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Stability of black holes in de Sitter space

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The theory of black-hole perturbations is extended to charged black holes in de Sitter space. These spacetimes have wormholes connecting different asymptotic regions. It appears that, at least in some cases, these holes are stable even at the Cauchy horizon. It follows that they violate cosmic censorship and an observer could in principle travel through the black hole to another universe. The stability of these spacetimes also implies the existence of a cosmological "no hair" theorem.

I. INTRODUCTION

The stability of black holes has long been an important issue in general relativity. Early perturbation calculations[1] began with the aim of understanding the final state of gravitational collapse, when it was not known how deviations from spherical symmetry would affect the final outcome. Later, with the discovery of the separability of the perturbation equations, it became possible to reduce the perturbation problem to the theory of potential scattering.[2,3] This can be used in many ways, including showing that black holes are stable in the sense that perturbations remain bounded as they evolve.

Besides these perturbative results there are a number of results that go by the name of "no-hair" theorems.[4] Broadly, these state that all stationary black holes are axially symmetric and that static (i.e. nonrotating) holes are characterized by their mass and charge alone. Together, these results give a picture of realistic gravitational collapse leading to black holes with the Reissner-Nordström metric or its rotating generalization.

With charged or rotating black holes it is now believed that perturbations grow without limit inside the event horizon as they approach a Cauchy horizon.[5] This has important consequences for the global structure of these spacetimes, which in the unperturbed state have wormholes or bridges between separate asymptotic regions. Any observer who tries to cross one of these bridges by traveling into the black hole would be stopped at the Cauchy horizon. This perturbation growth can be explained by the fact that an observer sees the entire future of the Universe outside in only a finite time interval. The radiation coming from outside becomes infinitely blue-shifted, compressed to an infinite energy density on the horizon.[6]

This argument has to be changed if the black hole is part of a closed universe with a cosmological event horizon, for then an observer at the Cauchy horizon sees only the limited number of events which lie within the cosmological horizon. We shall show how this affects the stability of a charged black hole in de Sitter space with a cosmological constant.

The reason for choosing this particular model is that all of the known black-hole spacetimes can be generalized to include a cosmological constant.[7] These metrics have been largely ignored because of a lack of physical significance, but there has been renewed interest in a cosmological constant because one is present in the inflationary scenario of the very early Universe. They also arise in the theory of quantum wormholes, though this is beyond the scope of the present paper.

In the special case that the black-hole mass and charge are set equal to zero these metrics reduce to de Sitter space in static coordinates. The stability analysis still applies to this case enabling us to use this in Sec. III to demonstrate the stability of the inflationary scenario. This leads us naturally to consider cosmological "no hair" theorems, which state that the universe approaches de Sitter space for infinite ranges of initial conditions.[8] These theorems underlie the ability of inflationary scenarios to explain the large-scale homogeneity of the universe. Most of them make assumptions which preclude the existence of black holes. In Sec. III we are able to demonstrate stability in the presence of black holes.

The behavior of perturbations near the Cauchy horizon of a charged black hole in de Sitter space depends on the blueshift effect but also upon the way in which modes are scattered between the two black-hole horizons. The existence of poles in the reciprocal transmission coefficient $T^{-1}(\sigma)$ has been used to show that the Cauchy horizon is unstable.[5] However, this argument fails to allow for the possibility that the perturbations entering the black hole from outside may have a zero in their amplitude which cancels the relevant pole in $T^{-1}(\sigma)$. On the face of it this seems to be a very unlikely occurrence, but this is exactly what happens to the black hole in de Sitter space. The only obstacle to the stability of the Cauchy horizon in the cosmological context would be the position of poles in the transmission coefficients of modes outside the hole. These poles are the frequencies of quasinormal modes, or "pure-ringing" tones of the black hole. In Sec. IV we give what we believe to be convincing numerical evidence that these modes do not hinder the stability of the Cauchy horizon for black holes whose charge and mass are equal or lie within a narrow range of each other.

The results on black-hole stability can also be applied to the case of gravitational collapse, where there is a black-hole metric outside the collapsing body. It therefore seems possible in principle to have a spacetime form from the collapse of a charged body and have the spacetime regular on the Cauchy horizon. In these spacetimes an observer who is beyond the Cauchy horizon can look back and see a spacetime singularity. We have to go beyond the classical theory of gravity to determine what boundary conditions to apply in this situation, because there is a breakdown in predictability of the classical theory. These spacetimes are counterexamples to the cosmic censorship hypothesis,[9] which states that naked singularities never form in real physical situations.

