PHYSICAL REVIEW D VOLUME 41, NUMBER 2 15 JANUARY 1990

Conserved quantities at spatial and null infinity: The Penrose potential

Joshua N. Goldberg
Department of Physics, Syracuse University, Syracuse, New York 13244-1130
(Received 19 June 1989)

We define a superpotential for energy-momentum and rotation momentum which is built out of the conformal tensor and a bivector. This superpotential is identified with that used by Penrose in his definition of quasilocal energy. It is applied to the definition of energy-momentum and rotation momentum at spatial and at null infinities. At spatial infinity the results are in agreement with those of Ashtekar and Hansen. At null infinity the results are unsatisfactory; they are tied to a specific Bondi frame. Thus, they are not in agreement with the results of Tamburino and Winicour, Geroch and Winicour, nor with those of Dray and Streubel. Some reasons for this failure are discussed.

I. INTRODUCTION

This work was begun with the object of understanding the Penrose construction of integrals for a quasilocal definition of energy-momentum and angular momentum.[1] In the Penrose construction, one uses surface integrals over the conformal tensor weighted with a bivector constructed from solutions of the twistor equation. The goal is to find a two-surface integral which could be given an interpretation in terms of the angular momentum twistor.[2] Tod[3] has shown that the construction works in a number of cases, where the two-surface can be embedded in a conformally flat space-time. However, there are still unresolved problems in the general case. Shaw[4,5] has studied the construction at spatial and null infinity. At spatial infinity he finds both the twistorial construction and agreement with the usual definitions. At null infinity one can carry out the construction, but the result is unsatisfactory.

I have used the word construction above because Penrose takes over to curved space-time an expression he derived using the linearized Einstein equations. There is no attempt to find a similar relationship in the nonlinear Einstein equations. However, we know that the general covariance of the Einstein equations results in differential conservation laws related to the field equations.[6,7] From Noether's theorem one shows that for an arbitrary diffeomorphism defined by the vector field ξ^a the invariance of the Lagrangian for the Einstein equations leads to identities of the form

$$2\sqrt{-g}\,G^a{}_b\xi^b \equiv (\sqrt{-g}\,U^{ab})_{,b} - \sqrt{-g}\,t^a \ . \tag{1.1}$$

When this expression is derived from the first-order Lagrangian of general relativity, U^{ab} is the superpotential of Freud[8] and t^a is essentially the Einstein pseudotensor, both transvected with the vector field ξ^a. In asymptotically flat space-times, we may choose ξ^a to be an asymptotic Killing vector. Equation (1) then allows one to identify the surface integral over the superpotential with the total energy and momentum of the system:

$$P[\xi,S] := \frac{1}{16\pi\kappa}\oint_S U^{ab}dS_{ab} = \frac{1}{16\pi\kappa}\int_\Sigma \sqrt{-g}\,(t^a + 16\pi\kappa T^a{}_b\xi^a)d\Sigma_a \ , \tag{1.2}$$

where the field equations $G^a{}_b = 8\pi\kappa T^a{}_b$ have been used. Σ is a hypersurface which extends out to spacelike or null infinity and S is its bounding two-surface which is taken in the limit to infinity. However, neither the Freud superpotential nor the Einstein pseudotensor are tensorial, so this expression is coordinate dependent and great care must be exercised in its use. In the following we shall be concerned with finding a tensorial superpotential.

Note that if one considers a four-dimensional domain which is bounded by two such surfaces and by a timelike or null surface at infinity, then the change in the "charges"

$$P[\xi;S] = \frac{1}{16\pi\kappa}\oint_S U^{ab}dS_{ab} \tag{1.3}$$

will be given by a flux integral over the timelike or null boundary at infinity. If we can assume that the matter tensor vanishes on that boundary, the flux integral involves only the gravitational stress-energy complex:

$$F[\xi,\Sigma] = \frac{1}{16\pi\kappa}\int_\Sigma t^a d\Sigma_a \ . \tag{1.4}$$

The first attempt to create a tensorial expression was made by Møller[9] who limited his work to a spacelike surface to overcome the problem of an infinite energy resulting from the use of spherical coordinates even in Minkowski space. This expression was generalized by Komar[10] to the superpotential

$$U^{ab} := \nabla^a\xi^b - \nabla^b\xi^a \ . \tag{1.5}$$

This expression leads to a constant of the motion in a vacuum space-time if ξ^a is a conformal Killing vector. In asymptotically flat space-time one gets a satisfactory result at spatial infinity except that the normalization for energy-momentum and for rotation momentum differ by a factor of 2. This problem persists at null infinity using the modification of the Komar superpotential defined by Tamburino and Winicour.[11,12] A further difficulty at null infinity is that gravitational radiation leads to supertranslation invariance which introduces terms in the expression for angular momentum which do not vanish in Minkowski space.[12]

On the other hand, using the symplectic structure of general relativity, Ashtekar and Streubel[13] have constructed flux integrals at null infinity which are to define the transport of energy-momentum and rotation momentum. They were able to find surface integrals which define energy-momentum consistent with their flux integrals. But, they were unable to do so for rotation momentum. Dray and Streubel[14] modified the Penrose construction at null infinity in a manner to be described later and Shaw[15] was able to show that their modification had the Ashtekar-Streubel flux. This result was independently obtained by Dray[16] with a more elegant and insightful calculation. Therefore, there now exist expressions at spatial infinity and at null infinity for energy-momentum and rotation momentum which transform appropriately under the respective asymptotic symmetry groups and whose flux vanish in Minkowski space.

However, there does not exist a superpotential, hence a tensorial differential conservation law, which leads to the Dray-Streubel expressions. In this paper we show that at spatial infinity there exists such a tensorial expression in terms of the Riemann tensor and a potential for the asymptotic Killing vectors. This leads to the Penrose integrals for energy-momentum and rotation momentum. At null infinity in the presence of gravitational radiation, the same expression does not lead to energy-momentum and rotation momentum definitions which transform appropriately with respect to the Bondi-Metzner-Sachs (BMS) group. Therefore, these do not agree with the Dray-Streubel integrals.

Shortly after Penrose introduced his spinor formulation of general relativity and the definition of energy-momentum in terms of the spinor components of the Weyl tensor,[17] Komar attempted to connect this work with the conformal tensor and the Bianchi identities at null infinity.[18] While Komar's use of the conformal tensor is similar to what is done in this paper, he does not construct a differential identity and the conditions he imposes on the tensors defining the asymptotic symmetry are different. This will be discussed in more detail in Sec. V.

In the following section we shall derive a superpotential using the uncontracted Bianchi identities and discuss its relation to the work of Penrose. This superpotential is similar to the one derived by Moreschi.[19] Section III discusses energy-momentum and rotation momentum at spatial infinity while Sec. IV does so at null infinity. In Sec. V we discuss the relation of this work to that of Dray and Streubel and Moreschi.

