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Integrable equation of state for noisy cosmic string

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It is argued that, independently of the detailed (thermal or more general) noise spectrum of the microscopic extrinsic excitations that can be expected on an ordinary cosmic string, their effect can be taken into account at a macroscopic level by replacing the standard isotropic Goto-Nambu-type string model by the nondegenerate string model characterized by an equation of state of the nondispersive "fixed determinant" type, with the effective surface stress-energy tensor satisfying $(T^\nu{}_\nu)^2 - T^\mu{}_\nu T^\nu{}_\mu = 2T_0^2$, where T_0 is a constant representing the null-state limit of the string tension T, whose product with the energy density U of the string is thereby held fixed: $TU = T_0^2$. It is shown that this equation of state has the special property of giving rise (in a flat background) to explicitly integrable dynamical equations.

The purpose of this work is to point out that the macroscopic behavior of a cosmic string under the influence of microscopic noise perturbations can be modeled by a special case within the uncoupled (and therefore self-dual) subcategory of the broad (in general charge-coupled) class of elastic string models that was originally developed[1] for the quite different purpose of generalizing the ordinary (intrinsically isotropic) Goto-Nambu model[2] to allow for various kinds of "superconduction" mechanisms.[3-7]

Before restricting attention to the physically interesting "noisy cosmic string" case, it is worth recalling the basic principles[1] of elastic string mechanics in a flat- or curved-spacetime background with metric $g_{\mu\nu}$. A covariant description of the kinematics of the string two-surface can conveniently be expressed in terms of the tangent bivector $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$, with $\epsilon^{\mu[\nu}\epsilon^{\rho\sigma]} = 0$, whose integrability is expressible by $\epsilon^{\mu\nu}\epsilon_{\nu\rho}\nabla_\mu\epsilon^{\rho\sigma} = 0$, and whose self-contraction specifies the orthogonal projector onto the string two-surface as $\eta^\mu{}_\nu = \epsilon^{\mu\rho}\epsilon_{\rho\nu}$, the normalization of the bivector and the timelike nature of the string two-surface being fixed by the trace requirement $\eta^\mu{}_\mu = 2$. When there is no external force on the string, its motion will be governed by a two-surface projected divergence equation of the form

$$\eta^\rho{}_\mu \nabla_\rho T^{\mu\nu} = 0 , \tag{1}$$

where $T^{\mu\nu} = T^{\nu\mu}$ is the two-surface stress-energy-density tensor, as restricted by the tangential condition $\gamma^\mu{}_\rho T^{\rho\nu} = 0$ with $\gamma^\mu{}_\nu = g^\mu{}_\nu - \eta^\mu{}_\nu$. For the evolution to be well determined by (1) some *equation of state* $F(T^\nu{}_\nu, T^\mu{}_\nu T^\nu{}_\mu) = 0$ is needed to restrict the number of independent invariants from two to one. Instead of the linear and quadratic invariants $T^\nu{}_\nu$ and $T^\mu{}_\nu T^\nu{}_\mu$, it is more suggestive to use the string tension T say and the string mass-energy per unit length U say (we use units in which the speed of light is unity so that mass and energy are equivalent) that are defined (modulo a sign adjustment) as eigenvalues of the stress-energy tensor by

$$T^{\mu\nu} = U u^\mu u^\nu - T v^\mu v^\nu , \tag{2}$$

where u^μ and v^ν are a mutually orthogonal timelike and spacelike unit tangent vector ($u_\mu u^\mu = -1 = -v_\mu v^\mu$, $u_\mu v^\mu = 0$, $u^\mu v^\nu - v^\mu u^\nu = \epsilon^{\mu\nu}$) specifying what generically will be a preferred reference frame. The exceptions, for which the eigenvalues coincide, are the *null states* (to which we shall return below) meaning states for which $T^\mu{}_\nu$ has a null eigenvector, as in the limit case of an ordinary cosmic-string model, which is specified by the isotropic expression $T^{\mu\nu} = -T\eta^{\mu\nu}$ corresponding to the relation $U = T = T_0$ say for the mass per unit length (whose combination with Newton's constant gives the dimensionless quantity GU taking the constant value GT_0 which must be small to allow neglect of the gravitational effect on the background). In recent literature on cosmic strings the energy per unit length is often denoted by the symbol μ, but the latter has a more traditional interpretation here as the relativistic chemical potential or *effective mass* per unit of the conserved number whose density we shall designate by ν, as defined (modulo multiplicative normalization constants) by

$$\ln\mu = \int \frac{dT}{T-U}, \quad \ln\nu = \int \frac{dU}{U-T} , \tag{3}$$

the normalization being adjusted so that $\mu\nu = U - T$.