The quantum properties of charged black holes in de Sitter space also have some interesting features. Both black holes and de Sitter space produce thermal radiation due to the Hawking process.[10] The temperature of the radiation is given by $\kappa/2\pi$ on the respective horizon. The black hole can only be in stable thermal equilibrium with de Sitter space when the temperatures are the same. This happens in the extreme case when the charge and mass of the black hole are equal.[11] In this case the spacetime is both classically and quantum-mechanically stable.

II. PERTURBATIONS

In our analysis of black-hole stability to gravitational perturbations, we shall adopt a procedure analogous to that of Chandrasekhar[3] in his study of the metric perturbations of the Reissner-Nordström solution. We shall generalize the calculation by the introduction of a nonzero cosmological constant, Λ. We therefore take as our model a charged black hole in de Sitter space, whose metric is given by

$$ds^2=e^{2\nu}dt^2-e^{2\psi}d\phi^2-e^{2\mu_2}dr^2-e^{2\mu_3}d\theta^2 , \tag{1}$$

where

$$e^{2\nu}=\frac{\Delta}{r^2},\quad e^{2\mu_2}=\frac{r^2}{\Delta},\quad e^{2\mu_3}=r^2,\quad e^{2\psi}=r^2\sin^2\theta \tag{2}$$

and

$$\Delta=r^2-2Mr+Q^2-\tfrac{1}{3}\Lambda r^3 . \tag{3}$$

The metric has coordinate singularities at the three positive roots of $\Delta=0$. However, by a series of suitable coordinate transformations, it is possible extend the spacetime through any of these horizons. We first transform to coordinates analogous to those of Eddington and Finkelstein. Thus, for the outer horizon, we introduce an outgoing null coordinate $u=t-r^*$ and an ingoing null coordinate $v=t+r^*$, where r^* is given by $dr^*=\Delta^{-1}r^2dr$. On the outer black-hole event horizons $u\to\infty$ or $v\to-\infty$. The metric can be extended across these horizons by using

$$V=e^{\kappa v},\quad U=-e^{-\kappa u} \tag{4}$$

where κ is the surface gravity on the horizon. We may use similar transformations for the inner horizon, but this time $u=r^*+t$ and $v=r^*-t$.

In this manner we may analytically continue the metric through all the horizons. It is then possible to represent the extended spacetime by a Penrose diagram, as shown in Fig. 1. From this it can clearly be seen that the extended spacetime possesses many asymptotic regions. Travel from one asymptotic region to another would necessitate an observer crossing both the black-hole event horizon and the Cauchy horizon. Since the black hole connects different asymptotic regions it may also be regarded as a wormhole.

Black holes are only properly defined in spacetimes which are asymptotically flat.[12] For a spacetime with multiple asymptotic regions, it is well to clarify what we mean by the "horizon." In our particular example there are three types of horizon. The black-hole event horizon we define to be the boundary of the past of a given asymptotic region. The Cauchy horizon forms the boundary of regions which can be reached only from some initial surface. In addition, there are cosmological event horizons defined along the boundary of the past of an observer's world line. These definitions can clearly be applied to any spacetime which has a global structure similar to Fig. 1.

The perturbed metric will be taken to be

$$ds^2=e^{2\nu}dt^2-e^{2\psi}(d\phi-wdt-q_2dr-q_3d\theta)^2 -e^{2\mu_2}dr^2-e^{2\mu_3}d\theta^2 . \tag{5}$$

By observing the effect of a change of sign in ϕ, it can be seen that the perturbations fall into two orthogonal classes. The axial perturbations are the quantities ω, q_2, and q_3; while the polar perturbations are the small increments $\delta\nu$, $\delta\mu_2$, $\delta\mu_3$ and $\delta\Psi$ of the functions ν_1, μ_2, μ_3, and Ψ, respectively.

Following Chandrasekhar, we shall consider the perturbations in the Ricci tensor together with the linearized Maxwell equations. We set $F_{\rm tr}=-Q/r^2$ as the only non-vanishing component of the unperturbed Maxwell tensor. For simplicity we adopt a tetrad formalism with signature $(+---)$. No explicit cosmological-constant term appears in the perturbed Einstein equations, which are given by

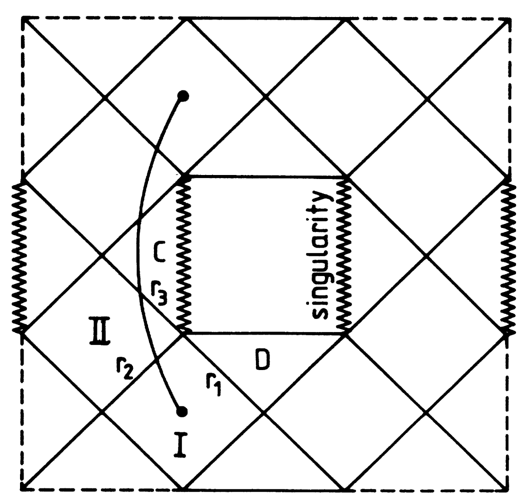

FIG. 1. Penrose diagram for a charged black hole in de Sitter space.

