Modification of Kaluza-Klein theory

J. Madore
Laboratoire de Physique Théorique et Hautes Energies, Université de Paris-Sud, Bâtiment 211, F-91405 Orsay, France
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A modification of the traditional formulation of Kaluza-Klein theory is proposed in which the internal structure is described by a noncommutative geometry based on a semisimple algebra. The classical theory of Yang-Mills fields and of Dirac fermions is developed in the resulting geometry. The generalized connection is written down which should describe the unification of the Yang-Mills fields with gravity, but the gravitational sector is not described in detail. As an example the mass and charge spectrum of the Yang-Mills-Dirac theory are given for a specific internal structure.

I. INTRODUCTION

The question of whether or not what appears to be a point at normal macroscopic length scales remains so at all length scales has been debated for many years. One of the first negative answers was given by Kaluza and Klein in their attempt to introduce extra dimensions in order to unify the gravitational field with electromagnetism. They suggested that at sufficiently small scales what appears as a point will in fact be seen as a circle. Later it was proposed that this internal manifold could be taken as a compact Lie group or even as a general compact manifold. The great disadvantage of these extra dimensions is that they introduce divergences in the quantum theory and an infinite spectrum of new particles. In fact the structure is strongly redundant and most of it has to be discarded. No use can be made at present of an infinite spectrum of particles. An associated problem is that of localization. We cannot, and indeed do not wish to have to, address the question of the exact position of a particle in the extra dimensions. We shall take this as motivation for describing the internal structure using a geometry in which the notion of a point does not exist in general. As particular examples of such a geometry we shall choose only internal structures which give rise to a finite spectrum of particles.

In other words we develop here the point of view that what one should do is modify the original idea of Kaluza and Klein by replacing a point in space-time by an internal structure which is not a manifold nor even a topological space. We recall the definition of the manifold on which Kaluza-Klein theory is usually based. Locally there is a projection

$$\begin{array}{c} \mathbb{R}^4\times F \\ \downarrow \\ \mathbb{R}^4 \end{array} , \tag{1.1}$$

where F is a manifold. We have therefore the embedding of associated algebras

$$0\rightarrow\mathcal{C}\rightarrow\mathcal{C}\otimes\mathcal{C}(F) . \tag{1.2}$$

Here $\mathcal{C}(F)$ designates an algebra of complex-valued functions on F. We have set $\mathcal{C}(\mathbb{R}^4)=\mathcal{C}$ and we have supposed that the algebra of a product manifold can be identified with the product of the algebras of the individual factors. Kaluza-Klein theory in the usual sense can be described equally well by referring to (1.1) or to (1.2). The internal structure is described by the manifold F or by the algebra of functions $\mathcal{C}(F)$.

We can modify Kaluza-Klein theory by replacing $\mathcal{C}(F)$ by an associative algebra M which is not necessarily an algebra of functions. We can then no longer refer to the diagram (1.1) as there is no internal manifold F. Diagram (1.2) is replaced by an embedding

$$0\rightarrow\mathcal{C}\rightarrow\mathcal{A} ,$$

where $\mathcal{A}$ is of the form

$$\mathcal{A}=\mathcal{C}\otimes M . \tag{1.3}$$

We shall choose as algebra M a finite direct sum of matrix algebras. This choice has the two advantages which we have mentioned. There is no valid notion of a point in the associated geometry and since the algebra of derivations is of finite dimension, the associated particle spectrum is finite.

In Sec. II we give a brief introduction to the geometry of matrices and in Sec. III the Kaluza-Klein construction is modified to include an internal structure described by this geometry. In the rest of this paper we shall abandon the gravitational sector and consider the Maxwell-Dirac equations in the geometry defined by $\mathcal{A}$. In Sec. IV gauge bosons are discussed and the different vacuua implied in general by the geometry are given. In Sec. V a modification of the Dirac equation is proposed. In Sec. VI the simplest model is discussed in some detail. The extensions to semisimple models are given in Sec. VII.

II. A NONCOMMUTATIVE GEOMETRY

The basic structure of the differential geometry of a manifold can be also expressed in terms of an algebra of functions defined on the manifold. Local coordinates are replaced by generators of the algebra; vector fields are replaced by derivations. Once this is realized, it is natural to generalize ordinary geometry to noncommutative geometry by replacing the algebra by an abstract associative algebra $\mathcal{A}$ which is not necessarily commutative. One of the main differences is the loss of the idea of a point. So there is no longer a well-defined notion of localization. It is this feature of noncommutative geometry which makes it particularly well suited to describe the internal structure of a Kaluza-Klein theory. We refer to Ref. 1 for a general introduction to the subject and for references to the previous literature.

