Noncommutativity of constraining and quantizing: A U(1)-gauge model

R. Loll
Physikalisches Institut, Rheinische Friedrich-Wilhelm-Universität Bonn, Nussallee 12, D-Bonn 1, West Germany
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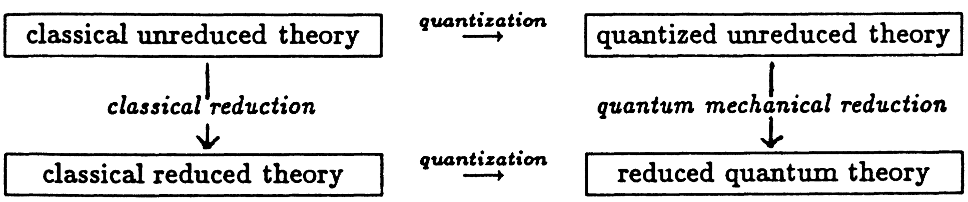
After discussing the general question of (non)commutativity of constraining and quantizing, we present a quantum-mechanical model with a U(1)-gauge constraint. The phase spaces before and after the symplectic reduction are topologically nontrivial, and hence the usual canonical quantization has to be modified. We show that Isham quantization can be used in both cases. The groups governing the quantizations are $Sp(4,\mathbf{R}) \simeq SO(3,2)$ for the unreduced and $Sp(2,\mathbf{R}) \simeq SO(2,1)$ for the reduced theory. The quantizations are nonunique and the analysis of the Dirac condition depends on the quantization chosen. In most cases quantization and constraining do not commute; in particular, we may find additional observables if we quantize before reduction.

I. INTRODUCTION

A foremost problem in physics today remains the quantization of theories with gauge degrees of freedom. Many interesting physical theories possess gauge invariances: for example, gauge field theories, string theories, and quantum gravity. The main problem lies in the treatment of the unphysical degrees of freedom, whose presence leads to ill-defined features in both canonical and path-integral quantization approaches.

If a theory possesses gauge degrees of freedom (or more general types of constraints), there are in principle two ways of proceeding: one can either reduce the classical system and then quantize the observables of the reduced system, or quantize first and then apply some quantum-mechanical reduction to arrive at the "true" quantum theory. This situation is depicted by the following diagram:



It is a subtle question whether or not one expects this diagram to be commutative, and this depends on the status given to the constraints. In particular cases there may be physical reasons for adopting commutativity as an axiom, and this will result in rather stringent conditions on the way both the quantization and the quantum-mechanical reduction are implemented. However, one does not expect this to be true in general since the constraints themselves may be subject to quantization. This can lead to quantum effects which are not present when the system is reduced classically before the quantization, and hence to the noncommutativity of the above diagram. A similar reasoning can be found in some theories about the creation of the Universe which rely on the existence of a phase-space tunneling (see Ref. 1 and references therein). The question may be rephrased in a quantum-mechanical context: does a particle "know" that it is constrained? Does it "remember" it is moving on a submanifold of some *bigger* phase space?

One example of noncommutativity is provided by Yang-Mills theory coupled to chiral fermions, as discussed by Faddeev and Slavnov.[2] There the algebra of the Gauss-law constraints acquires an anomaly in the quantization, and this way some of the first-class constraints are turned into second-class constraints. As a result the reduced quantum theory has more degrees of freedom if we quantize first instead of reducing the classical theory before quantization.

In this paper we will present a finite-dimensional Hamiltonian gauge system where noncommutativity arises in a different way. We choose the gauge group to be compact in order to avoid that solutions to the Dirac condition on physical state vectors are non-normalizable. Together with the requirements of freeness of the group action this leads to topologically nontrivial phase spaces for both the original and the reduced systems. The standard canonical quantization has to be modified, and we show that Isham quantization can be used in both cases. Then we compare the quantization of the classically reduced system with the theory obtained by quantizing first and then using the Dirac prescription[3] to reduce quantum mechanically. Neither the unreduced nor the reduced theory possess a unique quantization, and the analysis of the Dirac condition depends on the quantizations chosen. In general we get different results constraining before or after the quantization. In the latter case we may find fewer degrees of freedom or an additional gauge-invariant quantum observable, reflecting the noncommutativity of diagram 1.