$$\delta R_{ab}=-2[\eta^{nm}(\delta F_{an}F_{bm}+F_{an}\delta F_{bm})-\eta_{ab}Q\delta F_{tr}/r^2] . \tag{6}$$

For both the axial and polar perturbations, we let the time dependence vary as $e^{i\sigma t}$. In the case of the axial perturbations, we further separate variables by introducing

$$A(r,\theta)=\Delta(q_{2,\theta}-q_{3,r})\sin^3\theta=A(r)C_{l+2}^{-3/2}(\theta) , \tag{7}$$

$$B(r,\theta)=F_{t\phi}\sin\theta=3B(r)C_{l+1}^{-1/2}(\theta) , \tag{8}$$

where $C_{l+2}^{-3/2}(\theta)$ and $C_{l+1}^{-1/2}(\theta)$ are Gegenbauer functions.

Those Einstein-Maxwell equations governing the axial perturbations then reduce to

$$r^2e^{2\nu}\left[\frac{e^{2\nu}A_{,r}}{r^2}\right]_{,r}-\mu^2\frac{e^{2\nu}}{r^2}A+\sigma^2A=-\frac{4Q}{r}\mu^2e^{3\nu}B ,$$

$$[e^{2\nu}(re^{2\nu}B)_{,r}]_{,r}-(\mu^2+2)\frac{e^{\nu}}{r}B+\left[\sigma^2re^{-\nu}-\frac{4Q^2}{r^3}e^{\nu}\right]B=-Q\frac{A}{r^4} , \tag{9}$$

where

$$\mu^2=2n=(l-1)(l+2) . \tag{10}$$

These two equations may be decoupled by the substitutions

$$Z_1^-=p_1H_1^-+(-p_1p_2)^{1/2}H_2^- ,$$

$$Z_2^-=-(-p_1p_2)^{1/2}H_1^-+p_1H_2^- , \tag{11}$$

where

$$H_1^-=-2\mu re^{\nu}B, \quad H_2^-=A(r)/r \tag{12}$$

and

$$p_1=3M+(9M^2+4Q^2\mu^2)^{1/2}$$

$$p_2=3M-(9M^2+4Q^2\mu^2)^{1/2} . \tag{13}$$

This yields two one-dimensional Schrödinger-type wave equations of the form

$$\Lambda^2Z_i^-=V_i^-Z_i^- \quad (i=1,2) \tag{14}$$

with

$$V_i^-=\frac{\Delta}{r^5}\left[(\mu^2+2)r-p_j\left[1+\frac{p_i}{\mu^2r}\right]\right] \quad (i,j=1,2;\ i\neq j) \tag{15}$$

and

$$\Lambda^2=\frac{d^2}{dr^{*2}}+\sigma^2 . \tag{16}$$

We follow a similar series of steps for the polar perturbations. The variables r,θ may, in this case, be separated by the substitutions

$$\begin{aligned}
&\delta\nu=N(r)P_l(\theta) ,\\
&\delta\mu_2=L(r)P_l(\theta) ,\\
&\delta\mu_3=T(r)P_l+V(r)P_{l,\theta,\theta} ,\\
&\delta\psi=T(r)P_l+V(r)P_{l,\theta}\cot\theta ,\\
&\delta F_{tr}=\frac{r^2e^{2\nu}}{2Q}B_{tr}(r)P_l ,\\
&F_{t\theta}=-\frac{re^{\nu}}{2Q}B_{t\theta}(r)P_{l,\theta} ,\\
&F_{r\theta}=-\frac{i\sigma re^{-\nu}}{2Q}B_{r\theta}(r)P_{l,\theta} ,
\end{aligned} \tag{17}$$

where $P_l(\theta)$ are the Legendre polynomials. This yields the following set of equations for the radial functions:

