278)^{1/4}}. \tag{7.12}$$

We plot these approximants as a function of d in Fig. 3(c).

$f=\frac{1}{4}$: The approximants are

$$(\rho_4)_3(\infty)=\frac{16d}{(64d^2+4d+174)^{1/2}}, \tag{7.13}$$

$$(\rho_4)_4(\infty)=\frac{81d}{(66\,430d^3+1230d^2-1\,246\,172d+799\,003)^{1/3}}, \tag{7.14}$$

$$(\rho_4)_5(\infty)=\frac{256d}{(268\,435\,456d^4+2\,097\,152d^3-1\,042\,432\,d^2+8\,714\,186\,760d-2\,761\,330\,863)^{1/4}}. \tag{7.15}$$

These are plotted as a function of d in Fig. 3(d).

We conclude our numerical analysis by performing the complementary calculation of holding d fixed and letting f (or B) vary. We span the critical dimension $d=4$ by considering $d=5,4,3$ (all coefficients given to nearest integer).

$d=5$:

$$(\rho_4)_3(\infty)=\frac{16f}{(98f^2-19f+3)^{1/2}}, \tag{7.16}$$

$$(\rho_4)_4(\infty)=\frac{81f}{(94\,130f^3-11\,384f^2+2732f-1079)^{1/3}}, \tag{7.17}$$

$$(\rho_4)_5(\infty)=\frac{256F}{(394\,837\,591f^4+4\,473\,198f^3-38\,246\,487f^2+2\,121\,112f+1\,553\,011)^{1/4}}. \tag{7.18}$$

These are plotted as a function of f in Fig. 4(a). Note that, whereas the denominators of (7.16) and (7.18) are always real, the denominator of (7.17) vanishes for some positive f, and the convergent behavior (as the number of lines increases) that is seen for smaller f (e.g., $f=\frac{1}{4}$) breaks down. This might have been anticipated from the mean-field calculation,[10] usually valid in $d>4$ dimensions, that the theory does *not* exist for fixed $f<\frac{1}{2}$ [away from the large-N limit of the $O(N)$ generalization of the scalar theory] (note that, for $f\approx\frac{1}{2}$, the mean-field calculation shows that the field theory "approximately" exists):

$d=4$:

$$(\rho_4)_3(\infty)=\frac{16f}{(117f^2-29f+5)^{1/2}}, \tag{7.19}$$

$$(\rho_4)_4(\infty)=\frac{81f}{(121\,088f^3-29\,227f^2+8256f-2108)^{1/3}}, \tag{7.20}$$

$$(\rho_4)_5(\infty)=\frac{256f}{(557\,924\,232f^4-63\,708\,203f^3-32\,426\,221f^2-5\,634\,944f+3\,791\,531)^{1/4}}. \tag{7.21}$$

These approximants are plotted as a function of f in Fig. 4(b). As in the case $d=4$, the approximants begin to go wild for $f<\frac{3}{8}$ $(B>\frac{1}{4})$, confirming the suggestion from Padé calculations[7] and Monte Carlo simulations,[12] as well as the

