PHYSICAL REVIEW D VOLUME 41, NUMBER 12 15 JUNE 1990

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Naked singularities in self-similar gravitational collapse

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(Received 12 October 1989)

We formulate precise geometric criteria for the development of naked strong-curvature singularities in self-similar spacetimes.

It is now some twenty years since Penrose[1] first articulated the idea that, in the words of Israel,[2] classical general relativity contains a built-in safety feature that precludes the formation of naked singularities in generic gravitational collapse. This conjecture, which has become known as the cosmic censorship hypothesis, remains as the central unresolved issue in classical general relativity.[3] Whereas there is at present no precise mathematical formulation of this conjecture, there would appear to be a growing body of (spherically symmetric) evidence against it.[2] Recently, this counterevidence has been extended beyond dust[4] to include adiabatic perfect fluids[5] by way of the assumption of self-similarity. Indeed, the appearance of naked singularities in this type of collapse would appear to be a common feature with soft equations of state.[6]

There is a now traditional distinction between weak and strong versions of the cosmic censorship hypothesis.[1,7] In its weak form (asymptotic predictability[8]) the conjecture precludes causal influence of the singularity in the asymptotic regions of the spacetime. In its strong form the conjecture holds that the singularity must be spacelike.[2] In known examples, the strong form is violated directly, and the weak form is violated with the aid of a suitable truncation of the offending field, when the spacetime is not asymptotically flat.

Whereas the relation between the characterization of the strength of singularities and the cosmic censorship hypothesis is open to refinement, it is known that of the examples available the self-similar ones are distinguished by strong-curvature singularities.[9] Further, the examples are spherically symmetric.[10] It is important to show that the associated naked singularities do not arise out of this simplification. In this Rapid Communication we formulate precise geometric criteria for the development of naked strong-curvature singularities in self-similar spacetimes. In particular, we show that if a self-similar spacetime contains a null geodesic orbit of the self-similarity group, then as long as κ [defined by Eq. (5)] and c and d [defined by Eqs. (8) and (11)] are not zero the spacetime contains a strong-curvature singularity.

First we make our use of the term self-similar precise. We presume a four-dimensional manifold M with Lorentz metric g such that g admits a global one-parameter Lie group of homothetic motions (G) generated by a vector field ξ. That is, in local coordinates,

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = \tfrac{1}{2}\Theta g_{\alpha\beta}\,, \tag{1}$$

where ∇_α denotes covariant differentiation with respect to g, and $\Theta = \text{const} \neq 0$.[11] We designate by t ($-\infty < t < +\infty$) the natural group parameter for the integral curves of ξ so that in local coordinates x^α $\xi^\alpha = dx^\alpha/dt$. In this Rapid Communication we consider incomplete null geodesic orbits of G.[12]

We begin with some properties of homothetic orbits.[13] As a result of Eq. (1) we have the following intrinsic derivatives:

$$\frac{\delta \xi^\alpha}{\delta t} \equiv \xi^\beta \nabla_\beta \xi^\alpha = \tfrac{1}{2}[\Theta \xi^\alpha - \nabla^\alpha(\xi^\beta \xi_\beta)]\,, \tag{2}$$

$$\frac{\delta(\xi^\alpha \xi_\alpha)}{\delta t} = \frac{\Theta}{2}\xi^\alpha \xi_\alpha\,, \tag{3}$$

and

$$\frac{\delta[\nabla_\beta(\xi^\alpha \xi_\alpha)]}{\delta t} = \frac{\Theta}{2}\nabla_\beta(\xi^\alpha \xi_\alpha) - (\nabla_\beta \xi^\gamma)\nabla_\gamma(\xi^\alpha \xi_\alpha)\,. \tag{4}$$

According to Eq. (2) the orbit is a geodesic as long as there exists a function $\kappa(t)$ such that

$$\kappa(t)\xi^\alpha = \tfrac{1}{2}[\Theta \xi^\alpha - \nabla^\alpha(\xi^\beta \xi_\beta)]\,. \tag{5}$$

Further, along a geodesic orbit it follows from Eqs. (1)-(5) that

$$\frac{d\kappa}{dt} = 0\,. \tag{6}$$

Note that for $\xi^\alpha \xi_\alpha \neq 0$ it follows from Eqs. (3) and (5) that $\kappa = \Theta/4$. In the null case, however, κ is not restricted. If $\kappa = 0$ it follows from Eqs. (2) and (5) that t is an affine

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parameter and so the orbit is a complete null geodesic. In this Rapid Communication we will be concerned with incomplete null geodesics. For $\kappa \neq 0$ let $k^a \equiv e^{-\kappa t}\xi^a$. Then $\delta k^a/\delta t = 0$. Writing $k^a = dx^a/d\lambda$, where λ is an affine parameter, we have

$$\lambda = \frac{1}{\kappa} e^{\kappa t}, \tag{7}$$

where we have used the equivalence of affine parameters up to finite linear transformations.[14] It is clear from Eq. (7) that the null geodesics are incomplete.[15]