Also, independent of any consideration of gravity and the problem of its unification with other fields, an obvious thing to try to do is to develop a noncommutative version of classical field theory simply by replacing the objects in the commutative case which are used to describe the theory by the corresponding noncommutative objects. In a series of papers[2–7] this has been done for the geometry described by an algebra of the form (1.3) where M is the algebra M_n of $n\times n$ complex matrices. The essential part of this paper is to extend some of these results to the semisimple case of l copies of M_n:

$$M=\bigoplus_{i=1}^{l} M_n^{(i)} .$$

As before, we shall choose $\mathcal{C}$ to be the algebra of smooth complex-valued functions on $\mathbb{R}^4$ and as before the extra factor M is the origin of the extra structure which a point acquires. The elements of the algebra $\mathcal{A}$ are what replace the functions on $\mathbb{R}^4$. In particular the nearest object which we have to a coordinate is an element of this algebra. This means that the position of a particle, for example, no longer has a well-defined meaning. Since we certainly wish this to be so at macroscopic scales, we must require that to each $M_n^{(i)}$ there be associated a length scale $\kappa^{(i)}$ which is not much greater than a typical Compton wavelength. In other words, the fuzziness which the internal structures gives a point in space-time cannot be much greater than the quantum uncertainty in the position of a particle. There is no reason to suppose that the scales $\kappa^{(i)}$ are of the same order of magnitude but it would be natural to suppose that at least one of them is of the Planck scale. The Lorentz group acts on $\mathcal{A}$. This means in particular that directions are well defined. Our space-time then looks like a crystal which has a homogeneous distribution of dislocations but no disclinations. We can pursue this solid-state analogy and think of the ordinary Minkowski coordinates as macroscopic order parameters obtained by course graining over scales less than the $\kappa^{(i)}$. They break down and must be replaced by elements of the algebra $\mathcal{A}$ when one considers phenomena on these scales.

The Lie algebra $D(\mathcal{C})$ of vector fields on the manifold $\mathbb{R}^4$ can be identified with the algebra of derivations of $\mathcal{C}$, that is, with the algebra of linear maps of $\mathcal{C}$ into itself which satisfy the Leibnitz rule. This algebra is the most important mathematical object which one uses when one studies classical fields on $\mathbb{R}^4$ and their dynamics. To study these fields then in the noncommutative case we must consider the derivations $D(\mathcal{A})$ of the algebra $\mathcal{A}$.

For simplicity we choose $l=1$ and suppress the index in parentheses except in the last section. We recall the notation of Ref. 5. Let λ_a, for $1\le a\le n^2-1$, be a basis of the Lie algebra of the special unitary group in n dimensions, chosen so that the structure constants $C^a{}_{bc}$ are real. The Killing metric is given by $g_{ab}=-\mathrm{Tr}(\lambda_a\lambda_b)$. We shall raise and lower indices with this metric. The set λ_a is a set of generators of the matrix algebra M_n. It is not a minimal set but it is convenient because of the fact that the derivations

$$e_a=\kappa^{-1}\mathrm{ad}(\lambda_a) \tag{2.1}$$

form a basis over the complex numbers for the derivations of M_n. They satisfy the commutation relations

$$[e_a,e_b]=mC^c{}_{ab}e_c . \tag{2.2}$$

The mass scale m is defined to be the inverse of the length scale κ.

Let x^μ be coordinates of $\mathbb{R}^4$. Then the set (x^μ,λ^a) is a set of generators of the algebra $\mathcal{A}$. We define the exterior derivative of an element of $\mathcal{A}$ as usual. For example, if f is an element of M_n, then df is defined[8] by the formula

$$df(e_a)=e_a(f) . \tag{2.3}$$

This means in particular that

$$d\lambda^a(e_b)=m[\lambda_b,\lambda^a]=mC_{cb}{}^a\lambda^c .$$

We define the set of one-forms $\Omega^1(\mathcal{A})$ to be the set of all elements of the form $f\,dg$ or $dg\,f$, with f and g in $\mathcal{A}$ subject to the relations $d(fg)=(df)g+f\,dg$. We similarly define $\Omega^1(M_n)$ and $\Omega^1(\mathcal{C})$, the definition of the latter being of course the usual one. The n-forms $\Omega^n(\mathcal{A})$ and the generalization $\Omega(\mathcal{A})$ of the algebra of differential forms are defined, for example, in Refs. 8 and 3.