These quantities will be associated with corresponding tangential currents $j^\mu = \mu v^\mu$ and $n^\mu = \nu u^\mu$, which satisfy the string two-surface conservation laws

$$\eta_\mu{}^\nu \nabla_\nu j^\mu = 0, \quad \eta_\mu{}^\nu \nabla_\nu n^\mu = 0 , \tag{4}$$

that together are equivalent to the tangentially projected part of the system of equations of motion (1), the remaining part, determining the extrinsic motion, being given by

$$U\gamma^\mu{}_\nu u^\rho \nabla_\rho u^\nu = T\gamma^\mu{}_\nu v^\rho \nabla_\rho v^\nu . \tag{5}$$

It is useful[8] to introduce three more functions of state, the first two being the (preferred frame) propagation speeds c_L and c_T of, respectively, longitudinal and transverse perturbations, while the third is the (self-dual) "characteristic potential" Φ say, as defined up to an additive constant by

$$\Phi = \int c_L^{-1} \frac{dT}{U-T} . \tag{6}$$

It can be shown generally[8] that the "spasm" speed c_L and the "kink" speed c_T will be given by

$$c_L{}^2 = \frac{\nu}{\mu}\frac{d\mu}{d\nu} = -\frac{dT}{dU}, \quad c_T{}^2 = \frac{T}{U}. \tag{7}$$

Let $L_+{}^\mu$ and $L_-{}^\mu$ be "right-" and "left-" oriented longitudinal bicharacteristic vectors, and similarly let $T_+{}^\mu$ and $T_-{}^\mu$ be transverse bicharacteristic vectors. Imposing a unit normalization condition gives them explicitly as

$$L_\pm{}^\rho = \frac{u^\rho \pm c_L v^\rho}{\sqrt{1-c_L{}^2}}, \quad T_\pm{}^\rho = \frac{u^\rho \pm c_T v^\rho}{\sqrt{1-c_T{}^2}}, \tag{8}$$

the stress-energy tensor being expressible directly in terms of the latter as $T^{\rho\sigma} = \mu\nu T_+{}^{(\rho}T_-^{\sigma)}$.

The introduction of these quantities allows the equations of longitudinal motion (4) to be reassembled[8] as the characteristic combinations

$$L_\pm{}^\rho(\nabla_\rho\Phi \mp v^\sigma \nabla_\rho u_\sigma) = 0, \tag{9}$$

while the equations of extrinsic motion (5) can similarly be expressed in characteristic form as

$$\gamma^\mu{}_\nu T_+{}^\rho \nabla_\rho T_-{}^\nu = 0 = \gamma^\mu{}_\nu T_-{}^\rho \nabla_\rho T_+{}^\nu, \tag{10}$$

the equivalence of the left- and right-hand-side versions expressing the kinematic integrability condition for the string two-surface, so that the combined system of (12) and (13) provides the complete set of equations of motion of the string.

The most obviously simple kind of equation of state is what we shall refer to as the "fixed trace" type, meaning that $T^\nu{}_\nu$ is held to a constant value by fixing the sum of U and T, taking

$$T^\nu{}_\nu + 2T_0 = 0, \quad U + T = 2T_0, \tag{11}$$

which corresponds to

$$U = T_0 + \frac{\nu^2}{2a}, \quad T = T_0 - \frac{a\mu^2}{2}, \quad \nu = a\mu, \tag{12}$$

where T_0 and a are constants, the latter arising as the free constant of integration in (3). (Most of the earlier writings on superconducting cosmic-string mechanics,[3-5] and many more recent studies,[9-11] have in effect been based on the use of an equation of state of this "fixed trace" type, which of course makes no allowance for current saturation effects. The reader should be warned against possible confusion resulting from the failure, in nearly all of these writings, to distinguish clearly between the true tension T and the constant T_0.)