$$\begin{aligned}
&N_{,r}=aN+bL+c(nV-B_{r\theta}) ,\\
&L_{,r}=(a-1/r+\nu_{,r})N+(b-1/r-\nu_{,r})L\\
&\qquad+c(nV-B_{r\theta})-\frac{2}{r}B_{r\theta} ,\\
&nV_{,r}=-(a-1/r+\nu_{,r})N-(b+1/r-2\nu_{,r})L\\
&\qquad-(c+1/r-\nu_{,r})(nV-B_{r\theta})+B_{t\theta} ,\\
&B_{t\theta}=B_{r\theta_{,r}}+\frac{2}{r}B_{r\theta} ,\\
&r^4e^{2\nu}B_{tr}=2Q^2[2T-l(l+1)V]-l(l+1)r^2B_{r\theta} ,\\
&(r^2e^{2\nu}B_{t\theta})_{,r}+r^2e^{2\nu}B_{tr}+\sigma^2r^2e^{-2\nu}B_{r\theta}=2Q^2\frac{N+L}{r} ,
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
&a=\frac{n+1}{r}e^{2\nu} ,\\
&b=-\frac{1}{r}+\nu_{,r}+r\nu_{,r}^2+\sigma^2e^{-4\nu}r-2\frac{e^{-2\nu}}{r^3}Q^2-\frac{ne^{-2\nu}}{r} ,\\
&c=-\frac{1}{r}+r\nu_{,r}^2+\sigma^2e^{-4\nu}r-\frac{2e^{-2\nu}}{r^3}Q^2+\frac{e^{-2\nu}}{r} .
\end{aligned} \tag{19}$$

As in the case for the axial perturbations, it is possible to decouple these equations by a suitable choice of substitutions. By analogy with the axial equations we put

$$\begin{aligned}
&Z_1^+=p_1H_1^++(-p_1p_2)^{1/2}H_2^+ ,\\
&Z_2^+=-(-p_1p_2)^{1/2}H_1^++p_1H_2^+ ,
\end{aligned} \tag{20}$$

where p_1 and p_2 are as above, but H_1^+ and H_2^+ are given by

$$\begin{aligned}
&H_1^+=-\frac{1}{Q\mu}\left[r^2B_{r\theta}+2Q^2\frac{r}{\varpi}(L+nV-B_{r\theta})\right] ,\\
&H_2^+=rV-\frac{r^2}{\varpi}(L+nV-B_{r\theta}) ,
\end{aligned} \tag{21}$$

where

$$\varpi=nr+3M-\frac{2Q^2}{r} . \tag{22}$$

After some remarkable cancellations we again obtain two one-dimensional Schrödinger equations

$$\Lambda^2 Z_i^+ = V_i^+ Z_i^+ \quad (i=1,2) \tag{23}$$

with potentials

$$\begin{aligned} V_1^+ &= \frac{\Delta}{r}[U+\tfrac{1}{2}(p_1-p_2)W] , \\ V_2^+ &= \frac{\Delta}{r}[U-\tfrac{1}{2}(p_1-p_2)W] , \end{aligned} \tag{24}$$

where

$$\begin{aligned} U &= (2nr+3M)W+(\varpi-nr-M-\tfrac{2}{3}\Lambda r^3)-\frac{2nr^2}{\varpi}e^{2\nu} , \\ W &= \frac{re^{2\nu}}{\varpi^2}(2nr+3M)+\frac{1}{\varpi}(nr+M+\tfrac{2}{3}\Lambda r^3) . \end{aligned} \tag{25}$$

The remaining quantities in the perturbed metric can be obtained from the $Z_i^{(\pm)}$ by the transformations described in Ref. 3. The electromagnetic and gravitational perturbations are orthogonal linear combinations of $Z_i^{(\pm)}$. Therefore the evolution of perturbations is described completely by a set of one-dimensional scattering problems. In the region outside the holes, $r_2 < r < r_1$, the potentials are positive apart from V_2^+, which has a slight negative dip near $r=r_1$.

III. NO-HAIR THEOREMS

In the inflationary scenario, the universe undergoes a stage where it is geometrically similar to de Sitter space.[13] According to this picture, the universe has inherited its large-scale homogeneity and spatial flatness from its de Sitter ancestor. The approach to de Sitter space is brought about by an effective cosmological constant, usually resulting from the ground-state energy of a state of the universe different from that of today. Results pertaining to this approach to de Sitter space go by the name of cosmological "no-hair" theorems.[8]

It is possible to analyze perturbations of de Sitter space using conventional cosmological perturbation theory.[14] However, Boucher and Gibbons[15] have shown that the behavior of the perturbations can be seen most clearly in a coordinate system where the metric is static and has features in common with black-hole metrics. The perturbation equations which we derived in the previous section allow us to pursue this approach further. In our case, the universe would consist only of gravity and electromagnetic fields, whose energy density we take to be small. When the mass and charge of the black-hole metrics are set to zero they reduce to

$$ds^2 = -\left[1-\frac{r^2}{\alpha^2}\right]dt^2+\left[1-\frac{r^2}{\alpha^2}\right]^{-1}dr^2 + r^2(d\theta^2+\sin^2\theta d\phi^2) \tag{26}$$

where $\alpha^2 = 3/\Lambda$. A conformal diagram is shown in Fig. 2.