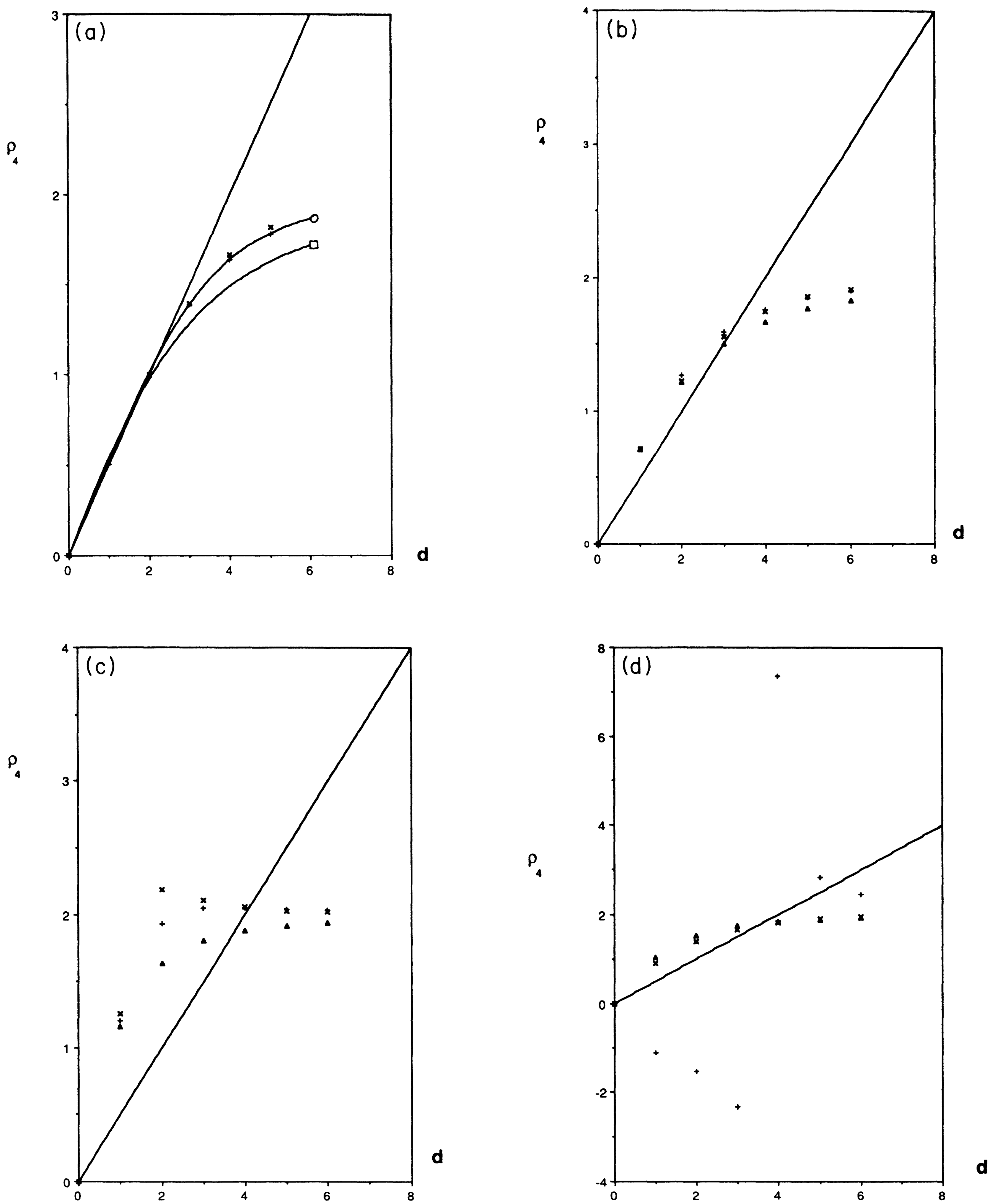

FIG. 3. Plot of $(\rho_4)_n(\infty)$ as a function of d values at (a) $B=-\infty$ where + denotes $(\rho_4)_4(\infty)$ and $\times$ denotes $(\rho_4)_5(\infty)$. This is overlaid by a continuous line plot of $(\rho_4)_n(\infty)$ as a function of d for the $\lambda=+\infty$ of the canonical $\lambda\phi^4$ theory results obtained from Ref. 18, where the line ended with a square denotes $(\rho_4)_4(\infty)$ and the line ended with a circle denotes $(\rho_4)_5(\infty)$. Here the solid line denotes $\rho_4=d/2$; (b) $B=-1$, where Δ denotes $(\rho_4)_3(\infty)$, + denotes $(\rho_4)_4(\infty)$, and $\times$ denotes $(\rho_4)_5(\infty)$, the solid line denotes $\rho_4=d/2$; (c) $B=\frac{1}{4}$, where Δ denotes $(\rho_4)_3(\infty)$, + denotes $(\rho_4)_4(\infty)$ and $\times$ denotes $(\rho_4)_5(\infty)$, the solid line denotes $\rho_4=d/2$ as before; (d) $B=\frac{1}{2}$, where Δ denotes $(\rho_4)_3(\infty)$, + denotes $(\rho_4)_4(\infty)$ and $\times$ denotes $(\rho_4)_5(\infty)$, the solid line denotes $\rho_4=d/2$ as before.

