PHYSICAL REVIEW D VOLUME 41, NUMBER 2 15 JANUARY 1990

Stability of the Schwarzschild–de Sitter model

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(Received 17 August 1989)

From a study of the shape of a spacelike, three-cylindrical transition layer separating a de Sitter future region from a Schwarzschild past region, we show that the uniform configuration (in the sense that the layer's radius is not a function of the layer's proper distance along its axis) considered by Frolov, Markov, and Mukhanov [Phys. Lett. B **216**, 272 (1989)] is a stable configuration (in the sense that, when perturbed, the three-cylinder does not tend to shrink down to a point as a cone). We also show that this model does not require fine-tuning since a variation of the layer's internal parameters from the uniform configuration does not destroy the three-cylinder but induces spatial oscillations in its radius.

I. INTRODUCTION

It is a general belief that singularities arising in general relativity signal the breakdown of the classical Einstein equations and that they should not appear once a quantum theory of gravitation becomes available. We still do not know what this quantum theory is, but this has not refrained various authors from wondering about the true nature of the classical singularities.[1] At the time being, one of the possible approaches to the problem is to investigate the quantum effects due to all kinds of fields (including the graviton) by constructing a renormalized expectation value of the stress-energy tensor (a quantum operator) $\langle T_{\mu\nu}\rangle$ which is used as the right-hand side of the Einstein field equations.[2] The geometry (i.e., the gravitational field) is then treated classically but small fluctuations around the background metric can still be quantized. This is the semiclassical approach. It is not yet clear what range of validity the semiclassical theory exactly possesses, but we can hope that it may provide valuable insights towards the predictions of a fully quantized theory of gravitation.

In the case of black-hole spacetimes, one expects quantum fields to begin to act dominantly on the geometry when the curvature reaches order unity in Planck units. Whether gravitation can still be treated classically at this level is far from being certain, but one can wonder if quantum polarization can provide a mechanism to slow down the infinite rise of the curvature and to maintain it bounded to Planckian magnitude. To show this, say, for the Schwarzschild black hole, one would need to quantize all types of fields in a general, spherically symmetric background (with two unspecified functions of r), to construct a suitable renormalized expectation value for the stress-energy tensor (adding up all the contributions), and then to solve self-consistently the semiclassical field equations. So far, nobody has succeeded in finding an expression for $\langle T_{\mu\nu}\rangle$ in such a general background geometry, and even for the fixed Schwarzschild metric only approximate analytic expressions have been obtained.[3]

It is nevertheless possible with a schematic analysis[4] to outline the possible behaviors for the corrected curvature when quantum effects are taken into account. Of critical importance is the sign of $\langle -T_t^t\rangle$ which is interpreted inside the Schwarzchild event horizon as a tension along the axis of the three-cylinder of constant time $r=$const (r and t have interchanged their roles as space and time). We can expect this component of the stress-energy tensor to be proportional to the curvature squared, with a coefficient of order unity that will depend on the number and types of the quantized fields. This coefficient can be of either sign.[2] Setting $\langle -T_t^t\rangle=(3/4\pi)a^2m^2(r)/r^6$ [we use Planck units; a^2 is the coefficient mentioned above, $m(r)$ the mass function at radius r] and solving Einstein's equation $dm/dr=4\pi r^2\langle -T_t^t\rangle$ we arrive to the following expression for the curvature:

$$\frac{m(r)}{r^3}=\frac{1}{a^2+(r/r_Q)^3}\ , \tag{1.1}$$

where $r_Q\equiv M^{1/3}$ is the radius at which quantum effects become important. We easily see that for $r\gg r_Q$ the curvature function recovers the usual Schwarzschild form M/r^3, but that its behavior for small radii depends critically on the sign of a^2. Should it be negative, the curvature would blow up at a nonvanishing value of r; should it be positive, the curvature would remain bounded (and constant) to order unity as r tends to zero.

Although very schematic, this analysis shows that a self-regulatory mechanism provided by the quantum fields is not implausible and that a nonrotating, uncharged black hole could be adequately described by the Schwarzschild solution down to a critical radius r_Q where quantum effects produce a smooth transition towards a constant curvature (de Sitter) region.

This picture is remarkably similar to the formulation of Markov's "new law of nature."[5] In 1982, Markov proposed that because of quantum corrections, spacetime curvature should always be subject to an upper bound of Planckian magnitude, and that when the limiting value is reached the effective stress-energy tensor of the quantum material would take the false-vacuum form $\langle T_{\mu\nu} \rangle = -\rho g_{\mu\nu}$, where ρ is a constant (positive) energy density. Although the validity of Markov's law is far from being established, it is fascinating to look at its consequences. Such an investigation has been carried by Frolov, Markov, and Mukhanov[6] (FMM) early in 1989. To construct the global structure of a nonsingular Schwarzschild black hole, the authors supposed that the transition region between the Schwarzschild and de Sitter regions was of short (timelike) extent and was modeled as a spacelike, spherical surface layer lying on a hypersurface of constant time $r = \text{const}$. The layer's intrinsic metric hence assumes the form

$$ds^2 = \text{const} \times dt^2 + r^2 d\Omega^2 , \tag{1.2}$$

and therefore possesses the topology of a three-cylinder with coordinate t running along the axis and whose "cross sections" are two-spheres, all of the same radius r. To make up a mental picture of this object, it is better to suppress one of the angular variables such that the layer would appear as an infinitely long cardboard tube of constant radius r. We will refer to this tube as the uniform (constant radius along the tube) FMM configuration. It is important to emphasize that the layer (tube) is a spacelike hypersurface: it therefore exists for a single instant of time.