A "fixed trace" equation of state arises naturally as a lowest-order approximation in the physical problem under consideration here, namely that of the macroscopically averaged effect of the microscopic excitations that will always be present to some degree in any physically realistic cosmic-string model. For a simple (isotropic) cosmic string the formula (7) just gives the well-known result that propagation occurs at the speed of light, i.e., $c_T = 1$ in the units we are using. For a simple purely "left" or purely "right" propagating microscopic wave packet, the underlying isotropy ensures that the corresponding macroscopically averaged contribution $\delta T^{\mu\nu}$ say to the modification of effective two-surface stress-energy tensor $T^{\mu\nu}$ of the string will have the form $\delta T^{\mu\nu} = l^\mu l^\nu$, with $\delta T_\nu{}^\nu = 0$, where l^μ is a suitably normalized null (lightlike) tangent vector field along the direction of propagation of the packet in the string two-surface. When added to the isotropic stress-energy contribution $T^{\mu\nu} = -T_0\eta^{\mu\nu}$ of the original unperturbed cosmic string this gives a net stress-energy tensor of the form $T^{\mu\nu} = -T_0\eta^{\mu\nu} + l^\mu l^\nu$, with $l^\nu l_\nu = 0$ which is the standard form for a null state [i.e., the degenerate alternative to the generic form (2)] in which (modulo a sign adjustment) T_0 is the repeated eigenvalue.

The combined effect $\Delta T^{\mu\nu} = \sum \delta T^{\mu\nu}$ produced by many such microscopic packets moving in both directions will retain the property of being trace-free while breaking the isotropy, thus determining a preferred rest frame characterized by timelike and spacelike unit vectors u^μ and v^μ as introduced above, in terms of which one will obtain expressions of the standard form

$$\eta^{\mu\nu} = -u^\mu u^\nu + v^\mu v^\nu, \quad \Delta T^{\mu\nu} = \varepsilon(u^\mu u^\nu + v^\mu v^\nu), \tag{13}$$

where ε is the corresponding rest-frame energy perturbation. Thus at a macroscopic level one obtains for the total effective two-surface stress-energy tensor $T^{\mu\nu}$ of the perturbed or "noisy" string an estimate expressible as

$$T^{\mu\nu} = -T_0\eta^{\mu\nu} + \Delta T^{\mu\nu} + O(\varepsilon^2), \tag{14}$$

which can be put in the standard form (2) by taking $U = T_0 + \varepsilon + O(\varepsilon^2)$, and $T = T_0 - \varepsilon + O(\varepsilon^2)$. To first order in ε this evidently corresponds to an equation of state of the "fixed trace" form (11), the corresponding conserved number density ν with its associated effective mass μ being definable, modulo a normalization constant a, by $\nu^2 = 2a\varepsilon + O(\varepsilon^2)$, and $\mu^2 = 2\varepsilon/a + O(\varepsilon^2)$, in accordance with (12).

The foregoing reasoning is independent of the form of the "noise" spectrum, but one can of course be more specific if the microscopic excitations are of purely thermal origin with an ordinary Bose spectrum, the two-dimensional adaptation of the standard derivation of a blackbody radiation distribution leading to the estimate

$$\varepsilon = \frac{ak^2\Theta^2}{2} = \frac{\sigma^2}{2ak^2}, \tag{15}$$

where Θ is the temperature and σ the entropy per unit length of string, k being Boltzmann's constant. In the case of a four-dimensional spacetime background, for which there are just two transverse polarization modes, the coefficient a in (15) works out as $a = 2\pi/3\hbar$, where $\hbar$ is the Dirac-Planck constant. In this thermal case, i.e., when the "noisy cosmic string" is interpretable more specifically as a "warm cosmic string," the number density ν and the associated effective mass μ will have a corresponding thermal interpretation as being respectively proportional to the entropy density and temperature in the form $\nu = \sigma/k$, $\mu = k\Theta$.

Regardless of the thermal or other (less unmusical?) nature of the microscopic noise spectrum, it is of interest to consider the nonlinear modification of the foregoing equation of state that will be needed when the amplitudes of the microscopic excitations are sufficiently large for corrections of higher order in the energy deviation $\varepsilon = (U-T)/2$ to be important. The need for a nonlinear correction to the naive "constant trace" equation of state can be seen following the consideration that this equation (11) is (uniquely) characterized by the property (which was effectively demonstrated from a different point of view in the analysis of Spergel, Piran, and Goodman[4]) that longitudinal perturbations always propagate at the speed of light, i.e., $c_L = 1$. However, the transverse perturbations have a reduced propagation speed, $c_T = 1 - 2\varepsilon + O(\varepsilon^2)$, so that here is a dispersion effect for the "fixed-trace" model which should not occur in a realistic treatment of the "noisy-string" application.