II. THE SUPERPOTENTIAL

To derive an expression such as (1.1) from the Riemann tensor we must introduce an antisymmetric tensor Q^{ab} to lower its rank. Write the Bianchi identities as $\epsilon^{mnrs}\nabla_n R_{rsab}\equiv 0$ (ϵ^{mnrs} is the totally antisymmetric tensor) and transvect with Q^{*ab} to obtain

$$\nabla_n({}^*R^{mn}{}_{ab}Q^{*ab})-{}^*R^{mn}{}_{ab}\nabla_n Q^{*ab}\equiv 0 \ . \tag{2.1}$$

The asterisk indicates a dual, for example,

$$Q^{*ab}:=\tfrac{1}{2}\epsilon^{abcd}Q_{cd} \ .$$

In the case of the Riemann tensor, the asterisk to the left (right) indicates that the dual is taken with respect to the first (last) two indices. If we shift the dual from Q^{*ab} to the Riemann tensor and expand the double dual on the right-hand side, we get

$$\nabla_a({}^*R^{*ma}{}_{bc}Q^{bc})-(-R^{ma}{}_{bc}+\delta^m{}_bR^a{}_c-\delta^m{}_cR^a{}_b)\nabla_a Q^{bc} -2G^m{}_b\nabla_n Q^{bn}\equiv 0 \ . \tag{2.2}$$

Note that in the term multiplying the second set of parentheses, the totally antisymmetric part of $\nabla^c Q^{ab}$ drops out and traces give additional contributions of the second term. Therefore, we define

$$\mathcal{P}^{abc}:=2(\nabla^{(a}Q^{b)c}-\nabla^{(a}Q^{c)b}+g^{a[b}\nabla_e Q^{c]e}) \ , \tag{2.3}$$

$\mathcal{P}^{[abc]}=\mathcal{P}_a{}^{ac}=0$. Using this in (2.2) we get our final result:

$$\nabla_a({}^*R^{*ma}{}_{bc}Q^{bc})-\tfrac{1}{3}(-R^m{}_{abc}+\delta^m{}_bR^a{}_c-\delta^m{}_cR_{ab})\mathcal{P}^{abc} \equiv 2G^m{}_c\xi^c \ , \tag{2.4}$$

where we have defined the vector ξ^a to be

$$\xi^a:=\tfrac{1}{3}\nabla_b Q^{ab} \ . \tag{2.5}$$

Comparing (2.4) with (1.1), we make the identification

$$\begin{aligned} U^{mn}&={}^*R^{*mn}{}_{ab}Q^{ab} \ , \\ t^m&=-\tfrac{1}{3}(C^m{}_{abc}-\delta^m{}_bR_{ac})\mathcal{P}^{abc} \ . \end{aligned} \tag{2.6}$$

Thus, using the Riemann tensor, we have obtained a tensorial superpotential related to the field equations as in (1.1). In the following, we shall refer to U^{mn} as the Penrose potential.

The importance of this result is that when

$$\mathcal{P}^{abc}=0 \ , \tag{2.7}$$

then

$$\nabla^a\xi^b+\nabla^b\xi^a=2R^{(a}{}_eQ^{b)e} \tag{2.8}$$

so that, when $R^{ab}=0$, ξ^a is a Killing vector. Moreschi[19] obtains a similar result. However, he does not identify $\mathcal{P}^{abc}$ nor does he emphasize the relation to the general form of conservation laws in general relativity.

Equations (2.4) and (2.7) are the basis of the Penrose derivation in linearized general relativity. His result omits t^a because he uses explicitly the Killing vectors of Minkowski space. In spinor language, the equation $\mathcal{P}^{abc}=0$ goes over to $\nabla_{A'}{}^{(A}\omega^{BC)}=0$, which is the twistor equation. To connect with the angular momentum twistor, Penrose suggests looking for solutions which are constructed from symmetric products of solutions of the rank-1 twistor equation, $\nabla_{A'}{}^{(A}\omega^{B)}=0$. In curved space-time, solutions of these equation do not exist because in general Killing vectors do not exist. Penrose then suggests that one take solutions on a closed two-surface and use these to construct the integrals of interest. This point of view derives from the desire to define the energy-momentum and rotation momentum in terms of the angular momentum twistor[1,2] as $Z^\alpha Z^\beta A_{\alpha\beta}$ where Z^α is defined by the pair $(\omega^A, -i\nabla_{AA'}\omega^A)$. Tod[3] uses this construction in his examples as does Shaw[4,5] in his investigations.

Here we are not interested in the twistor construction, but rather in obtaining the total energy-momentum and rotation momentum by means of surface integrals at infinity and the flux by means of a tensorial stress tensor. Therefore, we shall make use of the relationship between those Q^{ab} which are solutions of $\mathcal{P}^{abc}=0$ and Killing vectors, Eq. (2.5). At spatial infinity we shall look for asymptotic solutions of Eq. (2.7) and at null infinity for approximately asymptotic solutions. These bivectors will then be used in Eq. (2.10) to construct the corresponding charges. The calculations are done in the conformally related space-time introduced by Penrose[20] to describe null infinity and adapted by Ashtekar and Hansen[21,22] in their discussion of spatial infinity. One can show that under the conformal transformation $\hat{g}_{ab}=\Omega^2 g_{ab}$,

$$\mathcal{P}^{abc}=\Omega^3\hat{\mathcal{P}}^{abc}, \quad Q^{ab}=\Omega\hat{Q}^{ab} . \tag{2.9}$$

Combining these relations with the conformal invariance of the Weyl tensor, $\hat{C}^a{}_{bcd}=C^a{}_{bcd}$, we find, in a region where the vacuum Einstein equations are satisfied,

$$\begin{aligned} P[Q,S]&=\frac{1}{16\pi\kappa}\oint_S \Omega^{-1}\hat{C}^{mn}{}_{ab}\hat{Q}^{ab}dS_{mn} ,\\ F[Q,\Sigma]&=-\frac{1}{16\pi\kappa}\int_\Sigma \Omega^{-1}\hat{C}^{m}{}_{nab}\hat{\mathcal{P}}^{nab}d\Sigma_m . \end{aligned} \tag{2.10}$$

We shall make use of these integrals in examining the behavior at infinity.

III. SPACELIKE INFINITY

To examine the solutions of $\mathcal{P}^{abc}=0$ at spatial infinity we shall use the structure of i^0 described by Ashtekar and Hansen.[21] Therefore, we assume that there exists a space-time $(\hat{M},\hat{g}_{ab})$ which is conformally related to the physical space-time (M,g_{ab}), $\hat{g}_{ab}=\Omega^2 g_{ab}$. The manifold $\hat{M}$, the metric $\hat{g}_{ab}$, and the conformal factor Ω are smooth everywhere except at the point of infinity, i^0. At i^0, the manifold is $C^{>1}$, the metric $C^{>0}$, and the function Ω is C^2. [That a function $C^{>n}$ at a point p means that it is C^n at p and its $(n+1)$st derivative has a direction-dependent limit at the point.] At i^0 we have $\Omega=0$, $\nabla_a\Omega=0$, and $\nabla_a\nabla_b\Omega=2\hat{g}_{ab}$.