Using Israel's junction conditions,[7] it is possible to show that it is always possible to join the Schwarzschild and de Sitter solutions (except at the event horizon[4]) at a $r = \text{const}$ hypersurface by interposing a thin shell whose surface stress-energy tensor is related to the discontinuities in the first derivatives of the metric (in order for the shell to have a well-defined geometry, the metric itself

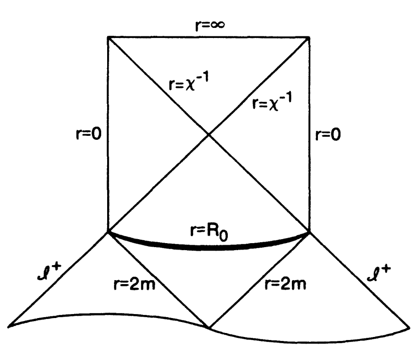

FIG. 1. The FMM model for a spherical, uncharged black hole. The Schwarzschild solution is joined at $r = R_0$ to the de Sitter solution with the help of a spacelike, three-cylindrical transition layer. The de Sitter solution describes a collapsing and then reexpanding closed universe.

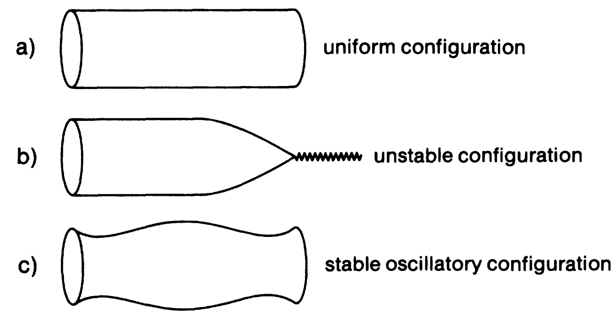

FIG. 2. Several configurations for the transition layer. Shown in (a) is the uniform configuration where the three-cylinder radius is constant along the cylinder's axis. Shown in (b) is an unstable configuration where the radius shrinks down to zero as would the radius of a cone. Not shown is an alternative unstable situation where the radius would increase to infinity. Shown in (c) is a stable configuration where the radius presents spatial oscillations along the cylinder's axis.

must be continuous). If the shell is located inside the Schwarzschild horizon and outside the de Sitter horizon (this is the case considered by FMM) then the shell is spacelike and its surface energy density vanishes while the surface pressures (axial and tangential) assume finite values which depend on the value of the shell's radius. The intriguing global black-hole structure constructed by FMM is that of an asymptotically flat universe connected by a black-hole interior to a collapsing and then reexpanding de Sitter universe (see Fig. 1).

Although the transition layer plays only the minor role of gluing together the Schwarzschild and de Sitter spacetimes, a more detailed study of its properties is not without interest. First, the physics of a spacelike thin shell has never been studied before. While the study of timelike shells is primarily concerned with the time evolution of a two-dimensional surface (such that its world sheet is a three-surface), the study of a spacelike shell will be concerned with the global shape (spatial structure) of a three-dimensional surface which exists at a single moment of time. While for a timelike shell one is interested in the equations of motion for the shell, for a spacelike shell one focuses attention on the differential equations ("shape differential equations") which describe the spatial structure of the shell. The philosophical point of view which one adopts when studying one type of shell is totally different from that adopted for the other type.

Second, it appears important to verify whether the FMM uniform configuration is stable in the following sense (see Fig. 2): upon a virtual displacement of the layer's radius from the uniform configuration, does the layer's topology tend to change from that of a three-cylinder (the cardboard tube) to that of cone where the cylinder's radius shrinks down to zero (or, alternatively, does the radius tend to expand to infinity); or does the topology remain pretty much the same but with (say) small spatial oscillations in the cylinder's radius as we move along the axis? We will refer to the first scenario as the unstable scenario, while the second scenario represents stability. The spacetime diagram depicting the spatial behavior of an unstable transition layer is given in Fig. 3. The fact that the layer's radius shrinks down to zero means in effect that the layer focuses to a singularity since the Schwarzschild region is dragged along with it. It would hence be something of a drawback if the FMM uniform configuration would be found to be unstable in the sense described above.

Third (but related to the second point), it is also of interest to evaluate whether the FMM configuration requires some fine-tuning. The junction conditions for the uniform layer provide definite values for the layer's surface pressures, once the radius of the three-cylinder is specified: what is the effect of varying these parameters? Is the model going to be destroyed completely or does it survive in some different form?