II. THE CLASSICAL GAUGE MODEL

The gauge system under consideration is a finite-dimensional first-class constrained system, with U(1) acting as a compact gauge group. The classical phase space is $S=\mathbf{R}^4/\{0\}$ with coordinates (x_1,x_2,p_1,p_2) and the symplectic form $\omega=dx_1\wedge dp_1+dx_2\wedge dp_2$ inherited from $\mathbf{R}^4(\approx \mathbf{T}^*\mathbf{R}^2)$. The Hamiltonian is that of the harmonic oscillator with unit mass and frequency, $H=\frac{1}{2}(p_1^2+p_2^2+x_1^2+x_2^2)$. The gauge group action is given by the lift to $\mathbf{R}^4\approx\mathbf{T}^*\mathbf{R}^2$ of the defining action of $SO(2)\approx U(1)$ on $\mathbf{R}^2$ [with coordinates (x_1,x_2)] by rotations about the origin. The corresponding quadratic phase-space constraint is $f=x_1p_2-x_2p_1=0$, with the associated Hamiltonian vector field

$$X_f=-x_2\frac{\partial}{\partial x_1}+x_1\frac{\partial}{\partial x_2}-p_2\frac{\partial}{\partial p_1}+p_1\frac{\partial}{\partial p_2} . \tag{2.1}$$

This vanishes only at the origin $\{0\}$ and hence generates a free Hamiltonian U(1) action on $\mathbf{R}^4/\{0\}$. We have to remove the origin from $\mathbf{R}^4$ in order for the quotient $S/U(1)$ to be a well-behaved manifold. This way the flat phase space S acquires a nontrivial topology, a fact which will have far-reaching consequences in the quantum theory. Note furthermore that S is not of the form of a cotangent bundle T^*Q of some configuration space Q (and, strictly speaking, we need now at least two coordinate charts to "patch around the hole" in $\mathbf{R}^4/\{0\}$). The Hamiltonian H is gauge invariant $\{H,f\}=0$; hence it will project down unambiguously to the quotient space.

After factoring out by the gauge degree of freedom and going to the constraint surface defined by $f=0$, we find the reduced phase space $S_{\rm red}=\mathbf{R}^2/\{0\}$. It is parametrized by a pair of canonical coordinates X and P, with $X^2+P^2>0$. Their explicit form in terms of the old variables is

$$\begin{aligned} X&=\frac{1}{\sqrt{2}r}(x_1^2+x_2^2-p_1^2-p_2^2) , \\ P&=\sqrt{2}\frac{1}{r}(x_1p_1+x_2p_2) . \end{aligned} \tag{2.2}$$

The reduced symplectic form on $S_{\rm red}$ induced by ω is given by $\omega_{\rm red}=dX\wedge dP$, i.e., $\{X,P\}=1$. Note that now $r^2=2(X^2+P^2)$; hence, for the reduced Hamiltonian we have $H_{\rm red}=X^2+P^2$. We only mention here that this model is an example of a constrained system that possesses a good gauge fixing only *after* the imposition of the constraint [the principal U(1) bundle becomes trivial only after restriction to the constraint surface].

III. QUANTIZATION

Our next task is to quantize the two classical systems (S,ω,H,f) and $(S_{\rm red},\omega_{\rm red},H_{\rm red})$ and compare the resulting quantum theories. The usual Schrödinger quantization in terms of canonical commutation relations $[x_i,p_j]=i\hbar\delta_{ij}$ and square-integrable functions on configuration space is not appropriate here, as will be explained in the following.

In the case of phase spaces $\mathbf{R}^{2n}$, the reason why one quantizes the $2n$ pairs (x_i,p_i), $i=1,2,\ldots,n$, is that they constitute a complete set of globally defined, smooth coordinate functions on phase space. Any phase-space function can be expressed in terms of this basis set. Thinking of the phase-space functions as generators of symplectic transformations on phase space, this means that by the flows of the Hamiltonian vector fields associated with the (x_i,p_i) we can transform any phase-phase point into any other.

However, as already mentioned earlier, when we remove the origin from $\mathbf{R}^{2n}$, the (x_i,p_i) are no longer globally defined smooth coordinates, and the previous translation invariance in both the x and p directions is broken, i.e., the flows corresponding to the x and p observables do not leave the origin invariant. In the quantum theory this finds expression in the fact that in the usual canonical quantization the spectrum of the operators $\hat{x}_i$ and $\hat{p}_i$ is all of the real line, which is incompatible with the removal of the origin from the phase space.

The quantization of a generic finite-dimensional phase space which is not of the form $\mathbf{R}^{2n}$ (setting aside the dynamics for the moment) is, of course, an unsolved problem. Such phase spaces are usually nonlinear, possess nontrivial topology, and may not even be of the form of a cotangent bundle T^*Q of some configuration space Q. Only a few attempts have been made to tackle this problem in any generality.