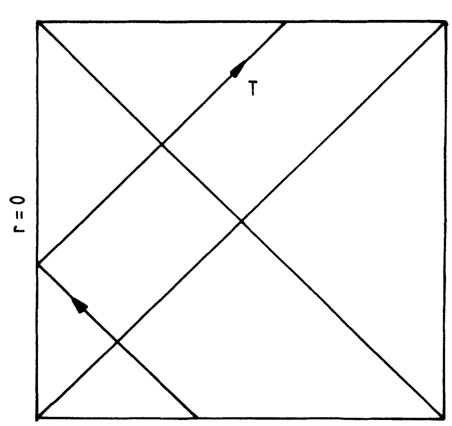

FIG. 2. de Sitter space with the wave vector of a perturbation.

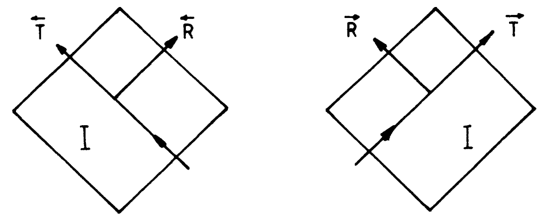

FIG. 3. The modes $\overleftarrow{Z}$ and $\overrightarrow{Z}$ in region I.

The perturbation equations for $Z_1 = H_1$ and $Z_2 = H_2$ take the same simple form for both axial and polar modes:

$$\frac{d^2Z}{dr^{*2}}+\sigma^2 Z = \frac{l(l+1)\Delta}{r^4}Z , \tag{27}$$

where

$$\Delta = r^2 - r^4/\alpha^2 \tag{28}$$

and

$$r = \alpha \tanh\frac{r^*}{\alpha} . \tag{29}$$

The exact solution of this equation which is bounded on the horizon is given by $Z = P_l^{i\alpha\sigma}(\alpha/r)$.

An observer at $r=0$ sees waves enter the region $0 \le r < \alpha$ through the past cosmological horizon, pass through $r=0$ and leave the future cosmological horizon. Asymptotically,

$$Z \to e^{i\sigma r^*} + Te^{-i\sigma r^*}, \quad r^* \to \infty . \tag{30}$$

The Wronskian can be used to show that $T = e^{i\delta_l}$, where δ_l is a phase shift, and the perturbations remain bounded. Their amplitude does not decrease in this coordinate system. However, this implies a cosmic no-hair theorem because as time proceeds the observer sees these perturbations on an exponentially smaller scale in physical coordinates and so the space to an observers past appears to approach de Sitter space.

The stability analysis of de Sitter space in the presence

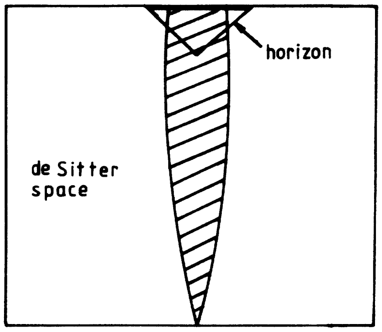

FIG. 4. A large perturbation of de Sitter space which collapses to a singularity.

of a black hole proceeds along similar lines. In this case the perturbations can cross the black hole or the cosmological event horizon, as shown in Fig. 3. The inward-going modes have asymptotic forms

$$\overleftarrow{Z}_i^{(\pm)} \to \begin{cases} e^{i\sigma r^*} + \overleftarrow{R}_i e^{-i\sigma r^*}, & r^* \to \infty \ , \\ \overleftarrow{T}_i^{(\pm)} e^{i\sigma r^*}, & r^* \to -\infty \end{cases} \tag{31}$$

with reflection and transmission coefficients $\overleftarrow{R}$ and $\overleftarrow{T}$. The outward going modes have the form

$$\overrightarrow{Z}_i^{(\pm)} \to \begin{cases} \overrightarrow{T}_i^{(\pm)} e^{i\sigma r^*}, & r^* \to \infty \ , \\ e^{-i\sigma r^*} + \overrightarrow{R}_i^{(\pm)} e^{i\sigma r^*}, & r^* \to -\infty \ . \end{cases} \tag{32}$$

This set of modes, with real σ, forms a complete set. Furthermore, these modes remain bounded on the horizons and therefore a general perturbation, given its mode expansion, also remains bounded.

We can apply this result on the stability of black-hole de Sitter spacetimes to the inflationary scenario. Consider a situation where there is a local enhancement of the radiation density sufficiently large to overcome the cosmological expansion. This density enhancement collapses to form a black hole, as shown in Fig. 4. Outside, the metric will have the form of a perturbed black-hole de Sitter spacetime and an observer sees the space approach de Sitter space. In the more general situation, there may be many such density enhancements on a variety of scales leading to many black holes separated by regions which approach de Sitter space.