To establish the correct fall-off behavior for physical fields, we can either introduce an orthonormal tetrad and examine the limiting values of the scalar components or we may make use of the existence of the metric at i^0, the Minkowski metric, the natural basis of which is an orthonormal tetrad. Although it will not be necessary to do so explicitly, we shall think in terms of the latter option.

We shall proceed by projecting all quantities and relations of interest onto the surface $\mathcal{H}$, $\Omega=\Omega_0$, and rescaling the quantities so that they have direction-dependent limits at i^0; that is, for the limit $\Omega_0\to 0$. Introduce the unit vector $\eta_a:=\alpha\nabla_a\Omega$, $\alpha^2=\nabla_b\Omega\nabla^b\Omega$. We assume that the magnetic part of the conformal tensor falls off faster than does the electric part. Thus we have that

$$\Omega^{1/2}C_{ambn}\eta^m\eta^n=E_{ab}, \quad C^*{}_{ambn}\eta^m\eta^n={}^1B_{ab} \tag{3.1}$$

have direction-dependent limits ${}^0\mathbf{E}_{\mathbf{ab}}$ and ${}^1\mathbf{B}_{\mathbf{ab}}$, respectively. Comparison with the results in Minkowski space indicate that $\Omega^{-1/2}\hat{Q}^{ab}$ for translations and $\hat{Q}^{ab}$ for rotations have direction-dependent limits. Furthermore, we find that for any quantity Φ which has a direction-dependent limit, $\Omega^{1/2}\nabla_a\Phi$ also has such a limit. In particular this means that

$$\lim_{\Omega\to 0}\Omega^{1/2}\eta^a\nabla_a\Phi=0 . \tag{3.2}$$

Thus, we have, for the connection induced on $\mathcal{H}$ $(\eta_b V^b=0)$,

$$D_a V^b := h^c{}_a h^b{}_d \Omega^{1/2}\nabla_c V^d , \tag{3.3}$$

which has a limit at i^0 whenever V^b has a limit. Here we have used $h^c{}_a=\delta^c{}_a+\eta^c\eta_a$. All the equations we write will have direction-dependent limits for $\Omega\to 0$. When the limit has been taken, the relevant quantities will be written in boldface.

Ashtekar and Hansen[21,22] have given the equations satisfied by ${}^0\mathbf{E}_{\mathbf{ab}}$. However, to complete the calculation as presented here, we shall also require ${}^1\mathbf{E}_{\mathbf{ab}}$, the $\Omega^{1/2}$ part of E_{ab}. To avoid cluttering the main argument with details, the relevant equations and the identity to be used will be derived in the Appendix. Here we shall obtain the equations for $\hat{Q}^{ab}$, $\hat{\mathcal{P}}^{nab}=0$, and the equations relating the asymptotic Killing vectors, ξ^a to $\hat{Q}^{ab}$. We begin by decomposing Q^{ab} relative to $\mathcal{H}$:

$$\Omega^{-1/2}\hat{Q}^{ab}=T^{ab}+T^a\eta^b-T^b\eta^a \quad \text{for translations} , \tag{3.4a}$$

$$\hat{Q}^{ab}=L^{ab}+L^a\eta^b-L^b\eta^a \quad \text{for rotations} . \tag{3.4b}$$

T^{ab}, T^a, L^{ab}, and L^a are each orthogonal to η_c; for example, $\eta_a T^a=0$. In the Appendix this decomposition of Q^{ab} is substituted into $\hat{\mathcal{P}}^{nab}=0$ which is similarly decomposed with respect to η_a. At i^0, the resulting equations are given below.

Translations:

$$\begin{aligned} &(\mathrm{i})\ \mathbf{T^{abc}}=0, \quad (\mathrm{iii})\ \mathbf{D^aT^b}-\mathbf{D^bT^a}=0 ,\\ &(\mathrm{ii})\ \mathbf{D_aT^{ab}}=0, \quad (\mathrm{iv})\ 3\mathbf{D^aT^b}-\mathbf{h^{ab}D_cT^c}=0 . \end{aligned} \tag{3.5}$$

T^{abc} is just $\mathcal{P}^{abc}$ with T^{ab} substituted for $\hat{Q}^{ab}$ and projected onto $\mathcal{H}$. From (iv) above, we have

$$\mathbf{T^a}=\mathbf{h^{ac}}D_c\mathbf{T} \quad\text{and}\quad 3\mathbf{D^aD^bT}-\mathbf{h^{ab}D_cD^cT}=\mathbf{0} . \tag{3.6}$$

Thus T^a is a hypersurface orthogonal conformal Killing vector on $\mathcal{H}$.

Rotations:

$$\begin{aligned} &\mathbf{L^{abc}}+2\mathbf{h^{a[b}L^{c]}}=0\ (\mathrm{i}) ,\\ &\mathbf{D_cL^{ac}}+2\mathbf{L^a}=0\ (\mathrm{ii}), \quad \mathbf{D^aL^b}=\mathbf{L^{ba}}(\mathrm{iii}) . \end{aligned} \tag{3.7}$$

L^{abc} has the same meaning as T^{abc} above. From (ii) and (iii) we note that L^a is a Killing vector on $\mathcal{H}$ and furthermore that

$$\mathbf{D}_c\mathbf{L}^c=0 \quad\text{and}\quad \mathbf{D}_c\mathbf{D}^c\mathbf{L}^a=-2\mathbf{L}^a . \tag{3.8}$$

Then we find that (i) becomes

$$\begin{aligned}&\mathbf{D}^a\mathbf{D}^b\mathbf{L}^c=\mathcal{R}^{bca}{}_d\mathbf{L}^d ,\\&\mathcal{R}^{bcad}=-(\mathbf{h}^{ba}\mathbf{h}^{cd}-\mathbf{h}^{bd}\mathbf{h}^{ca}) .\end{aligned} \tag{3.9}$$

Finally we get the following for the asymptotic Killing vectors.

$$\begin{aligned}&\Omega^{-1}h^a{}_c\xi^c=:\xi_T{}^a=-\mathbf{T}^a ,\\&\Omega^{-1}\eta_c\xi^c=:\xi_T=-\mathbf{D}_c\mathbf{T}^c .\end{aligned} \tag{3.10}$$

In the first line we have used $\mathbf{D}_c\mathbf{T}^{ac}=0$ from (3.5). Therefore, $\mathbf{T}^{ab}$ plays no role in defining the translational Killing vector and may be taken to be zero. Thus, in agreement with previously known results, the translational symmetries depends on an arbitrary smooth function **T** on $\mathcal{H}$. This gives rise to the supertranslations of which the rigid translations form a normal subgroup. One can show that for the translation subgroup, **T** satisfies

$$\mathbf{D}^a\mathbf{D}^b\mathbf{T}+\mathbf{h}^{ab}\mathbf{T}=0 . \tag{3.11}$$

The supertranslations mix the electric and magnetic parts of the conformal tensor. After a supertranslation, the magnetic part would no longer satisfy the fall-off condition assumed in Eq. (3.1). Therefore, preserving this condition, essentially no magnetic mass monopole, allows us to restrict the translation subgroup to the rigid translations. One can show, further, that **T** is of the form $\mathbf{T}=\mathbf{a}^a\eta_a$ and therefore depends on the four parameters $\mathbf{a}^a$.