We now turn to the behavior of scalar invariants along the incomplete null geodesic orbits. Suppose that at some point p along the geodesic

$$R_{\alpha\beta}\xi^\alpha\xi^\beta|_p = c \tag{8}$$

where $R_{\alpha\beta}$ is the Ricci tensor, and c is a nonzero constant. From Eq. (1) we have the Lie derivative

$$\mathcal{L}_\xi(R_{\alpha\beta}\xi^\alpha\xi^\beta) = 0 \tag{9}$$

from which we obtain

$$\lambda^2 R_{\alpha\beta}k^\alpha k^\beta = \frac{c}{\kappa^2}. \tag{10}$$

Further, since $\mathcal{L}_\xi K = -2\Theta K$, where K is the Kretschmann scalar ($\equiv R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, $R_{\alpha\beta\gamma\delta}$ the Riemann-Christoffel tensor) we have

$$K = d\exp(-2\Theta t), \tag{11}$$

where d is a nonzero constant. We conclude from Eqs. (10) and (11) that the incomplete null geodesic orbits terminate in a strong-curvature singularity at $\lambda = 0$ for $\kappa/\Theta > 0$.[16] Notice that this conclusion has been obtained without reference to any field equations.

It is instructive to consider a simple example.[17] A spherically symmetric spacetime in Bondi coordinates has the metric

$$ds^2 = 2c dv\, dr - c^2\left(1 - \frac{2m}{r}\right)dv^2 + r^2 d\Omega^2, \tag{12}$$

where $d\Omega^2$ is the metric of a unit sphere, and c and m are functions of r and v only. Suppose that $c = +1$ (v the "advanced time") and that $m/r \equiv M(J)$ where $J = v/r$. It then follows that the vector field $\xi = v\partial/\partial v + r\partial/\partial r$ obeys $\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha = 2g_{\alpha\beta}$ and so generates a one-parameter group of homothetic motions. Further, the orbits are null if and only if[18]

$$[1 - 2M(J)]J = 2. \tag{13}$$

Given the function $M(J)$, the algebraic equation (13) gives the number of null orbits of the homothetic group. For example, if $M = nJ$, where n is a positive constant, it follows that for $n > \frac{1}{16}$ there are no null orbits, but for $n = \frac{1}{16}$ there is a null orbit, and for $n < \frac{1}{16}$ there are two. Furthermore, since

$$\nabla_\alpha(\xi^\beta\xi_\beta) = -2(nJ^2 - 1)\xi_\alpha \tag{14}$$

for the null orbits, it follows that the null orbits are incomplete null geodesics which terminate in a strong curvature singularity.[19]

We have formulated precise geometric criteria for the development of naked strong-curvature singularities in self-similar spacetimes. As a result of this analysis, the most serious known counterexamples to cosmic censorship are not a result of their assumed spherical symmetry, but rather may be viewed as a byproduct of self-similarity. The dynamics of Einstein's equations does not play a role in the formulation of these criteria. One may view the dynamics as a filter between regular initial data and the end state in gravitational collapse. A central issue then is the effectiveness of the Einstein dynamics in filtering out self-similar end states from arbitrary (non-self-similar) initial data. Although cosmic censorship is usually restricted to classical general relativity, it is worth mentioning here that quantum effects are expected to break scale invariance.

This work was supported by the Natural Sciences and Engineering Research Council of Canada.

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[9]Regarding the strength of singularities, see F. J. Tipler, C. J. S. Clarke, and G. F. R. Ellis, in *General Relativity and Gravitation,* edited by A. Held (Plenum, New York, 1980); C. J. S. Clarke and A. Królak, J. Geom. Phys. **2**, 127 (1986); Newman, Ref. 4. The strength of singularities in spherically symmetric self-similar spacetimes has been examined by K. Lake, Phys. Rev. Lett. **60**, 241 (1988); B. Waugh and K. Lake, Phys. Rev. D **38**, 1315 (1988); **40**, 2137 (1989); V. Gorini, G. Grillo, and M. Pelizza, Phys. Lett. A **135**, 154 (1989); Ori and Piran, Ref. 6. See also Newman, Ref. 4, and Ref. 17.

[10]The extension to plane and hyperbolic symmetry is straightforward.

[11]If $\Theta = 0$ Eq. (1) reduces to Killing's equation.

[12]The behavior of scalar invariants along null orbits of G has been examined by C. B. Collins and J. M. Lang, Class. Quantum Grav. **3**, 1143 (1986). However, as pointed out by G. S. Hall, *ibid.* **5**, L77 (1988), their analysis is incomplete.

[13]For further discussion, see T. Zannias, Queen's Univ. report, 1989 (unpublished).

[14]Further specification of κ requires the examination of a congruence of homothetic orbits which we have not assumed. For a study of such congruences see C. C. Dyer and E. Honig, J. Math. Phys. **20**, 1 (1979); **20**, 409 (1979).

[15]Provided that the orbit can be extended far enough, it is clear that $\xi^a = 0$ at the terminal point.

[16]Naked singularities are those for which the orbit terminates in the *past*.

[17]If one imposes the Einstein equations, the example is an ingoing Vaidya spacetime. See K. Rajagopal and K. Lake, Phys. Rev. D **35**, 1531 (1987).

[18]The orbits with $J=0$ give $c=0$ in Eq. (8) and $\kappa=1$ in Eq. (5). $M(J)=0$ gives Minkowski space and here the orbits $J=2$ also give $c=0$ and $\kappa=1$. However, for both these orbits $d=0$ in Eq. (11). The orbits are incomplete, but extendable since they do not terminate in a singularity.

[19]The streamlined method given of course applies to more general situations. See B. Waugh, K. Lake, and T. Zannias, Queen's Univ. report, 1989 (unpublished).