It is the purpose of this paper to examine these questions by providing a space-dependent model for a three-cylindrical spacelike transition layer separating a de Sitter region from a Schwarzschild region (we allow the layer's radius R to become a function of proper distance s along the cylinder's axis). In this model, the original cardboard tube is replaced by some kind of rubber tube whose cross-sectional area $4\pi R^2$ can vary along the tube's axis. The uniform FMM configuration is then recovered as a limiting case of the general spacelike behavior of the three-cylinder radius $R(s)$. We will show that this uniform configuration is in fact stable (in the sense mentioned above) against virtual displacement of the layer's radius and that the FMM model does not require much fine-tuning: there exists a class of solutions to the shape differential equations which present an oscillatory behavior for $R(s)$. Perturbation of the original parameters (the surface pressures) will hence induce small spatial oscillations in the radius of the three-cylinder rather than making it shrink to zero and hence producing a pinching off of the cylinder into a singularity.

The paper is organized as follows: in the next section we will recall the basic formalism of surface layers in general relativity and derive the equations of motion (shape differential equations) for a timelike (spacelike) shell of perfect fluid. We apply this formalism in Sec. III to derive the shape differential equations of a spacelike, three-cylindrical transition layer separating a de Sitter future region from a Schwarzschild past region. We seek for the uniform FMM configuration in Sec. IV and then examine its stability in Sec. V. Section VI summarizes and concludes.

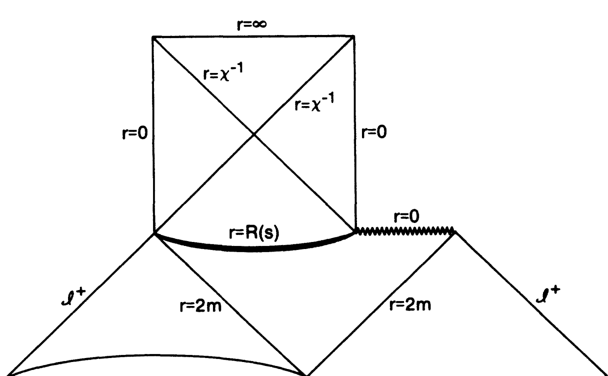

FIG. 3. Spacetime diagram depicting the behavior of an unstable transition layer: the layer's radius shrinking down to zero [as is Fig. 2(b)] drags the Schwarzschild region along with it. The layer therefore focuses into a singularity.

II. SURFACE LAYERS IN GENERAL RELATIVITY

Junction conditions for timelike or spacelike boundary surfaces have been well understood since the work of Israel.[7] The case of null shells has been treated by various authors.[8] The equations of motion for a timelike thin shell have also been developed by various people including Israel,[7] de la Cruz and Israel,[9] and Chase[10] (see also Blau, Guendelman, and Guth[11]). In this section we generalize these results and develop the equations of motion (shape differential equations) for a timelike (spacelike) thin shell. The equations derived below are therefore completely general, but to simplify the discussion, we will comment on the meaning of the equations as if the shell were timelike. There is usually no easy physical interpretation of the equations when the shell is spacelike. We use throughout the conventions of Misner, Thorne, and Wheeler.[12]

Let Σ be a (timelike or spacelike) hypersurface separating spacetime $\mathcal{V}$ into two parts $\mathcal{V}^{\pm}$. Let n^α be Σ's unit normal vector (pointing from $\mathcal{V}^-$ to $\mathcal{V}^+$) and define $\epsilon(=+1$ for timelike $\Sigma, -1$ for spacelike $\Sigma)$ such that $n^\alpha n_\alpha = \epsilon$. Let $x^\alpha_\pm$ be a system of coordinates in $\mathcal{V}^\pm$ (the coordinates do not have to join continuously on the hypersurface) and let ξ^a be a system of intrinsic coordinates on Σ. The vectors

$$e^\alpha_{(a)} = \frac{\partial x^\alpha}{\partial \xi^a}, \quad n_\alpha e^\alpha_{(a)} = 0 \tag{2.1}$$

are tangent to Σ and act as projectors from $\mathcal{V}$ onto the hypersurface; for clarity, we have suppressed for now the use of the $\pm$ indices.

The four-dimensional covariant derivative of any vector A^a tangent to Σ will have components along the tangent vectors $e^\alpha_{(a)}$ and also along the normal vector n^α. It is easy to show that

$$A^\alpha{}_{|\beta} e^\beta_{(b)} = A^a{}_{;b} e^\alpha_{(a)} + \epsilon A^a K_{ab} n^\alpha , \tag{2.2}$$

where we have defined $A^\alpha \equiv A^a e^\alpha_{(a)}$; the stroke denotes covariant differentiation with respect to the four-metric $g_{\alpha\beta}$ whereas the semicolon denotes the same for the three-metric $h_{ab} = e_{(a)} \cdot e_{(b)}$. We have introduced the extrinsic curvature

$$K_{ab} = -n_{\alpha|\beta} e^\alpha_{(a)} e^\beta_{(b)} , \tag{2.3}$$

which is a measure of how Σ is embedded in $\mathcal{V}$.

From Eqs. (2.2), (2.3), and the Ricci commutation relations, it is possible to express the four-dimensional curvature tensor in terms of the intrinsic curvature of Σ and its extrinsic curvature. The result (the Gauss-Codazzi equations) can be summarized in

$$-2\epsilon G_{\alpha\beta}n^{\alpha}n^{\beta}={}^{3}R+\epsilon(K^{ab}K_{ab}-K^{2}) , \quad (2.4)$$

$$G_{\alpha\beta}e^{\alpha}_{(a)}n^{\beta}=K_{;a}-K^{b}{}_{a;b} , \quad (2.5)$$

where $G_{\alpha\beta}$ is the Einstein tensor and K the trace $h^{ab}K_{ab}$. It is also possible to express $G_{\alpha\beta}e^{\alpha}_{(a)}e^{\beta}_{(b)}$ in a similar way.