Rotations:

$$\begin{aligned}&\Omega^{-1/2}\xi^a\eta a=:\xi_R=-\mathbf{D}_a\mathbf{L}^a=\mathbf{0} ,\\&\Omega^{-1/2}\xi^b h^a_b=:\xi_R{}^a=-2\mathbf{L}^a .\end{aligned} \tag{3.12}$$

The solutions of (3.7) which have finite direction-dependent limits at i^0 have the form $\mathbf{L}^a=\omega^{ab}\eta_b$, with $\omega^{ab}=-\omega^{ba}$. Hence, the rotational Killing vectors depend on an antisymmetric tensor corresponding to the rotations in Minkowski space.

The integrals for energy-momentum and rotation momentum charges can now be written (λ^a is the normal to S in $\mathcal{H}$ and $K_{mnrs}:=\Omega^{1/2}C_{mnrs}$). For energy-momentum we get

$$\begin{aligned}P[T,S]&=\frac{1}{8\pi\kappa}\oint_S K^{mn}{}_{rs}(T^r\eta^s-T^s\eta^r)\eta_m\lambda_n dS\\&=-\frac{1}{4\pi\kappa}\oint_S {}^0\mathbf{E}^a{}_b\mathbf{T}^b\lambda_a d\mathbf{S} .\end{aligned} \tag{3.13}$$

With $\mathbf{T}^b=-\mathbf{D}^b(\mathbf{a}^c\eta_c)=-\mathbf{h}^b{}_c\mathbf{a}^c$, we have

$$P[T,S]=\frac{1}{4\pi\kappa}\oint_S {}^0\mathbf{E}^a{}_b\mathbf{a}^b\lambda_a d\mathbf{S}=:P_b a^b , \tag{3.14}$$

where a^b is a constant vector in the physical space-time.

For the rotation momentum we get

$$\begin{aligned}P[L,S]&=\frac{1}{8\pi\kappa}\oint_S K^{mn}{}_{rs}(L^{rs}+L^r\eta^s-L^s\eta^r)\eta_m\lambda_n dS\\&=\frac{1}{4\pi\kappa}\oint_S(\epsilon_{ab[s}{}^1B^b{}_{r]}D^sL^r-{}^1E_{ab}L^b)\lambda^a dS ,\end{aligned}$$

$\epsilon_{abc}=\epsilon_{nabc}\eta^n$. Using the identity given in Eq. (A7) of the Appendix, we find, for $L^b=\frac{1}{4}D_c(\omega^{ij}h^b{}_i h^c{}_j)=-\frac{1}{2}\omega^{bj}\eta_j$,

$$P[L,S]=\frac{1}{4\pi\kappa}\oint_S(\epsilon^a{}_{b[r}{}^1\mathbf{B}^r{}_{s]}\omega^{rs})\lambda_a d\mathbf{S}=:M_{ab}\omega^{ab} . \tag{3.15}$$

If we introduce $\omega^{rs}=\epsilon^{rsc}\zeta_c$, this expression turns into that given by Ashtekar and Hansen.[21]

Since we have solved $\hat{\mathcal{P}}^{abc}=0$, the flux term vanishes and these charges are independent of the two-surface on $\mathcal{H}$ and hence are constants of the motion. Furthermore, P_a and M_{ab} transform appropriately under the asymptotic Poincaré group.

IV. NULL INFINITY

The situation is both simpler and more complicated at null infinity. In the presence of gravitational radiation, energy-momentum and rotation momentum cannot be constants of the motion. Therefore, the flux term cannot vanish. This means that Eq. (2.7), $\mathcal{P}^{abc}=0$, cannot be satisfied. Nonetheless, one knows that the asymptotic symmetry group is the BMS group, the semidirect product of the Lorentz group and the supertranslations at $\mathcal{I}^+$. Therefore, one might hope that enough of $\mathcal{P}^{abc}=0$ could be satisfied to yield the BMS generators and that the solutions for Q^{ab} would give the correct nonconserved energy-momentum and rotation momentum. Let me say at the outset that the former is true and the latter is not.

In considering the limit to null infinity, one takes the metric in the form[23,24]

$$\begin{aligned}ds^2=&\,e^{2b}\phi\, du^2+2e^{2b}du\, dr\\&-h_{AB}(dx^A-U^Adu)(dx^B-U^Bdu) .\end{aligned} \tag{4.1}$$

It is convenient to work with (anti)-self-dual components of the conformal tensor and the Killing potential. One can show that the equation for the Killing potential becomes simply

$$2\mathcal{P}^{+abc}:=\mathcal{P}^{abc}-\frac{i}{2}\epsilon^{bc}{}_{mn}\mathcal{P}^{amn} , \tag{4.2}$$

when we introduce $2Q^{+ab}:=Q^{ab}-iQ^{*ab}$ into $\mathcal{P}^{abc}$. Then $2U^{ab}=U^{+ab}+U^{-ab}$, where

$$\overline{U}^{-ab}=U^{+ab}=iC^{+ab}{}_{mn}Q^{+mn} . \tag{4.3}$$

We introduce the conformal completion as in the previous section, but here the asymptotic conditions on Ω and the differentiability conditions are different from those at i^0. At null infinity, $\mathcal{I}^+$, $\Omega=0$ is a null surface, $\nabla_a\Omega=n_a$ becomes a null vector tangent to the generators of $\mathcal{I}^+$ the metric becomes singular, but all quantities of physical interest are differentiable there. $\Omega^{-1}\hat{C}^{abcd}$ is finite on $\mathcal{I}^+$ as is $\hat{Q}^{ab}$. Therefore, we have

$$U^{+ab}:=\Omega^3\hat{C}^{+ab}{}_{mn}Q^{+mn} . \tag{4.3'}$$

so that at $\mathscr{I}^+$ the integrals for energy-momentum and rotation momentum have finite limits:

$$P[Q,S]=\frac{1}{16\pi\kappa}\oint_S \Omega^{-1}\hat{C}^{+ab}{}_{mn}Q^{+mn}dS_{ab}+\text{c.c.} \qquad (4.4)$$