In this paper we will follow the group-theoretical quantization approach by Isham[4] which uses some important ideas from the theory of geometric quantization (see, for example, Refs. 5 and 6). This method gives a quantization algorithm for a well-defined generalized class of nonlinear phase spaces. It applies to systems for which one can find a transitive symplectic group action by a finite-dimensional Lie group on phase space. This Lie group (called the *canonical group*) is the analogue of the Heisenberg group for the case of quantization on $\mathbf{R}^{2n}$, and determines a preferred class of observables [a finite-dimensional subalgebra of the Poisson-brackets algebra of the smooth functions $C^\infty(S)$] which are given operator status upon quantization. One then looks for inequivalent unitary irreducible representations of this algebra on some Hilbert space.

This scheme takes the nonlinear structure of phase space seriously, and as a consequence we neither end up with canonical commutation relations, nor usually with a *unique* quantum theory associated with a given classical theory. In an infinite-dimensional context this scheme has recently been successfully applied to the quantization of Hamiltonian gravity, leading to a complete set of observables in a loop-space representation.[7]

In the case at hand we need to find canonical groups for both $\mathbf{R}^4/\{0\}$ and $\mathbf{R}^2/\{0\}$ which encode the nontrivial topological structure of these phase spaces. This will lead to preferred sets of basic observables, i.e., phase-space functions whose Poisson-brackets algebra is isomorphic to the commutator algebra of the generators of the canonical group. Unlike in the quantization on $\mathbf{R}^{2n}$, these basic variables are not the x_i and p_i.

In the quantization the preferred classical observables are mapped into corresponding self-adjoint operators, whose commutator algebra is the same as the Poisson-brackets algebra of the underlying classical observables, with appropriate factors of $i\hbar$. The choice of a preferred set of observables is in accordance with the fact that we are not able to consistently quantize all classical phase-space functions, as is exemplified by the Groenewald-Van Hove theorem for the case of $\mathbf{R}^{2n}$ (see, for example, Ref. 8).

A. The group Sp(2n, R)

Let us now make some general remarks about the group $\mathrm{Sp}(2n,\mathbf{R})$ which will be needed in the following sections. The real symplectic group $\mathrm{Sp}(2n,\mathbf{R})$ is the noncompact simple Lie group of dimension $2n^2+n$ of linear transformations on $\mathbf{R}^{2n}$ preserving the standard symplectic form $\sum_{i=1}^{n} dx_i \wedge dp_i$. Its maximal compact subgroup is $\mathrm{U}(n)$. $\mathrm{Sp}(2n,\mathbf{R})$ is not simply connected, its fundamental group being $\pi_1(\mathrm{Sp}(2n,\mathbf{R}))=\mathbf{Z}$, and it has a unique twofold covering group $\widetilde{\mathrm{Sp}}(2n,\mathbf{R})$. The noncompactness of $\mathrm{Sp}(2n,\mathbf{R})$ implies that all unitary irreducible representations are necessarily infinite dimensional. Representations of $\mathrm{Sp}(2n,\mathbf{R})$ have been studied, for example, in connection with the theory of collective motion in nuclei (see, for example, Ref. 9 and references therein).

Since quantum-mechanical states are given by rays in Hilbert space, we are not only interested in unitary irreducible representations of $\mathrm{Sp}(2n,\mathbf{R})$, but also in those of $\widetilde{\mathrm{Sp}}(2n,\mathbf{R})$, giving rise to projective representations of $\mathrm{Sp}(2n,\mathbf{R})$. However, in the discussion below we will analyze in detail only irreducible representations of $\mathrm{Sp}(2n,\mathbf{R})$ and its twofold covering, which will be sufficient to illustrate the main point of this work.

B. The canonical group for S_{red}

In the present case we are lucky because canonical groups exist for both the original and the reduced systems.