IV. STABILITY OF THE CAUCHY HORIZON

We turn now to the stability of the black-hole spacetimes inside their event horizons. Initially, perturbations are set up outside the hole, in the region labeled I in Fig. 5. The relevant mode functions should therefore be set up on the past horizons. They can be divided into ingoing and outgoing modes $\overleftarrow{Z}$ and $\overrightarrow{Z}$, as shown in Fig. 5. The ingoing modes have unit amplitude on the past cosmological horizon and the outgoing modes have unit amplitude on the past black-hole horizon.

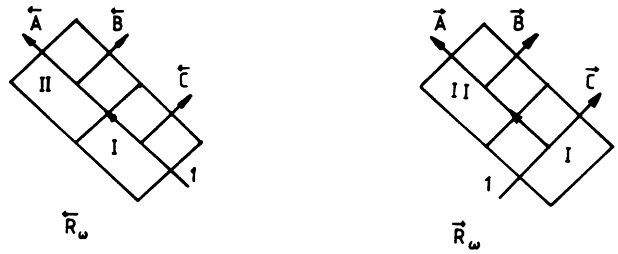

FIG. 5. The modes $\overleftarrow{Z}$ and $\overrightarrow{Z}$.

At the Cauchy horizon the four-velocity of any observer approaches $\partial/\partial V$ and he or she sees a flux which is basically

$$\frac{\partial z}{\partial V} = \frac{1}{\kappa_3} e^{\kappa_3 v} \frac{\partial z}{\partial v} \tag{33}$$

where $z(t)$ is constructed from the mode functions Z and the initial amplitude W,

$$z(t) = \frac{1}{2\pi} \int W Z e^{i\sigma t} d\sigma \ . \tag{34}$$

The exponential factor represents a blueshift which is diverging on the horizon where $v \to \infty$ ($v = r^* - t$ in region II).

Near the Cauchy horizon, the modes have an asymptotic form

$$Z \to A e^{-i\sigma r^*} + B e^{+i\sigma r^*}, \quad r^* \to +\infty \ . \tag{35}$$

Consequently,

$$\frac{\partial z}{\partial V} = \frac{i e^{\kappa_3 v}}{2\pi\kappa_3} \int_{-\infty}^{\infty} W(\sigma) A(\sigma) \sigma e^{-i\sigma v} d\sigma \ . \tag{36}$$

By closing the contour of integration in the lower half-plane we see that the flux near the Cauchy horizon is dominated by the pole in $A(\sigma)$ nearest to the real axis. For $\Lambda = 0$, according to Chandrasekhar and Hartle,[8] there is a pole located at $-i\kappa_2$, and the resulting flux is proportional to $e^{(\kappa_3 - \kappa_2)v}$. Because $\kappa_3 > \kappa_2$, the flux diverges on the horizon, and perturbations of the spacetime grow without limit there.

For the black hole in de Sitter space we have to match the modes through the event horizon. The u and v coordinates have been defined in regions I and II so that v has

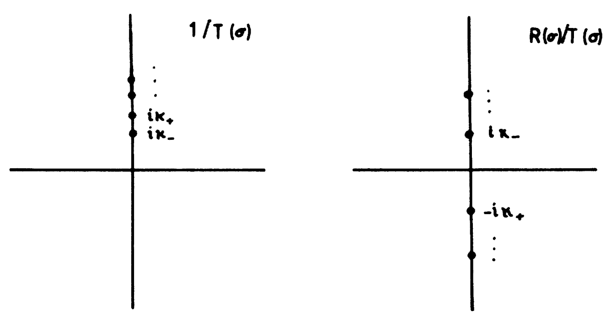

FIG. 6. The poles in $1/T(\sigma)$ and $R(\sigma)/T(\sigma)$.

the same meaning there. Consider the ingoing modes Z. Using Eq. (31) we have, in the interior region II,

$$\overleftarrow{Z} \to \overleftarrow{T}_I(-\sigma)e^{i\sigma r^*}, \quad r^* \to -\infty \ . \tag{37}$$

Consequently, $\overleftarrow{A}(\sigma)=\overleftarrow{T}_{\rm I}(-\sigma)/\overleftarrow{T}_{\rm II}(-\sigma)$. Similarly, for the outgoing modes we have $\vec{A}(\sigma)=\vec{R}_{\rm I}(\sigma)/\vec{T}_{\rm II}(-\sigma)$.

The analytic structure of $T_{\rm II}^{-1}$ and $T_{\rm I}^{-1}$ can be obtained by the arguments of Chandrasekhar and Hartle[8] with virtually no modification. They each have poles along the imaginary axis at multiples of $i\kappa_-$ and $i\kappa_+$, where κ_- and κ_+ are the surface gravities of the horizons on the incident and transmission sides of the potential barrier. The positions of these poles are shown in Fig. 6. Some explanation of these poles can be found in the Appendix.