In the Appendix we define the physical components of Q^{+ab} as

$$Q^{+ab}=Q_0U^{ab}+Q_1M^{ab}+Q_2V^{ab}\ . \qquad (\text{A}11)$$

The equations for these components on $\mathscr{I}^+$ are given in the Appendix:

$$\begin{aligned}&\dot{Q}^0{}_2=0\ (\text{i}),\quad \eth Q^0{}_2+2\dot{Q}_1=0\ (\text{iii})\ ,\\ &\bar{\eth}Q^0{}_2=0\quad (\text{ii}),\quad \eth Q^0{}_0+2\sigma^0Q^0{}_1=0\ (\text{iv})\ ,\end{aligned} \qquad (4.5\text{a})$$

and

$$\dot{Q}^0{}_0+2\eth Q^0{}_1+2\sigma^0Q^0{}_2=0\ (\text{v})\ . \qquad (4.5\text{b})$$

[$\eth$ is the angular differential operator *edth* (Ref. 25) and the overdot represents a u derivative.] As noted in the Appendix, when the magnetic part[25] of σ^0 is zero and when $\dot{\sigma}^0=0$, these equations have a ten-parameter solution which is isomorphic to the Poincaré group. If (v) above is omitted, the remaining four equations define the Killing vectors of the BMS group and allow us to obtain the correct energy-momentum expression—that is, the Bondi-Sachs expression. The general solution of these four equations is

$$\begin{aligned}&Q^0{}_2=2L^m{}_{-1}Y_{1m},\quad \dot{L}^m=0\ ;\\ &Q^0{}_1=-\tfrac{1}{2}u\eth Q^0{}_2+T(\theta,\phi),\quad T=\bar{T}\ ;\\ &\eth Q^0{}_0=-2\sigma^0Q^0{}_1\ .\end{aligned} \qquad (4.6)$$

$T(x^A)$ could have an imaginary part, but that does not contribute to the Killing vector. Furthermore, its contribution to the charge integrals would be a two-surface divergence, and would vanish. On the other hand, L^m is complex and defines the Lorentz transformations.

In terms of the potentials, the asymptotic Killing vectors have the simple expressions

$$\begin{aligned}&\xi^0=Q^0{}_1+\bar{Q}^0{}_1\ ,\\ &\Omega\xi^2=\bar{Q}^0{}_2\ ,\\ &\Omega\xi^3=Q^0{}_2\ .\end{aligned} \qquad (4.7)$$

We see that when $Q^0{}_2=0$, we have the supertranslations $\xi^0=2T(x^A)$. In the solution for the stationary case, the supertranslations are missing because the solution depends explicitly on the form of σ^0 and in general σ^0 is changed by a supertranslation. The rotations are given by $Q^0{}_2\neq 0$, $Q^0{}_1+\bar{Q}^0{}_1=0$ and the boosts by $Q^0{}_1-\bar{Q}^0{}_1=0$.

Thus, we see that although we have not satisfied all the equations (4.5) we do obtain the Killing vectors for the full BMS group. The charge integrals are then

$$P[Q,S]=\frac{1}{8\pi\kappa}\oint_S[Q^0{}_0\Psi_3-2Q^0{}_1\Psi_2+Q^0{}_2\Psi_1]d\hat{S}+\text{c.c.} \qquad (4.8)$$

With $\Psi_3=\eth\bar{\sigma}^0$ this becomes

$$P[Q,S]=\frac{1}{8\pi\kappa}\oint_S[-2Q^0{}_1(\sigma^0\bar{\sigma}^0+\Psi_2)+Q^0{}_2\Psi_1]d\hat{S}+\text{c.c.} \qquad (4.9)$$

This integral gives the correct expression for energy-momentum, but it is missing a term in $\eth^2\sigma^0$ for the supermomenta. Terms in σ are also missing in the expressions for rotation momenta.

To understand at least in part what is wrong, assume that we are using a Bondi frame—the surfaces $u=$const on $\mathscr{I}^+$ are connected by a rigid time translation. Consider two-surfaces S and $\tilde{S}$, separated by a supertranslation, that is they are defined by $u=u_0$ and $u=u_0+\alpha(x^A)$, respectively. Equation (4.9) gives the expression on S. The tetrad is adapted to the surface in the sense that the vectors $\hat{k}_A{}^a$ are tangent to S. This tetrad is not adapted to $\tilde{S}$ and so there are correction terms which involve $\eth\alpha$. However, if one carries out a null rotation to a tetrad adapted to $\tilde{S}$, we find

$$P[\tilde{Q},\tilde{S}]=\frac{1}{8\pi\kappa}\oint_{\tilde{S}}(\tilde{Q}^0{}_0\tilde{\Psi}_3-2\tilde{Q}^0{}_1\tilde{\Psi}_2+\tilde{Q}^0{}_2\tilde{\Psi}_1)d\tilde{S}+\text{c.c.} \qquad (4.10)$$

At first this looks like (4.8). However, the $\tilde{Q}$'s are solutions of (4.5a) in the frame adapted to S and not to $\tilde{S}$. They are not solutions of (4.5a) in the frame adapted to $\tilde{S}$ unless they are also solutions of (4.5b), which is possible only if $\sigma^0=0$. It is not only supertranslations which cause trouble, but also the rigid spatial translations. The energy-momentum and rotation momentum are defined only with respect to a given Bondi frame. That certainly is not satisfactory.

The flux term

$$\mathcal{F}[Q,\Sigma]=-\frac{1}{8\pi\kappa}\oint_\Sigma(\dot{Q}^0{}_0-2\eth Q^0{}_1+2\sigma^0Q^0{}_2)\Psi_3du\ dS\ , \qquad (4.11)$$

where Σ is the region on $\mathscr{I}^+$ bounded by S and $\tilde{S}$, does not give a hint about how the situation is to be corrected.

Let me emphasize that this result at null infinity does not depend on the choice of a particular Bondi frame. The point is that our goal was to express the asymptotic symmetries in terms of Killing vector potentials Q^{ab} which are defined as asymptotic solutions of $\mathcal{P}^{abc}=0$. Although such solutions do not exist in the presence of gravitational radiation, the hope was that one could solve enough of these equations to recover the BMS group and to obtain correct expressions for energy-momentum and rotation momentum. We have seen that by selecting an appropriate subset of the asymptotic equations $\mathcal{P}^{abc}=0$ the BMS generators can be recovered. However, the potentials so obtained have two disabilities: (1) If they satisfy the "appropriate subset" in one Bondi frame, they do not satisfy the same subset in another Bondi frame; (2) even in a specific Bondi frame, the charge integrals (4.8) do not give the accepted results. These two problems are related. Formally the Q^{ab} are tensors of rank 2. However, the solutions of the "selected subset" do not transform tensorially. Another algorithm for obtaining bivectors Q^{ab} which in some way are associated with the BMS group together with the Penrose superpotential may yet give us the correct expressions.