For S_{red} it is given by the three-dimensional real symplectic group $\mathrm{Sp}(2,\mathbf{R})$. $\mathrm{Sp}(2,\mathbf{R})$ acts on S_{red} by its defining representation in terms of real 2×2 matrices leaving invariant the skew-symmetric symplectic two-form $dX \wedge dP$ on $\mathbf{R}^2$. The phase-space functions generating this action are bilinear in the basic coordinate functions X and P and we will take them to be

$$h=\tfrac{1}{2}(P^2+X^2),\quad g=\tfrac{1}{2}(P^2-X^2),\quad c=XP\ , \tag{3.1}$$

with Poisson-brackets algebra

$$\{h,g\}=2c,\quad \{h,c\}=-2g,\quad \{g,c\}=-2h\ . \tag{3.2}$$

The one-parameter subgroups corresponding to the generators $\frac{1}{2}P^2$, $\frac{1}{2}X^2$, and XP are given by

$$\begin{aligned} l(a,0,0)(X,P)&=(X+aP,P)\ ,\\ l(0,b,0)(X,P)&=(X,P-bX)\ ,\\ l(0,0,c)(X,P)&=(e^{c}X,e^{-c}P)\ . \end{aligned} \tag{3.3}$$

To show that this group action is transitive on $\mathbf{R}^2/\{0\}$, we will give the explicit form of the transformation from a point (X,P) to a point (X',P). The complementary transformation from an arbitrary point (X,P) to a point (X,P') is found in a completely analogous manner and will be omitted here. For the transformation $(X,P)\rightarrow(X',P)$ we have to distinguish between two cases.

(a) $P\neq 0$: we simply have

$$(X',P)=l(1/P(X'-X),0,0)(X,P)\ . \tag{3.4}$$

(b) $P=0$: (in this case neither $X=0$ nor $X'=0$ since we have excluded the origin from $\mathbf{R}^2$). The explicit transformation is given by

$$(X',0)=l(0,-bX/X',0)l(1/b(1-X'/X),0,0)l(0,b,0)(X,0)\ , \tag{3.5}$$

for arbitrary $b\neq 0$. The action is not transitive on $\mathbf{R}^2$ because the Hamiltonian vector fields of all the generators vanish at the origin. Hence we have found a good canonical group for the quantization of the reduced phase space; moreover, the reduced Hamiltonian H_{red} is already contained in the linear span of the algebra of observables since $H_{\mathrm{red}}=2H$.

However, this action is only one of a whole one-parameter family of possible $\mathrm{Sp}(2,\mathbf{R})$ actions on $\mathbf{R}^2/\{0\}$. Note that in terms of polar coordinates $(\mathbf{R},\Phi)$, where

$$X=R\cos\Phi,\quad P=R\sin\Phi\ , \tag{3.6}$$

the generators (3.1) read

$$h=\tfrac{1}{2}R^2,\quad g=-\tfrac{1}{2}R^2\cos 2\Phi,\quad c=\tfrac{1}{2}R^2\sin 2\Phi\ , \tag{3.7}$$

and the symplectic form is $\omega=R\,dR\wedge d\Phi$. Now, the set of phase-space functions

$$\begin{aligned} h_n&=(1/n)R^2\ ,\\ g_n&=-(1/n)R^2\cos n\Phi\ ,\\ c_n&=(1/n)R^2\sin n\Phi\ , \end{aligned} \tag{3.8}$$

for arbitrary real $n>0$, forms an $\mathrm{sp}(2,\mathbf{R})$ algebra under Poisson brackets, as can easily be checked. The Hamiltonian vector field $X_{h_n}=(2/n)\partial/\partial\Phi$ corresponding to the observable h_n generates the U(1) subgroup of $\mathrm{Sp}(2,\mathbf{R})$. We can integrate these relations to get a symplectic and transitive group action whenever $2/n$ is an integer; i.e., the possible values for n are $n=2/k$, $k=1,2,3,\ldots$.

For a given value of k the Sp(2,**R**) action winds k times around the origin in $\mathbf{R}^2$ as the U(1) parameter varies between 0 and 2π. The reduced Hamiltonian is given by $H_{\rm red}=nH_n$. The above "natural" Sp(2,**R**) action corresponds to $k=1$.

The action on $S_{\rm red}$ we will have to use in order to compare the quantum theories is the one with $n=1$, induced from the canonical group action on S. Only for this choice of n the level spacing for the quantum Hamiltonians $\hat{H}$ and $\hat{H}_{\rm red}$ is the same.