In the presence of a surface layer, the four-metric $g_{\alpha\beta}$ will be continuous at Σ, but its normal derivative will not. It follows from (2.3) that K_{ab} will suffer a jump at the boundary surface. It can be shown that this jump $[K_{ab}]\equiv K_{ab}|^{+}-K_{ab}|^{-}$ (the difference of the limits when we approach Σ from both sides) is related to the shell surface stress-energy tensor by

$$8\pi S_{ab}=\epsilon([K_{ab}]-h_{ab}[K]) . \quad (2.6)$$

Inversion of (2.6) yields

$$[K_{ab}]=8\pi\epsilon(S_{ab}-\tfrac{1}{2}h_{ab}S) . \quad (2.7)$$

If we define $T^{\alpha\beta}$ to be the "background" stress-energy tensor of $\mathcal{V}$ (in general different from one side of Σ to the other) then the full stress-energy tensor will be composed of $T^{\alpha\beta}$ plus a distributionlike contribution proportional to $S^{ab}e^{\alpha}_{(a)}e^{\beta}_{(b)}$. The conservation equation for this stress-energy tensor then leads to [using (2.5) and the field equations]

$$S^{a}{}_{b;a}+\epsilon[T_{bn}]=0 , \quad (2.8)$$

where $T_{bn}\equiv T_{\alpha\beta}e^{\alpha}_{(b)}n^{\beta}$.

We now specialize to the case where the shell is composed of a perfect fluid. It is then possible to write the shell stress-energy tensor as

$$S_{ab}=\sigma u_{a}u_{b}+P(h_{ab}+\epsilon u_{a}u_{b}) , \quad (2.9)$$

where u^{a} is the three-velocity of the shell ($u^{a}u_{\alpha}=-\epsilon, u^{\alpha}n_{\alpha}=0$), $P_{ab}\equiv h_{ab}+\epsilon u_{a}u_{b}$ is the projector to the two-space orthogonal to u^{a} (rest frame). If the shell is timelike, σ can be interpreted as a surface energy density; if the shell is spacelike, however, σ becomes a (longitudinal) pressure. In both cases, P is interpreted as a surface (transverse) pressure. If we now contract the conservation equation (2.8) with u^{a} and P^{ab}, we get, respectively,

$$(\sigma u^{a})_{;a}+\epsilon P u^{a}{}_{;a}=[T_{\alpha\beta}u^{\alpha}n^{\beta}] , \quad (2.10)$$

which is in effect a continuity equation for the inertial mass of the shell with an external source of work found on the right-hand side, and

$$(\sigma+\epsilon P)a^{a}+P^{ab}(P_{,b}+\epsilon[T_{bn}])=0 , \quad (2.11)$$

which are Euler's equations describing the internal motion of the constituting particles of the shell ($a^{a}\equiv u^{a}{}_{;b}u^{b}$ is the transverse acceleration).

The shell's equations of motion now follow simply by applying Eq. (2.2) to the shell four-velocity $u^{\alpha}=u^{a}e^{\alpha}_{(a)}$ (as measured from either side of Σ):

$$a^{\alpha}\equiv u^{\alpha}{}_{|\beta}u^{\beta}=a^{a}e^{\alpha}_{(a)}+\epsilon u^{a}u^{b}K_{ab}n^{\alpha} . \quad (2.12)$$

While a^{a} describes the shell's internal motion, it is the normal component $n_{\alpha}a^{\alpha}$ which describes the motion of the shell as a whole. We therefore find $n_{\alpha}a^{\alpha}|^{\pm}=u^{a}u^{b}K_{ab}|^{\pm}$ (the notation is self-explanatory). Upon adding and subtracting we then have

$$\begin{aligned} n_{\alpha}a^{\alpha}|^{+}+n_{\alpha}a^{\alpha}|^{-}&=2u^{a}u^{b}\tilde{K}_{ab} ,\\ n_{\alpha}a^{\alpha}|^{+}-n_{\alpha}a^{\alpha}|^{-}&=u^{a}u^{b}[K_{ab}] , \end{aligned} \quad (2.13)$$

where we have defined $2\tilde{K}_{ab}=K_{ab}|^{+}+K_{ab}|^{-}$ the averaged extrinsic curvature. The remainder of the exercise consists in expressing the right-hand sides of (2.13) in convenient forms. From Eqs. (2.4) and (2.6) and the field equations, it is straightforward to show that $\tilde{K}_{ab}S^{ab}=-\epsilon[T_{\alpha\beta}n^{\alpha}n^{\beta}]$; using Eq. (2.9) to express $u^{a}u^{b}$ in terms of S^{ab} and h^{ab}, we find

$$2u^{a}u^{b}\tilde{K}_{ab}=-\frac{2}{\sigma+\epsilon P}(\epsilon[T_{\alpha\beta}n^{\alpha}n^{\beta}]+P\tilde{K}) , \quad (2.14)$$