To see why a satisfactory "noisy-string" model should be dispersion-free, meaning $c_L = c_T$, it suffices to consider that if one zooms in to a microscopic perspective, what had appeared to be longitudinal perturbations in the nondegenerate macroscopic string model will become resolved as packets of very-short-wavelength transverse oscillations. At the microscopic level all the perturbations, whether of short or long wavelength, will move along the same null characteristics of the isotropic string model. What changes at the macroscopic level is that as a result of twisting and turning through the microscopic wrinkles of the isotropic string two-surface, the unresolved smoothed average paths of these microscopically null characteristics will be timelike, but being slowed down to subluminal averaged speeds should not change the fact that *all propagation modes share the same characteristics.*

It can be seen from (7) that the crucial condition $c_L = c_T$ leads uniquely to the replacement of the "fixed trace" equation of state by a "fixed determinant" equation of state, meaning that the determinant of the mixed form T^{ν}_{μ} of the two-surface stress-energy tensor, i.e., the scalar $\frac{1}{2}\varepsilon^{\mu\nu}\varepsilon_{\rho\sigma}T^{\rho}_{\mu}T^{\sigma}_{\nu}$, is held to a constant value, T_0^2 say, by fixing the product of U and T:

$$(T^{\mu}_{\nu})^2 - T^{\mu}_{\nu}T^{\nu}_{\mu} = 2T_0^2, \quad UT = T_0^2 . \tag{16}$$

This "fixed determinant" equation of state has a parametric representation in terms of the characteristic potential Φ in the form

$$U = T_0 \coth\Phi, \quad T = T_0 \tanh\Phi , \tag{17}$$

so that $c_L = c_T = \tanh\Phi$ with $\nu = (aT_0)^{1/2}\mathrm{csc}\Phi$, and $\mu = (T_0/a)^{1/2}\mathrm{sech}\Phi$, where a is normalized so that in the relativistic limit as ν and μ tend to zero, this formulation is consistent with the "fixed trace" limit as given by (12), whose analogue for the "fixed determinant" case will be given by $U = T_0(1 + \nu^2/aT_0)^{1/2}$, and $T = T_0(1 - a\mu^2/T_0)^{1/2}$. In the opposite (Newtonian) limit as Φ tends to zero this equation of state goes over to a form describable as that of a nonrelativistic polytrope with index equal to minus one.

The "no-dispersion" property $T_{\pm}{}^{\mu} = L_{\pm}{}^{\mu}$ that thus uniquely characterizes the "fixed determinant" equation of state gives rise to another elegant special property by enabling (9) to be used to extend (10) by removal of the orthogonal projection, giving a *full* set of equations of motion in the single expression

$$L_{\pm}{}^{\nu}\nabla_{\nu}L_{\mp}{}^{\mu} = 0 . \tag{18}$$

This means that the unit tangent to the "left"-moving characteristics is parallel propagated along the "right"-moving characteristics and vice versa.

All the numbered results above apply to a spacetime background of arbitrary curvature and any dimension greater than two. If the background is now restricted to be *flat* then the most general solution to (19) can be obtained explicitly as the locus of midpoints of straight lines between points on an arbitrary given pair of timelike generating curves, the corresponding characteristic curves being obtained by restricting one or the other end point to be fixed. While knowning the Minkowski coordinates, x^{μ} say, of points on the string two-surface is not quite sufficient by itself, specification of the characteristic curves provides all that is needed for a complete dynamical description, since from $L_{+}{}^{\mu}$ and $L_{-}{}^{\mu}$ one gets the state parameter Φ [since $L_{+\mu}L_{-}{}^{\mu} = -\cosh(2\Phi)$] and the preferred rest frame [since $2u^{\mu} = \mathrm{sech}\Phi(L_{+}{}^{\mu} + L_{-}{}^{\mu})$]. Let us express the two freely chosen timelike generating curves as $x^{\mu} = 2f_{\pm}{}^{\mu}(\lambda)$, where λ is an arbitrary, in general, improper, time parameter, and let us use a dot for differentiation with respect to an affine time parameter, τ say, as defined by $\dot{f}_{\pm}{}^{\mu} = \dot{\lambda}\, df_{\pm}{}^{\mu}/d\lambda$ with the normalization condition $\dot{f}_{\pm\mu}\dot{f}_{\pm}{}^{\mu} = -1$. Then the corresponding string two-surface locus and the associated unit bicharacteristic vectors will be given by

$$x^{\mu} = f_{+}{}^{\mu}(\lambda_{+}) + f_{-}{}^{\mu}(\lambda_{-}), \quad L_{\pm}{}^{\mu} = T_{\pm}{}^{\mu} = \dot{f}_{\pm}{}^{\mu}(\lambda_{\pm}) , \tag{19}$$

the two independent parameters $\lambda_{\pm}$ acting as characteristic coordinates. It is always possible to use an affine parametrization, identifying λ with τ, but it is often preferable to use a nonaffine gauge condition identifying λ with the coordinate time, x^0 say, along each generating curve, giving the constant values $df^0_{\pm}/d\lambda = 1$, so as to obtain $\lambda_{+} + \lambda_{-} = x^0$.