C. The canonical group for S

Similarly, for $S=\mathbf{R}^4/\{0\}$ the canonical group is the ten-dimensional real symplectic group Sp(4,**R**), acting by real 4×4 matrices leaving the symplectic form $\omega=dx_1\wedge dp_1+dx_2\wedge dp_2$ on $\mathbf{R}^4$ invariant. Again it is straightforward to show that the action is transitive if we exclude the origin. The generators can be written as bilinear functions in x_1, x_2, p_1, and p_2. We can choose a basis of the algebra in such a way that both f and H appear explicitly among the generators:

$$\begin{aligned}
H&=\tfrac{1}{2}(p_1^2+p_2^2+x_1^2+x_2^2)\ ,\\
G&=\tfrac{1}{2}(p_1^2+p_2^2-x_1^2-x_2^2)\ ,\\
C&=x_1p_1+x_2p_2\ ,\\
H'&=\tfrac{1}{2}(p_1^2-p_2^2+x_1^2-x_2^2)\ ,\\
G'&=\tfrac{1}{2}(p_1^2-p_2^2-x_1^2+x_2^2)\ ,\\
C'&=x_1p_1-x_2p_2\ ,\\
D&=\tfrac{1}{2}x_1x_2\ ,\\
E&=\tfrac{1}{2}p_1p_2\ ,\\
f&=x_1p_2-x_2p_1\ ,\\
F&=x_1p_2+x_2p_1\ .
\end{aligned}\tag{3.9}$$

One checks that the observables H, G, C, and F are gauge invariant and that there are no linear combinations K of the other generators with $\{K,f\}=0$ (strong gauge invariance) or $\{K,f\}\approx0$ (weak gauge invariance). The functions H, G, and C form an sp(2,**R**) algebra under Poisson brackets and they project down to observables on the reduced phase space:

$$\begin{aligned}
H_{\rm red}&=X^2+P^2\ ,\\
G_{\rm red}&=-X\sqrt{X^2+P^2}\ ,\\
C_{\rm red}&=P\sqrt{X^2+P^2}\ ,
\end{aligned}\tag{3.10}$$

satisfying

$$\begin{aligned}
\{H_{\rm red},G_{\rm red}\}&=2C_{\rm red}\ ,\\
\{H_{\rm red},C_{\rm red}\}&=-2G_{\rm red}\ ,\\
\{G_{\rm red},C_{\rm red}\}&=-2H_{\rm red}\ ,
\end{aligned}\tag{3.11}$$

with respect to the Poisson brackets on $S_{\rm red}$.

Similar to the case of Sp(2,**R**), we find a two-parameter family of possible Sp(4,**R**) actions on $\mathbf{R}^4/\{0\}$. This is easily verified in terms of polar coordinates (R_1,Φ_1,R_2,Φ_2). There are now two U(1) subgroups, we can introduce real parameters n_1 and n_2 exactly as before and again get the condition of integer valuedness for $2/n_1$ and $2/n_2$ for possible Sp(4,**R**) actions. However, only if we choose $n_1=n_2=2$, do the observables H and f have a polynomial relation with the set of basic Sp(4,**R**) generators (if the relation is nonpolynomial we do know how to quantize these observables), so we are forced to adopt the original "natural" Sp(4,**R**) action.

D. Quantization of the reduced system

The unitarity irreducible representations of Sp(2,**R**) are well known and have been classified a long time ago by Bargmann[10] (see also Ref. 11). Note that we have local group isomorphisms Sp(2,**R**)$\approx$SL(2,**R**)$\approx$SO(2,1) $\approx$SU(1,1). We will discuss only representations in which the spectrum of the Hamiltonian is bonded from below, the so-called positive discrete series.[9,11]

We recall that unitary irreducible representations in this series are labeled by a positive integer $w\geq2$, with Hilbert space $H_w^{(2)}$ given by the holomorphic complex-valued functions $f(z)$ on the upper half plane S_1 which are square integrable with respect to the measure $d\mu_w=y^{w-2}dx\,dy$, with $z=x+iy$. The scalar product on $H_w^{(2)}$ is

$$(f,g)=\int_{S_1}d\mu_w\overline{f(z)}g(z),\quad f,g\in H_w^{(2)}\ .\tag{3.12}$$

The quantization of the basic classical observables h, g, and c of the reduced phase space yields the following self-adjoint operators, acting on holomorphic functions on S_1:

$$\begin{aligned}
\hat{h}&=(1/i)(wz+\partial/\partial z+z^2\partial/\partial z)\ ,\\
\hat{g}&=(1/i)(wz-\partial/\partial z+z^2\partial/\partial z)\ ,\\
\hat{c}&=i(w+2z\partial/\partial z)\ .
\end{aligned}\tag{3.13}$$