while combining Eqs. (2.7) and (2.9) yields

$$u^{a}u^{b}[K_{ab}]=8\pi(P+\tfrac{1}{2}\epsilon\sigma) . \quad (2.15)$$

Substituting (2.14) and (2.15) into (2.13) finally gives the shell's equations of motion

$$n_{\alpha}a^{\alpha}|^{+}+n_{\alpha}a^{\alpha}|^{-}=-\frac{2}{\sigma+\epsilon P}(\epsilon[T_{\alpha\beta}n^{\alpha}n^{\beta}]+P\tilde{K}) , \quad (2.16)$$

$$n_{\alpha}a^{\alpha}|^{+}-n_{\alpha}a^{\alpha}|^{-}=8\pi(P+\tfrac{1}{2}\epsilon\sigma) . \quad (2.17)$$

Together with Eq. (2.10), Eqs. (2.16) and (2.17) determine the motion of any type of shell (spacelike or timelike) in any background $\mathcal{V}$. These results generalize the work of Chase.[10]

III. THE SCHWARZSCHILD–de SITTER MODEL

We now proceed and apply the general formalism developed above to the case of a spacelike ($\epsilon=-1$), three-cylindrical transition layer separating a de Sitter region ($\mathcal{V}^{+}$) from a Schwarzschild region ($\mathcal{V}^{-}$) (see Fig. 4).

We suppose for definiteness that the layer is restricted to lie outside the de Sitter horizon and within the Schwarzschild event horizon. This means that we are free to use the usual Schwarzschild coordinates (r,t,θ,ϕ)

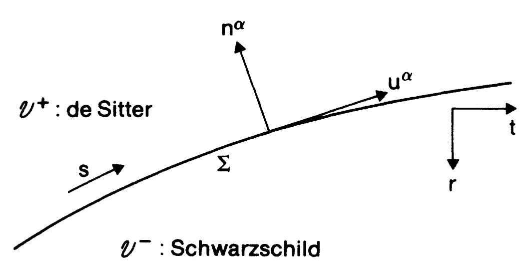

FIG. 4. A spacelike transition layer Σ separating a de Sitter future region from a Schwarzschild past region. Shown are the normal vector n^{α}, the axial vector u^{α}, as well as the different coordinates. The angular dimensions are suppressed.

(r now being a time coordinate; t a space coordinate) in which the metric on both sides takes the form

$$ds^2=-f^{-1}dr^2+f\,dt^2+r^2d\Omega^2 \,, \tag{3.1}$$

where $f=f_d\equiv\chi^2r^2-1, t=t_d$ in the de Sitter region, whereas $f=f_S\equiv 2m/r-1, t=t_S$ in the Schwarzschild region. The parameter χ^{-1} represents the size of the de Sitter horizon; it is of the order of the Planck length but not necessarlly equal to it. The layer is described by the relation $R(s)$ (which expresses that the three-cylinder radius can vary with proper distance along its axis) and possesses the intrinsic three-metric

$$(ds^2)_\Sigma=ds^2+R^2d\Omega^2 \tag{3.2}$$

(we stress once more that the layer occurs only at a single moment of time; we are interested in finding the global shape of the layer, not its time evolution).

The axial vector of Σ, as viewed from either side, can be expressed as

$$u^\alpha=(u^r,u^t,u^\theta,u^\phi)=(\dot R,f^{-1}\beta,0,0) \,, \tag{3.3}$$

where $\beta\equiv(\dot R^2+f)^{1/2}$ and where the overdot denotes a differentiation with respect to proper length s. This ensures that Eq. (3.1) reduces to the intrinsic line element (3.2) on the path of Σ. The choice of sign for β means that t increases monotonically with s. From the orthogonality condition $u^\alpha n_\alpha=0$ and our convention that n^α points towards the de Sitter region, we get

$$n_\alpha=(f^{-1}\beta,-\dot R,0,0) \,. \tag{3.4}$$

A straightforward calculation now shows that

$$n_\alpha a^\alpha=K^s{}_s=\dot\beta/\dot R, \quad K^\theta{}_\theta=K^\phi{}_\phi=\beta/R \,. \tag{3.5}$$

By noting that $(T_{\alpha\beta})_{\text{de Sitter}}=-(3\chi^2/8\pi)g_{\alpha\beta}$ whereas $(T_{\alpha\beta})_{\text{Schwarzschild}}=0$, we also find

$$[T_{\alpha\beta}n^\alpha n^\beta]=3\chi^2/8\pi, \quad [T_{\alpha\beta}n^\alpha u^\beta]=0 \,. \tag{3.6}$$

If we substitute Eqs. (3.5) and (3.6) into Eq. (2.17) we obtain

$$\dot\beta_d-\dot\beta_S=8\pi\dot R(P_\perp-\tfrac{1}{2}P_s) \,, \tag{3.7}$$

where we have redefined our notation from P to $P_\perp$ (tangential pressure), and from σ to P_s (axial pressure). Using Eqs. (3.2) and (3.6), the conservation equation (2.10) reduces to

$$(R^2P_s)^{\cdot}-P_\perp(R^2)^{\cdot}=0 \,, \tag{3.8}$$

which implies $\dot R(P_\perp-\frac{1}{2}P_s)=\frac{1}{2}(RP_s)^{\cdot}$. Equation (3.7) can then be integrated to

$$\beta_d-\beta_S=4\pi RP_s+\text{const} \,. \tag{3.9}$$

The constant can be evaluated by considering the limit $\chi=m=0$ (flat spacetimes on both sides, which implies that there is no shell). The left-hand side then vanishes, together with the first term of the right-hand side; the constant therefore has to be zero.