mean-field result previously mentioned, that in $d=4$ dimensions there is a value of $f, f^* \approx \frac{1}{2}$, above which the continuum theory does not exist. (Note that the mean-field result, although it seems qualitatively correct, does not predict the boundary between trivial and nonexistent theories wholly accurately.[12])

$$(\rho_4)_3(\infty)=\frac{16f}{(157f^2-52f+9)^{1/2}}\ , \tag{7.22}$$

$$(\rho_4)_4(\infty)=\frac{81f}{(197\,401f^3-85\,851f^2+26\,489f-4998)^{1/3}}\ , \tag{7.23}$$

$$(\rho_4)_5(\infty)=\frac{256f}{(1\,123\,716\,318f^4-437\,253\,017f^3+90\,146\,040f^2-51\,985\,016f+11\,983\,109)^{1/4}}\ . \tag{7.24}$$

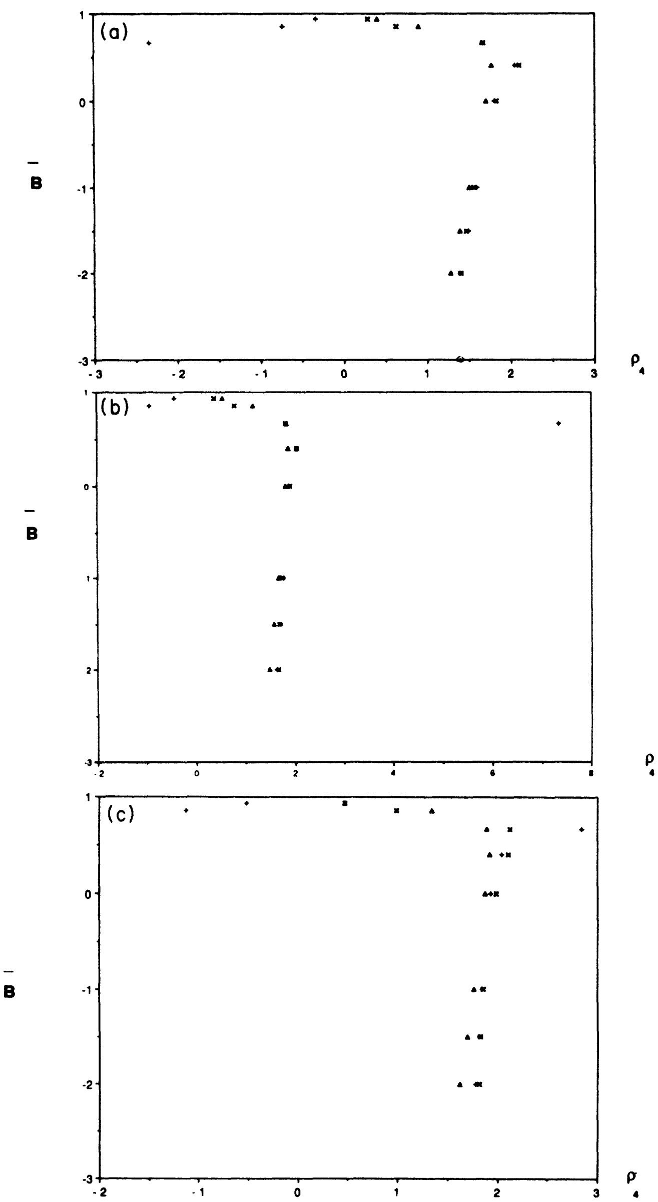

FIG. 4. Plot of $\bar{B}=2B/(1+|B|)$ as a function of $(\rho_4)_n(\infty)$ at (a) $d=5$, where Δ denotes $(\rho_4)_3(\infty)$, $+$ denotes $(\rho_4)_4(\infty)$ and $\times$ denotes $(\rho_4)_5(\infty)$; (b) $d=4$, where Δ denotes $(\rho_4)_3(\infty)$, $+$ denotes $(\rho_4)_4(\infty)$ and $\times$ denotes $(\rho_4)_5(\infty)$; (c) $d=3$, where Δ denotes $(\rho_4)_3(\infty)$, $+$ denotes $(\rho_4)_4(\infty)$ and $\times$ denotes $(\rho_4)_5(\infty)$.