A basis of $H_w^{(2)}$ is given by a complete set of (unnormalized) eigenstates of $\hat{h}$:

$$f_n(z)=\frac{(z-i)^n}{(z+i)^{w+n}},\quad n=0,1,2,\ldots\ .\tag{3.14}$$

The spectrum of $\hat{h}$ is discrete and the eigenspaces are nondegenerate:

$$\hat{h}f_n(z)=(w+2n)f_n(z)\ .\tag{3.15}$$

The quantum commutation relations are

$$\begin{aligned}
[\hat{h},\hat{g}]&=2i\hat{c}\ ,\\
[\hat{h},\hat{c}]&=-2i\hat{g}\ ,\\
[\hat{g},\hat{c}]&=-2i\hat{h}\ .
\end{aligned}\tag{3.16}$$

There also exist other unitary realizations of the positive discrete series: e.g., in terms of square-integrable holomorphic functions on the unit disc or in terms of a subspace of functions $L^2(\mathrm{Sp}(2,\mathbf{R}))$ in the regular representation. There is also a lowest-weight representation corresponding to $w=1$ (called mock discrete representation by Lang), which however does not have a realization in terms of holomorphic functions on the upper half plane.

A tool that will help us in identifying certain irreducible representations is the Casimir operator $\hat{K}$, whose eigenvalues characterize inequivalent irreducible representations of Sp(2,**R**). It is given here by[11]

$$\hat{K}=\hat{h}^2-\hat{g}^2-\hat{c}^2 . \tag{3.17}$$

For the positive discrete series we have $\hat{K}=w^2-2w$: i.e.,

w	$\hat{K}$	Spectrum of $\hat{h}$				
1	-1	1	3	5	7	$\cdots$
2	0	2	4	6	8	$\cdots$
3	3	3	5	7	9	$\cdots$
4	8	4	6	8	10	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

(3.18)

There also exist similar representations for nonintegral $w>0$, which are lowest-weight representations of the covering group of Sp(2,**R**) (Ref. 12) [topologically Sp(2,**R**)$\approx S^1\times \mathbf{R}^2$, hence its covering group has the topology of $\mathbf{R}^3$].

E. Quantization of the unreduced system

First of all note that the group Sp(4,**R**) is isomorphic with SO(3,2), the group of symmetries of anti–de Sitter space. Again we will only discuss representations in which the spectrum of the Hamiltonian is bounded from below, and from the positive discrete series[9] we will only consider those with Hilbert spaces of holomorphic functions valued in **C**, and not in $\mathbf{C}^n$, for $n>1$.

These unitary irreducible representations are labeled by a positive integer $w'\geq 4$, with Hilbert space $H_w^{(4)}$, given by the holomorphic complex-valued functions $f(z)$ on the Siegel upper half space S_2. This is the space of symmetric complex 2×2 matrices $z=x+iy$ for which $\det(y)\equiv y_{11}y_{22}-y_{12}^2>0$. The explicit form of the wave functions and the self-adjoint operators can be found in Ref. 9. The quantum commutators of the gauge-invariant observables are

$$[\hat{H},\hat{G}]=2i\hat{C},\quad [\hat{H},\hat{C}]=-2i\hat{G}\ ,$$
$$[\hat{G},\hat{C}]=-2i\hat{H},\quad [\hat{H},\hat{f}]=0\ , \tag{3.19}$$
$$[\hat{G},\hat{f}]=0,\quad [\hat{C},\hat{f}]=0\ .$$

The eigenvalues of $\hat{H}$ are

$$N=2w'+2r,\quad r=0,1,2,\ldots, \tag{3.20}$$

and the degeneracy of the rth eigenspace is $\frac{1}{2}(r+1)(r+2)$, i.e., 1,3,6,10,15,. . . .

In Fronsdal's classification[13] of the discrete series of irreducible representations of SO(3,2) according to $D(E_0,s)$, with E_0 the lowest eigenvalue of the energy (which coincides with $\frac{1}{2}N$ here) and spin s, these are the representations $D(w',0)$. Again, representations for $w'=2$ and $w'=3$ exist, but they do not possess the above realization in terms of square-integrable functions.

The other case in which we are interested is the spinor representation of Sp(4,**R**) (Ref. 14). It is induced by the Schrödinger representation of the phase space $\mathbf{R}^4$, and as such is defined on functions $f(x_1,x_2)$ in $L^2(\mathbf{R}^2)$. The generators (3.9) have their usual coordinate representation in terms of self-adjoint operators. The subset of gauge-invariant operators again satisfies algebra (3.19) by construction.