V. DISCUSSION

The main point of this work has been to emphasize the need for a global tensorial differential conservation law which connects to the Einstein equations by Eq. (1.1). At this time there is no completely satisfactory expression.[26] Ashtekar and Winicour[27] have given a list of properties which conserved quantities defined at i^0 or $\mathcal{J}^+$ should have:

(1) P should be linear in the BMS vector field ξ^a—and by extension in this discussion, in the potential Q^{ab}.

(2) The expression for $P[Q,S]$ should involve only those fields which can be constructed from the knowledge of $\xi^a(Q^{ab})$, Ω, and $\hat{g}_{ab}$ in an arbitrarily small neighborhood of S.

(3) When $\xi^a(Q^{ab})$ is a BMS translation, P should be the corresponding component of the Bondi four-momentum evaluated at S.

(4) If $\xi^a(Q^{ab})$ is the restriction to $\mathcal{J}^+(\mathcal{H})$ of a physical space Killing vector (potential), P should be proportional to the corresponding Komar integral.

(5) In Minkowski space, P should vanish for all BMS vector fields (potentials) and cross sections S.

(6) There should exist a local flux integral $[\mathcal{F}|\xi(Q),\Sigma]$, which is linear in $\xi^a(Q^{ab})$ and which gives the difference $P[\xi(Q),S']-P[\xi(Q),S]$, for S and S' closed two-surfaces on $\mathcal{J}^+(\mathcal{H})$.

(7) $\mathcal{F}$ should vanish in the absence of gravitational radiation.

As already noted, the Komar tensor as applied by Winicour with Tamburino[11] and with Geroch[12] fails on conditions (5) and (7) with respect to rotations. In addition, there are problems with the relative normalization of the energy-momentum and the rotation momentum. The Penrose potential, which we have been discussing here, fails only the very important condition (2) at null infinity. The quantities can be defined by integration over an arbitrary surface, but they belong to a particular Bondi frame. Therefore, corresponding quantities defined on different surfaces, in general, are not related to one another by a BMS transformation. Some correction terms are needed which take into account the shear of the null rays incident on two-surface cross sections of $\mathcal{J}^+$.

As noted previously, Komar[18] used the conformal tensor in vacuum space-time to understand the spinor form of the Penrose energy-momentum at null infinity. He also transvects a bivector with the conformal tensor to define the superpotential U^{ab}. However, his bivector has a totally antisymmetric covariant derivative, $\nabla_a r_{bc}=\nabla_{[a}r_{bc]}$. One might think that r_{bc} is the dual of Q^{ab} defined here. But Q^{*ab} satisfies the same equation as Q^{ab} and therefore does not have a totally antisymmetric covariant derivative. The author also used similar constructions in an attempt to describe multipoles[28] and energy-momentum[29] at $\mathcal{J}^+$. How these different uses of the conformal tensor and a bivector are related is yet to be understood.

However, work by Streubel,[30] Dray and Streubel,[14] Shaw,[15] and Dray[16] has resulted in expressions constructed on $\mathcal{J}^+$ which do satisfy all seven of the Ashtekar-Winicour criteria. But, they are not derived from a (tensorial) differential conservation law. The construction is based on work by Streubel similar to the discussion at the end of Sec. IV. Instead of taking the asymptotic solutions of $\mathcal{P}^{abc}$, Streubel takes the Minkowski space solutions. He then applies a supertranslation to the frame and identifies terms which could be added to the Penrose tensor which would yield supertranslation invariance. Then, he and Dray were able to show that the expressions derived from the Penrose tensor are related to the modified form by a complex supertranslation. Then Shaw and Dray independently showed that the flux of energy-momentum and rotation momentum so defined is given by the flux integrals, previously defined by Ashtekar and Streubel,[13] which have the properties (6) and (7) above. Moreschi is able to carry out a similar construction.[19]

Therefore, it appears that satisfactory expressions for energy-momentum and rotation momentum exist at both i^0 and $\mathcal{J}^+$. However, the problem remains to complete our understanding at null infinity. This requires the construction of a new rule for obtaining the bivectors Q^{ab} and possibly also determining additional terms for a tensorial differential conservation law leading to the expressions of Dray and Streubel. Then we would understand the origin of the shear terms in the supermomenta and the rotation momentum. Work on this is proceeding.

ACKNOWLEDGMENTS

I have been stimulated to think about this problem by listening to Roger Penrose talk about quasilocal energy. My understanding of the problem developed through many discussions with him, Abhay Ashtekar, and Jeff Winicour. I am also grateful to William Shaw for sending me a copy of his dissertation and reprints of the papers which followed. This research has been supported in part by the National Science Foundation under Grant No. PHY-8700651.

APPENDIX

1. Spatial infinity

In this section we shall first derive the equations satisfied by the conformal tensor in the form needed in Sec. III. Then we shall indicate how the equations $\hat{\mathcal{P}}^{abc}=0$ are decomposed. This will be done in the vicinity of spatial infinity when $R_{ab}=0$.

From the Bianchi identities we have

$$\begin{aligned} 3\nabla_{[a}C_{bc]r}{}^{s} &= 3\hat{\nabla}_{[a}\hat{C}_{bc]r}{}^{s}-3(C^{s}{}_{k[a}\hat{C}_{bc]r}{}^{k}-C^{k}{}_{r[a}\hat{C}_{bc]k}{}^{s}) \\ &=0 \\ &=3\hat{\nabla}_{[a}\hat{C}_{bc]r}{}^{s}+6\Omega^{-1/2}(\hat{g}_{r[a}\hat{C}_{bc]}{}^{s}{}_{k}-\delta^{s}{}_{[a}\hat{C}_{bc]rk})\eta^{k} \\ &=0\ ; \end{aligned} \tag{A1}$$

$$C^{s}{}_{ka}=2\Omega^{-1/2}(2\delta^{s}{}_{(k}\eta_{a)}-\hat{g}_{ka}\eta^{s})\ ,$$

where we have used $\eta_a=\frac{1}{2}\Omega^{-1/2}\hat{\nabla}_a\Omega$. Introduce $K_{bcr}{}^s:=\Omega^{1/2}\hat{C}_{bcr}{}^s$ which has a direction-dependent limit at i^0. Multiplying by $\Omega^{1/2}$ we have

$$3\Omega^{1/2}\hat{\nabla}_{[a}K_{bc]r}{}^s-3\eta_{[a}K_{bc]r}{}^s+6(\hat{g}_{r[a}K_{bc]}{}^s{}_k-\delta^s{}_{[a}K_{bc]rk})\eta^k=0\ . \tag{A2}$$