The shape differential equations ρf a three-cylindrical transition layer separating a de Sitter region from a Schwarzschild region can therefore be integrated to give

$$\beta_d-\beta_S=4\pi RP_s \tag{3.10}$$

without making any assumption on the layer's equation of state $P_s(P_\perp)$. This is a consequence of the fact that we have a conservative system, i.e., that the right-hand side of Eq. (2.10) is identically zero. By multiplying Eq. (3.10) by $(\beta_d+\beta_S)$ one gets

$$\beta_d+\beta_S=(f_d-f_S)/4\pi RP_s \tag{3.11}$$

which upon differentiating with respect to s reproduces Eq. (2.16). The only independent shape equations for the system are therefore Eqs. (3.8) and (3.10), together with an equation of state relating the axial pressure P_s to the transverse pressure $P_\perp$.

IV. THE UNIFORM CONFIGURATION

In this section we will recover from the general shape Eqs. (3.8) and (3.10) the uniform configuration [in the sense that $R(s)=R_0=\text{const}$] considered by Frolov, Markov, and Mukhanov.[6] It is convenient at this point to introduce dimensionless variables to replace the original ones. Since we are interested in a transition layer with radius of order $m^{1/3}$ (in Planck units), we define a new radial coordinate x such that

$$R=\chi^{-1}(\chi m)^{1/3}x \,. \tag{4.1}$$

We also define $\bar s$, π_s, and $\pi_\perp$ according to

$$s=\chi^{-1}\bar s, \quad P_s=(\chi/4\pi)\pi_s, \quad P_\perp=(\chi/4\pi)\pi_\perp \,. \tag{4.2}$$

It is then simple algebra to show that Eqs. (3.8) and (3.10) reduce to

$$(\dot x^2+x^2-\alpha^2)^{1/2}-(\dot x^2+2/x-\alpha^2)^{1/2}=x\pi_s \,, \tag{4.3}$$

$$(x^2\pi_s)^{\cdot}-\pi_\perp(x^2)^{\cdot}=0 \,, \tag{4.4}$$

where the overdot now denotes a differentiation with respect to the rescaled proper distance $\bar s$ and where $\alpha^2\equiv(\chi m)^{-2/3}$ is a negligible quantity for a macroscopic black hole.

Differentiation of equation (4.3) (but neglecting α^2 with respect to the other terms; we consider a situation where the layer's radius is such that $\chi^{-1}\ll R_0\ll 2m$) and use of Eq. (4.4) yields

$$(\ddot x+x)(\dot x^2+x^2)^{-1/2} -(\ddot x-1/x^2)(\dot x^2+2/x)^{-1/2}=2\pi_\perp-\pi_s \,. \tag{4.5}$$

A uniform solution is obtained if one puts $\dot x=\ddot x=0$ in Eqs. (4.3) and (4.5). This provides values for the pressures π_s and $\pi_\perp$ once the transition radius x_0 is specified:

$$\pi_s^{(0)}=1-X_0, \quad \pi_\perp^{(0)}=1-X_0/4 \,, \tag{4.6}$$

where $X_0\equiv 2^{1/2}x_0^{-3/2}$. These values agree with those found by FMM.

It is possible to fix x_0 by requiring that the curvature invariant $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ be continuous (and equal to the limiting value) at R_0. We then have $48m^2R_0^{-6}=24\chi^4$, or $2x_0^{-6}=1$, so we find

$$x_0 = 2^{1/6} \simeq 1.1225 . \tag{4.7}$$

If the value of x_0 is uniquely specified, it is not so for the actual layer's radius R_0 since the parameter χ is left undetermined. This is also true for the exact value of the limiting curvature

$$R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \le 24\chi^4 = \beta / l^4 , \tag{4.8}$$

where l is the Planck length and β a free parameter of order unity.

If we substitute Eq. (4.7) into Eq. (4.6) and translate back to the original variables, we obtain

$$P_s^{(0)} = -\chi(2^{1/4}-1)/4\pi ,$$
$$P_\perp^{(0)} = \chi(1-2^{-7/4})/4\pi . \tag{4.9}$$

The axial pressure $P_s^{(0)}$ is negative (the stress along the axis of the three-cylinder is a tension) whereas $P_\perp^{(0)}$ is positive (the tangential stress is a pressure). Equation (4.9) expresses the values that the pressures must possess in order to support a uniform three-cylindrical transition layer with radius $R_0 = 2^{1/6}\chi^{-1}(\chi m)^{1/3}$. It is also possible to find such values for arbitrary radii, as shown in Eq. (4.6); it is also possible (and easy) to generalize to the case where the inequality $\chi^{-1} \ll R_0 \ll 2m$ is not satisfied.[6]

V. STABILITY OF THE UNIFORM CONFIGURATION

We have found so far that a spacelike, three-cylindrical transition layer between a de Sitter and Schwarzschild regions can have a uniform behavior (in the sense that the cylinder's radius is constant along its axis) provided that its principal pressures be given by Eq. (4.6). The space-dependent equations now allow us to verify whether the uniform configuration is stable [in the sense mentioned in the Introduction (Fig. 2): when perturbed, an unstable layer would tend to have its radius shrinking down to zero as the radius of a cone, while a stable layer would remain cylindrical with perhaps the presence of small spatial oscillations in its radius] and what the effects of slightly changing the layer's internal parameters from the uniform configuration are.