We get a basis for $L^2(\mathbf{R}^2)$ by forming binomials $\Psi_m(x_1)\Psi_n(x_2)\equiv\Psi_{m,n}(x_1,x_2)$, $m,n=0,1,2,\ldots,$ from two sets of harmonic-oscillator eigenfunctions. The spectrum of $\hat{H}$ is discrete:

$$\hat{H}\Psi_{mn,}(x_1,x_2)=(m+n+1)\Psi_{m,n}(x_1,x_2)\ ; \tag{3.21}$$

the degeneracy of the rth eigenspace (with eigenvalue $r+1$) is $r+1$. The Hilbert space splits into two irreducible parts $H=H^{\text{even}}+H^{\text{odd}}$, spanned by the subspaces of functions with even and odd parity, with spectrum of the Hamiltonian

$$\begin{aligned} &N=1,3,5,7,\ldots \quad \text{(even spinor representation)}\ ,\\ &N=2,4,6,8,\ldots \quad \text{(odd spinor representation)}\ . \end{aligned} \tag{3.22}$$

In Fronsdal's classification these are the so-called singleton representations, $D(\frac{1}{2},0)$ and $D(1,\frac{1}{2})$.

F. Reduction of the quantized system

The next step is that of quantum-mechanical reduction, i.e., implementation of the Dirac condition to project out physical states. The spectrum of the quantum constraint $\hat{f}$ is discrete and hence we do not encounter any problems with non-normalizable eigenvectors of the condition $\hat{f}\Psi_{\text{phys}}=0$. For the set of quantum theories considered above we find three qualitatively different cases.

(i) Spinor representation on H^{odd}. None of the elements in this irreducible representation is annihilated by the constraint:

Spectrum of $\hat{H}$	2	4	6	8	$\cdots$
Degeneracy of eigenspace	2	4	6	8	$\cdots$
No. of gauge-invariant functions	0	0	0	0	$\cdots$

i.e., the physical state space is empty, $H_{\text{phys}}^{\text{odd}}=\{0\}$.

(ii) Spinor representation on H^{even}. There is exactly one gauge-invariant eigenfunction in each eigenspace of $\hat{H}$:

Spectrum of $\hat{H}$	1	3	5	7	...
Degeneracy of eigenspace	1	3	5	7	...
No. of gauge-invariant functions	1	1	1	1	...

The operators $\hat{H}$, $\hat{G}$, and $\hat{C}$ act irreducibly on $H^{\text{even}}_{\text{phys}}$, and we identify this representation with the quantum theory of the reduced system with value $K=-1$ for the Casimir operator $\hat{K}$.

(iii) Discrete series representations. Since we know the explicit realization of the irreducible representation labeled by w', we can construct the gauge-invariant linear combinations for each eigenspace of $\hat{H}$. The result is

Spectrum of $\hat{H}$	$2w'$	$2w'+2$	$2w'+4$	$2w'+6$	...
Degeneracy of eigenspace	1	3	6	10	...
No. of gauge-invariant functions	1	1	2	2	...

The number of gauge-invariant functions in the rth eigenspace of $\hat{H}$ (starting with $r=0$) is $[r/2]+1$. The gauge-invariant functions for given w' are conveniently labeled by a pair of integers $(r,l), r,l=0,1,2,3,\ldots,2l\leq r$:

$$F^{(r,l)}_{w'}:=\frac{(z_{11}z_{22}-z_{12}z_{21}+1)^{r-2l}}{[z_{11}z_{22}+i(z_{11}+z_{22})-z_{12}z_{21}-1]^{w'+r}}[-4z_{12}z_{21}-(z_{11}-z_{22})^2]^l \;. \tag{3.23}$$

Since the spectrum of the Hamiltonian is degenerate, it is clear that $H_{w',\text{phys}}$ does not carry an *irreducible* representation of Sp(2,**R**).

The quantum observables $\hat{H}$, $\hat{G}$, and $\hat{C}$ act on the $F^{(r,l)}_{w'}$; in particular, the action of the Hamiltonian is given by

$$\hat{H}F^{(r,l)}_{w'}=(2w'+2r)F^{(r,l)}_{w'} \;. \tag{3.24}$$

In order to determine the Sp(2,**R**) representation we have constructed by this Dirac prescription, we compute the Casimir operator $\hat{K}=\hat{H}^2-\hat{G}^2-\hat{C}^2$ on the physical state space $H_{w',\text{phys}}$. Its action on physical wave functions is

$$\hat{K}F^{(r,l)}_{w'}=[4w'^2-4w'-8l(1-2l-2w')]F^{(r,l)}_{w'} +8(r-2l)(r-2l-1)F^{(r,l+1)}_{w'} \;. \tag{3.25}$$

We see that in general the $F^{(r,l)}_{w'}$ are not eigenfunctions of $\hat{K}$. Since $\hat{K}$ commutes with $\hat{H}$, $\hat{G}$, and $\hat{C}$, we know that in an irreducible representation it can only be a multiple of the identity operator. Hence we conclude that $H_{w',\text{phys}}$ decomposes into irreducible subspaces under the action of Sp(2,**R**), according to different eigenvalues k of $\hat{K}$.