The general flat-background solution given above is akin to the well-known Eisenhart flat-background solution for the degenerate (ultrarelativistic) case, which is also given by an expression of the form $x^{\mu} = f_{+}{}^{\mu}(\lambda_{+}) + f_{-}{}^{\mu}(\lambda_{-})$ but with the restriction that the separate generating curves $x^{\mu} = 2f_{\pm}{}^{\mu}(\lambda)$ are not timelike but null. Although it has the advantage that the gauge choice $\lambda_{+} + \lambda_{-} = x^0$ ceases to be inconsistent with affine parametrization, the null tangency condition is a restriction whose implementation[12] raises nontrivial problems.[13] The much discussed phenomenon of cusps (where $T_{+}{}^{\mu}$ and $T_{-}{}^{\mu}$ coincide) that are well known to occur as a generic (though transient) feature in the ultrarelativistic "silent" cosmic-string case in a background with dimension less than the critical value five[14] (and as durable rather than transient features when the dimension is less than four, as in the effectively three-dimensional case of a string motion within a plane in ordinary spacetime such as is illustrated in Fig. 1) will have its critical dimensions reduced in the "noisy" string case due to the extra degree of freedom gotten from removal of the null tangency restriction on the functions $f_{\pm}{}^{\mu}(\lambda)$. For a "noisy" cosmic-string model, the occurrence of cusps (corresponding, in this case, not to motion at the speed of light but to singular concentrations of infinite energy density U) will, as a generic feature, be entirely absent in a four- (or higher-) dimensional space-

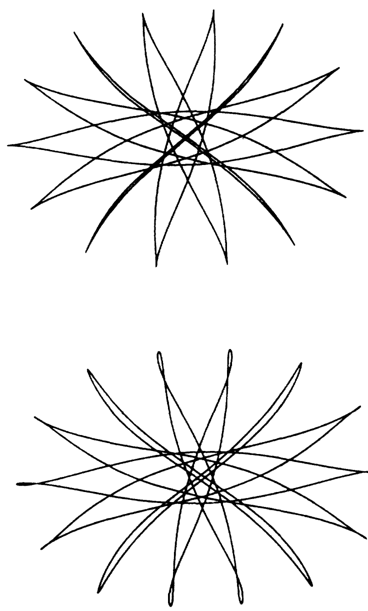

FIG. 1. The figure shows successive space configurations for two neighboring (periodic but nonstationary) spinning-loop solutions obtained by subjecting the spinning-line solution for an ordinary cosmic string to a small perturbation within the plane of rotation: first while retaining the use of the ordinary ("silent") cosmic-string model so that the cusps remain as a permanent feature at the extremities; and second with allowance for some "noise" energy in the perturbation by the use of the "fixed determinant" string model so that the cusps are smoothed out except for a transient occurrence (which would disappear altogether if the motion were allowed to deviate from the plane).

time background. Even in the effectively three-dimensional case of motion in a plane, cusps will generically occur only as transient features, as is apparent in the example illustrated in Fig. 1, which represents a slight deviation from the solution that (apart from a stationary ring, which is possible[15,16] except in the Goto-Nambu limit) is the simplest, namely a *spinning line,* stationary with respect to the *corotating* frame, with tension determined by the angular frequency ω as a function of radius by $T^2/T_0{}^2=(r_0{}^2-r^2)(\omega^{-2}-r^2)$, diminishing to zero at cusps at the extremities, $r=r_0$, where the string turns back on itself. [The spinning-line solution has as a nonstationary but axisymmetric analogue the *radially bouncing circle* solution with harmonically varying radius $r=r_0\cos\omega t$, in which the tension is given by $T^2/T_0{}^2=r^2/(\omega^{-2}-r_0{}^2+r^2)$.]

It is to be remarked that (16) is an example of a "self-dual" equation of state that also turns up[16] in another physical context as the outcome of the Nielsen mechanism[17,18] (a prototype for the more realistic "superconducting string" mechanisms introduced by Witten[3]) whereby a Kaluza-Klein procedure is applied to a simple (isotropic) cosmic string, starting from a spacetime with an extra dimension in the form of a bundle structure whose fibers are Killing vector trajectories.

Note added. The validity of the model presented above has recently been confirmed using a different approach by Vilenkin.[19]

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