A fortunate cancellation takes place in the amplitude A_σ because there is a zero in $T_{\rm I}$ at $i\kappa_2$ which cancels the poles in $T_{\rm II}^{-1}$. Similarly, there is a zero in $R_{\rm I}$, also at $i\kappa_2$. The leading pole in A_σ from $T_{\rm II}^{-1}$ is now at $i\kappa_3$ and this pole results in a flux on the Cauchy horizon which is now finite.

The only other poles in $A(\sigma)$ come from poles in $\overleftarrow{T}_{\rm I}$ or $\vec{R}_{\rm I}$. Stability of the Cauchy horizon requires that neither of these amplitudes have poles with $0<\Im m\,\sigma<\kappa_3$. Such a pole would correspond to modes with growing exponential behavior,

$$Z \to \begin{cases} e^{-i\sigma r^*}, & r^* \to \infty \ , \\ e^{i\sigma r^*}, & r^* \to -\infty \ . \end{cases} \tag{38}$$

This kind of mode is known as an unbound state or a quasinormal mode.[3] Their complex frequencies dominate the late time behavior of perturbations outside the hole in a similar way that pure tones dominate in the ringing of a bell.

The frequencies of the first few quasinormal modes have been tabulated in Tables I and II. These results have been obtained by numerical integration of the perturbation equations and minimizing $1/T_{\rm I}$. They are reasonably stable to changes in the range of integration with variations of around 10% in the least accurate cases. The values depend only upon the ratios α/M and Q/M.

In Table I, with $\alpha=8M$, the values of σ all lie in the range $0<\Im m\,\sigma<\kappa_3$, and result in instabilities in the Cauchy horizon. With $\alpha=20M$ on the other hand, the values

TABLE I. The complex characteristic frequencies of the quasinormal modes for Z_1 and Z_2 with $\alpha=8M$. The values of the surface gravity κ_3 on the Cauchy horizon are also indicated.

Q/M	$l=1$	$l=2$	$l=3$	$l=4$	κ_3
			Z_1		
0	0.191 0.068i	0.352 0.075i	0.502 0.072i	0.648 0.075i	∞
0.2	0.192 0.069i	0.357 0.072i	0.507 0.075i	0.658 0.073i	2400
0.4	0.203 0.071i	0.376 0.074i	0.534 0.073i	0.691 0.074i	131.5
0.6	0.222 0.075i	0.404 0.078i	0.581 0.076i	0.750 0.076i	20.00
0.8	0.266 0.088i	0.473 0.080i	0.663 0.080i	0.846 0.081i	3.771
1.0	0.373 0.073i	0.610 0.076i	0.835 0.076i	1.058 0.076i	0.1531
			Z_2		
0	0.087 0.090i	0.285 0.069i	0.455 0.071i	0.616 0.071i	
0.2	0.089 0.091i	0.286 0.069i	0.461 0.072i	0.622 0.073i	
0.4	0.091 0.090i	0.294 0.070i	0.470 0.073i	0.635 0.074i	
0.6	0.096 0.091i	0.305 0.073i	0.495 0.075i	0.669 0.075i	
0.8	0.106 0.092i	0.327 0.076i	0.533 0.077i	0.725 0.080i	
1.0	0.094 0.092i	0.371 0.098i	0.609 0.074i	0.834 0.076i	

TABLE II. The complex characteristic frequencies of the quasinormal modes for Z_1 and Z_2 with $\alpha=20M$. The values of the surface gravity κ_3 on the Cauchy horizon are also indicated.

Q/M	$l=1$	$l=2$	$l=3$	$l=4$	κ_3
			Z_1		
0	0.252 0.063i	0.440 0.064i	0.654 0.068i	0.813 0.068i	∞
0.2	0.255 0.064i	0.445 0.065i	0.662 0.069i	0.822 0.068i	2400
0.4	0.263 0.064i	0.460 0.066i	0.682 0.070i	0.848 0.070i	131.5
0.6	0.275 0.065i	0.510 0.069i	0.743 0.075i	0.915 0.072i	20.00
0.8	0.292 0.067i	0.569 0.072i	0.782 0.072i	0.995 0.076i	1.552
1.0	0.447 0.075i	0.649 0.079i	0.853 0.080i	0.999 0.079i	0.0548
			Z_2		
0	0.099 0.068i	0.355 0.068i	0.605 0.076i	0.803 0.070i	
0.2	0.099 0.068i	0.355 0.068i	0.606 0.076i	0.811 0.072i	
0.4	0.100 0.068i	0.357 0.069i	0.610 0.075i	0.834 0.075i	
0.6	0.101 0.069i	0.360 0.069i	0.617 0.074i	0.865 0.078i	
0.8	0.102 0.069i	0.403 0.072i	0.630 0.072i	0.901 0.080i	
1.0	0.103 0.069i	0.419 0.069i	0.690 0.064i	0.979 0.067i	

of σ lie outside the range $0<\Im m\sigma<\kappa_3$ when $Q=M$ (more precisely, when $Q>0.999M$). This numerical approach cannot show rigorously that all of the modes fall outside the range but it does show up a clear trend. There is strong evidence that the Cauchy horizon is stable in this case.