Yet again the approximants break down when $f \lesssim \frac{3}{8}$, more violently than for the case of $d=4$ dimensions.

VIII. CONCLUSIONS

Although the series approximants have been based on few terms, we can already draw some conclusions. First, we see from (6.6)–(6.8) that the $d \to \infty$ and $f \to \infty$ limits commute, term by term. Thus the values for ρ_{2n} in (6.9) are also the values that we would obtain for the pseudofree theory at large f in large dimension.

From (1.7) the $f \to \infty$ $(B \to -\infty)$ limit, with its suppression of small fields, is equivalent to the spin-$\frac{1}{2}$ Ising limit, which is also the $\lambda \to \infty$ strong-coupling limit of the canonical $\lambda\phi^4$ theory. The agreement with the strong-coupling results of Ref. 18 are total, as can be seen in Fig. 3(a). For $d>4$ we seem to have a trivial theory. For $d>4$ we anticipate a nontrivial theory, with ν_4 tending to zero uniformly in d as the number of internal lines is increased.

At the other extreme, for small *fixed* $f \approx 0$,

$$\rho_4 = O(f)\ , \tag{8.1}$$

if we ignore problems of definition [see (7.16) onwards] in all dimensions d. In turn, from (6.4),

$$\nu_4 \sim \frac{d}{2} + O(f) > 0\ . \tag{8.2}$$

This corresponds to the four-point function *vanishing* in the limit $a \to 0$. That is, for small f, the *noncanonical* theory looks trivial. This is superficially at variance with the $d=4$ results based on the high-temperature series,[12] although the Padé approximants break down for small f. However, we cannot infer that we would obtain the same result if f were driven to zero as $f=O(a^d)$ in the continuum limit, since semianalytic calculations suggest otherwise.[12] This will be the study of a further paper.

In between these two extremes, Figs. 3 show that the theory is inevitably *trivial* for large dimension, in accord with (6.9). This is interesting because noncanonical quantization had the potential for enabling theories that would be perturbatively nonrenormalizable (if canonical) to be defined and nontrivial[9] when expressed as an expansion about a nontrivial pseudo-free theory. The IVM provided an extreme example of this idea, being maximally nonrenormalizable in one sense [with propagators $O(k^0)$ for large moments k]. However, in the more usual sense of nonrenormalizability due to large dimension we see that noncanonical quantization of the form considered here does not help, away from $f \simeq 0$ at least.

However, the approximant scheme begins to break down visibly by the time that $f \simeq \frac{1}{4}$ for $d < 4$ dimensions. As we have noted, the $d > 4$ dimension mean-field calculations indicated that the pseudofree theory is nonexistent for all $f < \frac{1}{2}$. The results of Figs. 3 show this to be even more likely in $d < 4$ dimensions (although we already had a partial case for the nonexistence of the $d = 2$ pseudo-free theory for such f).[12] This reinforces the original argument for noncanonical quantization based on perturbative nonrenormalizability. Only for $d \geq 4$ dimensions is the introduction of a nontrivial ($f \neq \frac{1}{2}$) measure justified (even if, ultimately, it does not help).

Finally, for $f = 1$ ($B = -1$) the results of Fig. 3(b) are sufficiently different from those of Fig. 3(a) to make it unclear as to whether the theory can exist in small dimensions in this case, although this looks unlikely. This is compatible with the $d = 2$ result of Ref. 12. [The case $f = 1$ is interesting in its own right because it corresponds to a conformally invariant measure which, in the context of gravity, gives rise to a natural mechanism whereby $\langle g_{\mu\nu} \rangle \neq 0$ (Ref. 4). This choice will also be considered separately elsewhere.] The work of Ref. 12 indicates that, in $d = 4$ dimensions at least, the noncanonical theory for $f > \frac{1}{2}$ is trivial (in the same equivalence class as the canonical theory). This series is too short here to do more than make this plausible. In summary, the noncanonical quantization of a free theory that we have considered here has been of no help in obtaining nontrivial scalar theories when the canonical theory is trivial and is a positive hindrance when the canonical theory already exists. If noncanonical quantization is to be successful, a different approach is needed, except perhaps for $B \rightarrow 1^-$, which will be considered elsewhere.

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