Transvect with $\eta_s\eta^c$ to obtain

$$\Omega^{1/2}(\hat{\nabla}_aE_{br}-\hat{\nabla}_bE_{ar})+\Omega^{1/2}\eta^c\hat{\nabla}_cK_{abr}{}^s\eta_s-\Omega^{1/2}\eta^c\hat{\nabla}_c\eta_sK_{abr}{}^s=0\ , \tag{A3}$$

$$E_{br}=K_{bcrs}\eta^c\eta^s\ .$$

To the order required, we can neglect the third term in the above equation and treat the second term as follows. Write

$$K_{abr}{}^s={}^0K_{abr}{}^s+\Omega^{1/2}\,{}^1K_{abr}{}^s+O(\Omega)$$

and

$$E_{br}={}^0E_{br}+\Omega^{1/2}\,{}^1E_{br}+O(\Omega)\ . \tag{A4}$$

Project (A3) onto the hyperboloid $\mathcal{H}$ and pass to the limit $\Omega\to 0$ to obtain

$$2\mathbf{D}_{[\mathbf{a}}{}^0\mathbf{E}_{\mathbf{b}]\mathbf{r}}=0\ . \tag{A5}$$

Subtract (A5) from (A3), divide by $\Omega^{1/2}$, and pass to the limit $\Omega\to 0$ to obtain

$$2\mathbf{D}_{[\mathbf{a}}{}^1E_{\mathbf{b}]\mathbf{r}}+{}^1\mathbf{K}_{\mathbf{abr}}{}^{\mathbf{s}}\eta_{\mathbf{s}}=0\ . \tag{A6}$$

In addition we need the following identity:

$$D_c[2E^{[a}{}_bh^{c]}{}_j\omega^{bj}]\equiv D_{[c}E^a{}_{b]}\omega^{bc}-D_cE^c{}_b\omega^{ba}-E^a{}_b\omega^{bj}\eta_j\ , \tag{A7}$$

which holds for both ${}^0\mathbf{E}_{\mathbf{br}}$ and ${}^1\mathbf{E}_{\mathbf{br}}$. This identity together with (A5) and (A6) is needed to show that the rotation momentum integrals are finite and depend only on the magnetic part of the conformal tensor.

The equations $\hat{\mathcal{P}}^{abc}=0$ also must be decomposed relative to η^a and projected onto $\mathcal{H}$. From Eqs. (3.4) we have that

$$\Omega^{-1/2}\hat{Q}^{ab}=T^{ab}+T^a\eta^b-T^b\eta^a\quad\text{for translations}\ , \tag{3.4a}$$

$$\hat{Q}^{ab}=L^{ab}+L^a\eta^b-L^b\eta^a\quad\text{for rotations}\ . \tag{3.4b}$$

T^{ab}, T^a, L^{ab}, and L^a are tangent to $\mathcal{H}$. For example, $\eta_aT^{ab}=0$. Also, each term has a direction-dependent limit at i^0.

For the translations we find

$$\hat{\nabla}^a\hat{Q}^{bc}=\mathbf{D}^\mathbf{a}\mathbf{T}^{\mathbf{bc}}+2\mathbf{T}^{[\mathbf{b}}\mathbf{h}^{\mathbf{c}]\mathbf{a}}+2(\mathbf{D}^\mathbf{a}\mathbf{T}^{[\mathbf{b}}+\mathbf{T}^{\mathbf{a}[\mathbf{b}})\eta^{\mathbf{c}]}+\eta^\mathbf{a}(\mathbf{T}^{\mathbf{bc}}+2\eta^{[\mathbf{b}}\mathbf{T}^{\mathbf{c}]})\ .$$

From this we have, for $\hat{\mathcal{P}}^{abc}$,

$$\mathcal{P}^{abc}=\mathbf{T}^{\mathbf{abc}}+\eta^\mathbf{a}(\mathbf{D}^\mathbf{b}\mathbf{T}^\mathbf{c}-\mathbf{D}^\mathbf{c}\mathbf{T}^\mathbf{b})+2\eta^{[\mathbf{c}}(2\mathbf{D}^{|\mathbf{a}|}\mathbf{T}^{\mathbf{b}]}-\mathbf{D}^{\mathbf{b}]}\mathbf{T}^\mathbf{a}-\mathbf{h}^{\mathbf{b}]\mathbf{a}}\mathbf{D}_\mathbf{e}\mathbf{T}^\mathbf{e})+2\eta^\mathbf{a}\eta^{[\mathbf{b}}\mathbf{D}_\mathbf{e}\mathbf{T}^{\mathbf{c}]\mathbf{e}}\ ,$$

$$\mathbf{T}^{\mathbf{abc}}:=2\mathbf{D}^{(\mathbf{a}}\mathbf{T}^{\mathbf{b})\mathbf{c}}+2\mathbf{D}^{(\mathbf{a}}\mathbf{T}^{\mathbf{c})\mathbf{b}}-2\mathbf{h}^{\mathbf{a}[\mathbf{b}}\mathbf{D}_\mathbf{e}\mathbf{T}^{\mathbf{c}]\mathbf{e}}\ .$$

$\hat{\mathcal{P}}^{abc}=0$ then gives Eq. (3.5). A similar calculation for the rotations gives

$$\Omega^{1/2}\hat{\nabla}^a\hat{Q}^{bc}=\mathbf{D}^\mathbf{a}\mathbf{L}^{\mathbf{bc}}+2\mathbf{h}^{\mathbf{a}[\mathbf{c}}\mathbf{L}^{\mathbf{b}]}+2\eta^{[\mathbf{c}}(\mathbf{D}^{|\mathbf{a}|}\mathbf{L}^{\mathbf{b}]}-\mathbf{L}^{\mathbf{b}]\mathbf{a}})$$

and

$$\Omega^{1/2}\hat{\mathcal{P}}^{abc}=\mathbf{L}^{\mathbf{abc}}+2\mathbf{L}^{[\mathbf{b}}\mathbf{h}^{\mathbf{c}]\mathbf{a}}-\eta^\mathbf{a}(\mathbf{D}^\mathbf{b}\mathbf{L}^\mathbf{c}-\mathbf{D}^\mathbf{c}\mathbf{L}^\mathbf{b}+2\mathbf{L}^{\mathbf{bc}})+2\eta^{[\mathbf{c}}(\mathbf{D}^{|\mathbf{a}|}\mathbf{L}^{\mathbf{b}]}-\mathbf{D}^\mathbf{b}\mathbf{L}^\mathbf{a}-\mathbf{L}^{\mathbf{b}]\mathbf{a}}-\mathbf{h}^{\mathbf{b}]\mathbf{a}}\mathbf{D}_\mathbf{e}\mathbf{L}^\mathbf{e})+2\eta^\mathbf{a}\eta^{[\mathbf{b}}(\mathbf{D}_\mathbf{e}\mathbf{L}^{\mathbf{c}]\mathbf{e}}+2\mathbf{L}^{\mathbf{c}]})\ ,$$

where L^{abc} is defined by substituting L^{ab} for T^{ab} in the above equation for T^{abc}. Setting the above equation to zero gives Eqs. (3.7).