It would be interesting to explore many values of α and Q, but this requires more computing resources than we have available. Increasing α further has only a small effect upon the frequencies, suggesting stability for sufficiently large α. The instability for small α therefore seems to be an effect of large curvature which is not related to the instability of Cauchy horizons for black holes in flat space.

In the limit that $\alpha\to\infty$ the spacetime outside the hole becomes flat. In this limit the cancellation between the zeros and poles in A leave an increasingly large residue, and this results in a large flux at the Cauchy horizon. From the Appendix we find that there is an amplification factor

$$\frac{c_3}{c_1}\frac{\kappa_1-\kappa_2}{\kappa_2-\kappa_3}, \tag{39}$$

where $c_1\sim\alpha^{-2}$ and $c_3\sim M^{-2}$. If α is the present cosmological scale, and the radiation entering the black hole is in the form of starlight, then the flash of radiation near the Cauchy horizon would be an obstacle to passing through the black hole. As $\alpha\to\infty$, the radiation flux diverges and the Cauchy horizon becomes unstable.

APPENDIX

In this appendix we provide a rough guide to the residues of the poles in the inverse transmission coefficients $T^{-1}(\sigma)$. The mode functions $f(x)$ satisfy equations of the form

$$\frac{d^2f}{dr^2}+\sigma^2 f=Vf, \tag{A1}$$

where

$$V\to\begin{cases} c_-e^{-2\kappa_- x}, & x\to\infty,\\ c_+e^{2\kappa_+ x}, & x\to-\infty.\end{cases} \tag{A2}$$

Consider two solutions f_1 and f_2 with the asymptotic behavior

$$\begin{aligned} &f_1(x,\sigma)\to e^{-i\sigma x}, \quad x\to\infty,\\ &f_2(x,\sigma)\to e^{i\sigma x}, \quad x\to-\infty. \end{aligned} \tag{A3}$$

Because (f_1,f_1^*) and (f_2,f_2^*) are pairs of independent solutions, it follows that there is an expansion

$$f_2(x,\sigma)=\frac{R(\sigma)}{T(\sigma)}f_1(x,\sigma)+\frac{1}{T(\sigma)}f_1(x,-\sigma), \tag{A4}$$

where $R(\sigma)$ and $T(\sigma)$ are the usual reflection and transmission coefficients. We can then deduce that

$$\frac{1}{T(\sigma)}=-\frac{1}{2i\sigma}[f_1(x,\sigma),f_2(x,\sigma)], \tag{A5}$$

where

$$[f,g]=fg'-gf'. \tag{A6}$$

Approximate solutions can be constructed by keeping only the leading terms in V, then

$$\begin{aligned} &f_1\sim e^{-i\sigma x}+\frac{1}{4\kappa_-}\frac{c_-}{\kappa_-+i\sigma}e^{-(2\kappa_-+i\sigma)x}, \quad x\to\infty\\ &f_2\sim e^{i\sigma x}+\frac{1}{4\kappa_+}\frac{c_+}{\kappa_++i\sigma}e^{(2\kappa_++i\sigma)x}, \quad x\to-\infty. \end{aligned} \tag{A7}$$

A rough guide to the behavior of the transmission coefficient can be found by using these results at $x=0$, then

$$\begin{aligned} \frac{1}{T(\sigma)}\approx&-\frac{c_+}{4\kappa_+i\sigma}-\frac{c_-}{4\kappa_-i\sigma}\\ &+\frac{(\kappa_++\kappa_-+i\sigma)c_+c_-}{16i\sigma\kappa_+\kappa_-(\sigma-i\kappa_+)(\sigma-i\kappa_-)}. \end{aligned} \tag{A8}$$

This shows the origin of the poles at $i\kappa_\pm$. For the cases of interest, $\kappa_\pm=(\kappa_1,\kappa_2)$ in region I and (κ_2,κ_3) in region II. Thus

$$\frac{T_{\rm I}(i\kappa_2)}{T_{\rm II}(i\kappa_2)}\approx\frac{c_3}{c_1}\frac{\kappa_1-\kappa_2}{\kappa_2-\kappa_3}. \tag{A9}$$

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