By squaring Eq. (4.3) twice, it is possible to bring it to the form

$$\dot{x}^2 + V(x) = \alpha^2 , \tag{5.1}$$

$$V(x) = -\left[\frac{x^2 - 2/x - (x\pi_s)^2}{2x\pi_s} \right]^2 + 2/x , \tag{5.2}$$

which is formally similar to the equation of motion of a particle with energy α^2 moving in a potential V. [Although Eqs. (5.1) and (5.2) do not describe a time evolution in any way (they describe the spatial variation of the three-cylinder radius along its axis), we shall nevertheless continue to use the terms "energy" and "potential" in order to discuss Eq. (5.1). We will have to remember that it is only a formal analogy.] Now, a radius x for which $\dot{x}$ is zero satisfies the equality $V(x) = \alpha^2$, whereas a radius for which $\ddot{x}$ vanishes corresponds to an extremum of the potential. A uniform configuration would therefore correspond to a radius x for which $V(x) = \alpha^2$ and $V'(x) = 0$. If we take the limit where α^2 is taken to be vanishingly small, it is straightforward (if tedious) to check that $V(x_0) = V'(x_0) = 0$ if π_s and $\pi_\perp$ are given by Eq. (4.6).

The stability of the uniform configuration depends on the sign of $V''(x_0)$; if it is positive, $V(x)$ has a local minimum at $x = x_0$ and the layer is stable, if it is negative, then $V(x)$ has a local maximum at $x = x_0$ and a small perturbation in the layer's radius will provoke an irreversible expansion or contraction of the cylinder's radius. Unfortunately, we cannot proceed any further with this analysis until we specify an equation of state $\pi_s(\pi_\perp)$ for the layer's pressures, which after integration of Eq. (4.4) will yield π_s as a function of x. It is clearly important to have this information in order to draw the general shape of $V(x)$. Unfortunately, we do not know anything about the internal physics of the transition layer (which is, we recall, a model for a smooth transition region between the Schwarzschild and de Sitter regions). For the time being, therefore, we have complete freedom on the choice of the equation of state and this represents the only major difficulty of this analysis. A look at Eq. (4.9) tells us that the layer's pressures both depend linearly upon χ (and only χ). It is not physically unreasonable to require that the equation of state be independent of the free parameter χ such that one natural choice for the equation of state is a linear relationship

$$P_s = -cP_\perp , \tag{5.3}$$

where

$$c = (2^{1/4}-1)/(1-2^{-7/4}) \simeq 0.2692 .$$

Alternatively, Eq. (5.3) is clearly the simplest relationship one can try. With this choice, we can now integrate Eq.

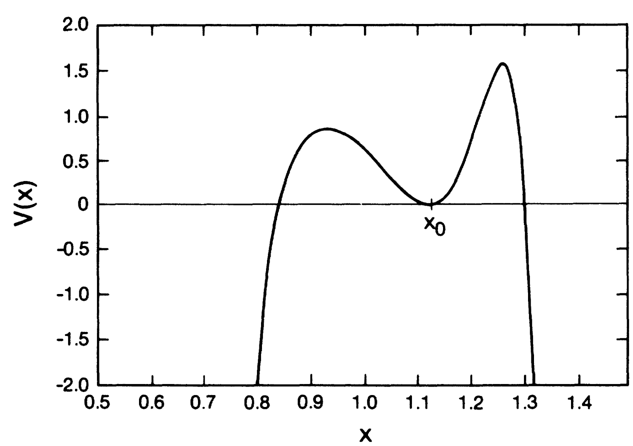

FIG. 5. The potential function $V(x)$, as defined by Eqs. (5.2) and (5.4). Values of x for which $V(x) = \alpha^2 \to 0$ are points for which $\dot{x} = 0$; values of x for which $V'(x) = 0$ are points for which $\ddot{x} = 0$. The point x_0 therefore represents a uniform configuration (constant radius for the three-cylinder as we move along its axis). Since the potential presents a local minimum there, the uniform configuration is stable.

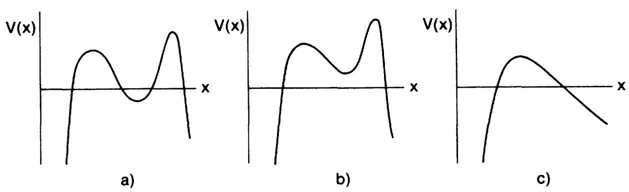

FIG. 6. Shown here are different behaviors for the potential function corresponding to different choices for the parameters c and a. Found in (a) is a potential for which oscillatory configurations are generally possible. In (b), a uniform or oscillatory solution is possible only if the energy parameter α^2 is large enough. In (c), the local minimum of $V(x)$ has disappeared; the possible uniform configuration is therefore unstable.