One can show that for given $w', H_{w',\text{phys}}$ decomposes into an infinite number of irreducible representations of Sp(2,**R**), labeled by an even integer $m=0,2,4,6,\ldots$, with Casimir eigenvalue $k=4(w'+m)^2-4(w'+m)$.

In other words, on the representation space $H_{w',\text{phys}}$ we find a new nontrivial gauge-invariant quantum observable $\hat{K}$, commuting with $\hat{H}$, $\hat{G}$, $\hat{C}$, and $\hat{f}$. This new observable does not have a classical analogue, and it does not appear if we solve for the physical degrees of freedom before the quantization.

G. Planck's constant

So far no mention has been made of how Planck's constant enters into our considerations. On the one hand, $\hbar$ introduces a scale in the quantum theory, on the other hand, it allows quantities to become physically dimensionful. Although we have argued earlier that the usual Schrödinger quantization on $\mathbf{R}^2$ with self-adjoint operators $\hat{x}=x$ and $\hat{p}=(1/i)\hbar d/dx$ is not appropriate here, it nevertheless helps us in determining the quantum scale. Regarding the spinor representation of Sp(2,**R**) as being induced by the Schrödinger quantization on $\mathbf{R}^2$ leads to the following modification of the commutation relations (3.16):

$$[\hat{h}',\hat{g}']=2i\hbar\hat{c}' \;,\quad [\hat{h}',\hat{c}']=-2i\hbar\hat{g}' \;,\quad [\hat{g}',\hat{c}']=-2i\hbar\hat{h}' \;. \tag{3.26}$$

The spectrum of $\hat{h}'$ becomes $(n+1/2)\hbar$, $n=0,1,2,\ldots$, but, like in the spinor representation of Sp(4,**R**), this representation splits into two irreducible ones with spectra

$$N=\tfrac{1}{2}\hbar,\tfrac{5}{2}\hbar,\tfrac{9}{2}\hbar,\ldots,$$

and (3.27)

$$N=\tfrac{3}{2}\hbar,\tfrac{7}{2}\hbar,\tfrac{11}{2}\hbar,\ldots;$$

i.e., successive energy levels differ by the constant $2\hbar$. If we assume that this is also true for the eigenvalues of the Hamiltonian in the other Sp(2,**R**) representations, the scale in the quantum theories for both S and S_{red} becomes fixed.

IV. CONCLUSIONS

In the example presented above, due to the nontriviality of the phase spaces S and S_{red}, the corresponding quantum theories are not unique, and therefore the question of commutativity of constraining and quantizing becomes slightly more involved.

Since in many physical applications one has problems with the quantization of the classically reduced system, it is interesting to ask what kind of reduced theory we may obtain by following Dirac's prescription. In order to illustrate this point, we looked at various possible quantum theories, without aiming at a complete analysis of the representation theory.

If we reduce the gauge system classically and quantize the reduced system, possible quantizations correspond to different unitary irreducible representations of Sp(2,**R**), labeled by the value of the Casimir operator $\hat{K}$.

If we quantize first and reduce at the quantum level, we find that the analysis of the Dirac condition $\hat{f}\Psi_{\text{phys}}=0$ is representation dependent. In our investigation three qualitatively different cases occurred: the physical subspace $H_{\text{phys}} \subset H$ (i) is empty (there are no physical states), (ii) coincides with one of the quantum theories for the reduced system, or (iii) contains infinitely many quantum theories for the reduced system.

Only in case (ii) we can speak of commutativity of constraining and quantizing. In case (i) we end up with a different number of degrees of freedom, and in case (iii) we find a new gauge-invariant quantum observable which does not have a classical analogue.

We only mention here that it is physically consistent to use the weaker Dirac condition on matrix elements, $(\Psi'_{\text{phys}}, \hat{f}\Psi_{\text{phys}})=0$, instead of $\hat{f}\Psi_{\text{phys}}=0$, to determine a subset of physical states. Using this condition for the spinor representations of H^{odd} and H^{even} again leads to infinitely reducible representations of Sp(2,**R**).

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