2. Null infinity

Introduce a null tetrad[24] $k^\mathbf{a}$ and the conformally related tetrad $\hat{k}^\mathbf{a}$ ($\mathbf{a}=0,\ldots,3$; $\mathbf{A}=2,3$)

$$\begin{aligned}&k^0=du=\hat{k}^0\ ,\\&k^1=\tfrac{1}{2}e^{2b}(\phi du+2dr)=\Omega^{-2}\hat{k}^1\ ,\\&k^\mathbf{A}=\xi^\mathbf{A}{}_A(dx^A-U^Adu)=\Omega^{-1}\hat{k}^\mathbf{A}\ .\end{aligned} \tag{A8}$$

Introduce a self-dual bivector basis as

$$\begin{aligned}&V^{ab}=2k_1{}^{[a}k_3{}^{b]}=\Omega^3\hat{V}^{ab},\quad U^{ab}=2k_2{}^{[a}k_0{}^{b]}=\Omega\hat{U}^{ab}\ ,\\&M^{ab}=2k_1{}^{[a}k_0{}^{b]}-2k_3{}^{[a}k_2{}^{b]}=\Omega^2\hat{M}^{ab}\ .\end{aligned} \tag{A9}$$

Then

$$\begin{aligned}\Omega^{-1}\hat{C}^{ab}{}_{cd}=&-\Psi_0\hat{U}^{ab}\hat{U}_{cd}+\Psi_1(\hat{U}^{ab}\hat{M}_{cd}+\hat{M}^{ab}\hat{U}_{cd})\\&-\Psi_2(\hat{M}^{ab}\hat{M}_{cd}-\hat{U}^{ab}\hat{V}_{cd}-\hat{V}^{ab}\hat{U}_{cd})\\&+\Psi_3(\hat{V}^{ab}\hat{M}_{cd}+\hat{M}^{ab}\hat{V}_{cd})-\Psi_4\hat{V}^{ab}\hat{V}_{cd}\end{aligned} \tag{A10}$$

and

$$\hat{Q}^{+ab}=Q_0\hat{U}^{ab}+Q_1\hat{M}^{ab}+Q_2\hat{V}^{ab}\ . \tag{A11}$$

As defined, the Ψ's and Q's have finite limits on $\mathcal{I}^+$ and give the leading behavior of the conformal tensor and Killing potential.

There are eight complex equations $\hat{\mathcal{P}}^{abc}=0$, three of which tell us that the Q's have finite limits on $\mathcal{I}^+$. The remaining five equations become, on $\mathcal{I}^+$ (the overdot represents a time derivative),

$$\begin{aligned}&\dot{Q}^0{}_2=0\ \text{(i)},\quad \eth Q^0{}_2+2\dot{Q}^0{}_1=0\ \text{(iii)}\ ,\\&\bar{\eth}Q^0{}_2=0\ \text{(ii)},\quad \eth Q^0{}_0+2\sigma^0Q^0{}_1=0\ \text{(iv)}\ ,\\&\dot{Q}^0{}_0+2\eth Q^0{}_1+2\sigma^0Q^0{}_2=0\ \text{(v)}\ .\end{aligned} \tag{A12}$$

In stationary space-times, $\dot{\sigma}^0=0$, these equations have solutions[4,18] which, in general, depend explicitly on the form of $\sigma^0=\eth^2\alpha$. These solutions do not contain the supertranslations which generate a change in the shear σ^0.

shear σ^0. When $\dot{\sigma}^0 \neq 0$, in the presence of gravitational radiation, the integrability conditions for these equations cannot be satisfied.

The generality of the tetrad in Eq. (A8) is limited by the omission of a lapse function in k^0 and the choice of the dual vector k_1 as a tangent vector to the generators of the null surface u =const. However, the calculation of the equations in (A12) was carried out in a Bondi frame.

[1]R. Penrose, Proc. R. Soc. London **A381**, 53 (1982).

[2]R. Penrose and W. Rindler, *Spinors and Space-time* (Cambridge University Press, Cambridge, England, 1986), Vol. 2.

[3]K. P. Tod, Proc. R. Soc. London **A388**, 457 (1983).

[4]W. T. Shaw, Proc. R. Soc. London **A390**, 191 (1983).

[5]W. T. Shaw, Class. Quantum Grav. **3**, 1069 (1986).

[6]J. N. Goldberg, Phys. Rev. **111**, 315 (1958).

[7]P. G. Bergmann, in *Encyclopedia of Physics,* edited by S. Flugge (Springer, Berlin, 1962).

[8]P. Freud, Ann. Math. (Princeton) **40**, 417 (1939).

[9]C. Møller, Ann. Phys. (N.Y.) **4**, 347 (1958).

[10]A. Komar, Phys. Rev. **113**, 934 (1959).

[11]L. Tamburino and J. Winicour, Phys. Rev. **150**, 1039 (1966).

[12]R. Geroch and J. Winicour, J. Math. Phys. **22**, 803 (1981).

[13]A. Ashtekar and M. Streubel, Proc. R. Soc. London **A376**, 585 (1981).

[14]T. Dray and M. Streubel, Class. Quantum Grav. **1**, 15 (1984).

[15]W. T. Shaw, Class. Quantum Grav. **1**, L33 (1986).

[16]T. Dray, Class. Quantum Grav. **2**, L7 (1985).

[17]R. Penrose, Ann. Phys. (N.Y.) **10**, 171 (1960).

[18]A. Komar, Phys. Rev. **127**, 955 (1962).

[19]O. Moreschi, Class. Quantum Grav. **3**, 503 (1986).

[20]R. Penrose, Proc. R. Soc. London **A284**, 159 (1965).

[21]A. Ashtekar and R. O. Hansen, J. Math. Phys. **19**, 1542 (1978).

[22]A. Ashtekar, in *General Relativity and Gravitation,* edited by A. Held (Plenum, New York, 1980), Vol. 2.

[23]H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, Proc. R. Soc. London **A269**, 21 (1962).

[24]R. K. Sachs, Proc. R. Soc. London **270**, 103 (1962).

[25]E. T. Newman and R. Penrose, J. Math. Phys. **7**, 863 (1967).

[26]J. Winicour, in *Highlights in Gravitation and Cosmology,* proceedings of the International Conference, Goa, India, 1987, edited by B. R. Iyer, A. Kembhavi, J. V. Narlikar, and C. V. Vishveshiwara (Cambridge University Press, Cambridge, England, 1988).

[27]A. Ashtekar and J. Winicour, J. Math. Phys. **23**, 2410 (1982).

[28]J. N. Goldberg, Phys. Rev. **131**, 1367 (1963).

[29]J. N. Goldberg, in *General Relativity and Gravitation,* edited by A. Held (Plenum, New York, 1980), Vol. 1.

[30]M. Streubel, Gen. Relativ. Gravit. **9**, 551 (1978).