(4.4) $d(x^2\pi_s)-\pi_\perp d(x^2)=0$ to obtain π_s as a function of x:

$$\pi_s=-ax^{-\gamma}\ , \tag{5.4}$$

where $\gamma\equiv 2(1/c+1)\simeq 9.4294$ and where the constant of integration $a\equiv(2^{1/4}-1)2^{\gamma/6}\simeq 0.5624$ has been chosen such that $\pi_s(x_0)=\pi_s^{(0)}$.

Equation (5.4) now allows us to draw the general behavior of the potential function $V(x)$, see Fig. 5. The potential possesses a local minimum at $x=x_0\simeq 1.1225$ so we immediately see that indeed, this point corresponds to a stable uniform configuration. We can then conclude that the FMM model is stable against a virtual displacement of the layer's radius from uniform configuration, as well as against small perturbations in the global parameters of the system (the black-hole mass m and the de Sitter horizon radius χ^{-1}). Variation of the latter induces only small spatial oscillations in the layer's radius.

Figure 5 also shows that two other kinds of behavior for the three-cylinder radius are possible: one can have a cylinder (pictured as the rubber tube) whose radius is zero at both ends and increases monotonically to a maximum value in the middle; the cylinder can also have an infinitely large radius at both ends (now infinitely far apart) which decreases monotonically to minimum value in between. The situation where the layer's radius would be infinitely large at one end and monotonically decreasing to zero at the other end is not allowed since the energy parameter $\alpha^2\equiv(\chi m)^{-2/3}$ is restricted to be roughly less than one in order for the Schwarzschild horizon to lie outside the de Sitter horizon. These two new configurations clearly have nothing to do with the FMM model, so we shall not discuss them any further.

It is important to note that the shape of the potential and in particular the existence of a local minimum (i.e., the existence of a stable uniform configuration), crucially depend on the value assigned to the parameters c and a defined in Eqs. (5.3) and (5.4). There exists a region in the (c,a) plane around the values given above for which an oscillatory behavior for the layer's radius is still present, as shown in Fig. 6(a). However, there also exists values of c and a corresponding to a potential of the general shape depicted in Fig. 6(b). Here, one needs to carefully adjust the value of the energy parameter in order to recover either a stable uniform configuration or an oscillatory behavior. Worse things can also happen if one chooses negative values for c (e.g., $c=-1$, considered by Blau, Guendelman, and Guth[11] in a different context), as shown in Fig. 6(c). In such cases, the local minimum of $V(x)$ ceases to exist and therefore the uniform configuration found there is unstable. The region of the (c,a) plane which produces uniform or oscillatory solutions is nevertheless broad enough to allow us to conclude that the FMM Schwarzschild–de Sitter model does not require fine-tuning. In its generic form, the model consists of a three-cylindrical transition layer whose radius presents spatial oscillations along the axis.

VI. CONCLUSION

We have presented in this paper a study of one aspect of the black-hole interior model suggested by Frolov, Markov, and Mukhanov[6] according to which a Schwarzschild black hole would give rise to an inflationary universe via a sudden transition from the Schwarzschild solution to a de Sitter solution occurring at the "quantum radius" $r\sim m^{1/3}$. While these authors considered a situation where the transition layer is a uniform (constant radius) three-cylinder occurring at one instant of time, we have allowed the radius to become a function of proper distance along the cylinder's axis.

Our space-dependent shape equations for a spacelike, three-cylindrical transition layer separating a de Sitter future region from a Schwarzschild past region (together with a reasonable equation of state relating the axial pressure to the transverse pressure) succeeded in showing that the FMM uniform configuration is stable in the sense that a variation of the layer's global parameters (the black-hole mass m and the de Sitter horizon radius χ^{-1}), as well as of its internal parameters (the surface pressures) from the uniform configuration does not force the three-cylinder radius to shrink down to zero (as the radius of a cone), but rather induces spatial oscillations in the radius as we move along the axis. This oscillating three-cylinder would then represent the FFM model in its generic form. There is a broad range of values for the parameters c and a for which an oscillatory behavior is possible; it is, however, also possible to vary them in such a way that the model would be destroyed.

The main assumption in the analysis regards our choice for the equation of state $P_s(P_\perp)$. Nothing being known about the internal physics of the transition layer, any choice is in practice as good as the other. We have chosen a linear relationship mainly for simplicity and because it removes the dependence on the free parameter χ (the uniform configuration values $P_s^{(0)}$ and $P_\perp^{(0)}$ depend linearly upon it).

The transition layer is, of course, only a crude model for a smooth transition from Schwarzschild to de Sitter phases. It nevertheless remains interesting to study the physics and "architecture" of such a layer if only for the intriguing properties of spacelike objects in general relativity.

ACKNOWLEDGMENTS

The authors would like to warmly thank Esteban Calzetta, Werner Israel, and Hans Kunzle for many discussions that helped clarify the concepts presented here. E.P. would like to acknowledge generous support from the Natural Sciences and Engineering Research Council of Canada, and Fonds pour la Formation de Chercheurs et l'Aide à la Recherche, Québec, Canada. R.B. wishes to acknowledge the hospitality of the Department of Physics of the University of Alberta, where most of this work was completed.

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