The set of $d\lambda^a$ forms a system of generators of $\Omega^1(M_n)$ as a left or right module, but it is not a convenient one. For example, $\lambda^a d\lambda^b \neq d\lambda^b\lambda^a$. However because of the particular structure of M_n there is another system of generators completely characterized by the equation

$$\theta^a(e_b)=\delta^a_b . \tag{2.4}$$

It is related to the $d\lambda^a$ by the equations

$$d\lambda^a=mC^a{}_{bc}\lambda^b\theta^c, \quad \theta^a=\kappa\lambda_b\lambda^a d\lambda^b , \tag{2.5}$$

and it satisfies the same structure equations as the components of the Maurer-Cartan form on the special unitary group SU(n):

$$d\theta^a=-\tfrac{1}{2}mC^a{}_{bc}\theta^b\theta^c . \tag{2.6}$$

The product on the right-hand side of this formula is the product in $\Omega(M_n)$. Although this product is not in general antisymmetric, because of the relation (2.4) we have

$$\theta^b\theta^a=-\theta^a\theta^b .$$

Also the θ^a commute with the elements of M_n and $\Omega^1(M_n)$ can be identified with the tensor product of M_n and the dual of the vector space of derivations. The subalgebra $\Omega^*(M_n)$ of $\Omega(M_n)$ generated by the θ^a is an exterior algebra. Formula (2.6) means that it is a differential subalgebra. Since we shall only use $\Omega^*(M_n)$ in what follows we shall write the product as a wedge product:

$$\theta^a\theta^b=\theta^a\wedge\theta^b .$$

Choose a basis $\theta^\alpha_\lambda dx^\lambda$ of $\Omega^1(\mathcal{C})$ over $\mathcal{C}$ and let e_α be the Pfaffian derivations dual to θ^α. Set $i=(\alpha,a)$, $1\le i\le 4+n^2-1$, and introduce $\theta^i=(\theta^\alpha,\theta^a)$ as generators of $\Omega^1(\mathcal{A})$ as a left or right $\mathcal{A}$ module and $e_i=(e_\alpha,e_a)$ as a basis of $D(\mathcal{A})$ over $\mathcal{C}$. We can write $\Omega^1(\mathcal{A})$ as a direct sum

$$\Omega^1(\mathcal{A})=\Omega^1_H\oplus\Omega^1_V , \tag{2.7}$$

where we define

$$\Omega^1_H=M_n\otimes\Omega^1(\mathcal{C}), \quad \Omega^1_V=\mathcal{C}\otimes\Omega^1(M_n) .$$

The horizontal part Ω^1_H has basis θ^α and the vertical part Ω^1_V has basis θ^a. We shall decompose the exterior derivative into horizontal and vertical parts also:

$$d=d_H+d_V .$$

The generators θ^a of $\Omega^1(M_n)$ can be considered as a sort of moving frame. If we compare Eq. (2.6) with the first structure equations for this frame,

$$d\theta^a+\omega^a{}_b\wedge\theta^b=\Theta^a ,$$

we see that if we require the torsion form Θ^a to vanish then the internal structure is like a curved space with a linear connection given by

$$\omega^a{}_b=-\tfrac{1}{2}mC^a{}_{bc}\theta^c . \tag{2.8}$$

We have then a covariant derivative D_a which differs from e_a. Alternatively we could require that the connection $\omega^a{}_b$ vanish. We would have then a torsion form given by

$$\Theta^a=-\tfrac{1}{2}mC^a{}_{bc}\theta^b\wedge\theta^c . \tag{2.9}$$

We have in this case a covariant derivative D_a which is equal to e_a in the absence of gauge couplings but we have

$$[D_a,D_b]=mC^c{}_{ab}D_c . \tag{2.10}$$

In the rest of this paper we shall choose the solution (2.8), to avoid having an extra torsion form. If α is a one-form $\alpha=\alpha_a\theta^a$ then we have by definition, in the absence of torsion,

$$d\alpha=D_a\alpha_b\theta^a\wedge\theta^b \tag{2.11}$$

with

$$D_a\alpha_b=e_a\alpha_b-\tfrac{1}{2}mC^c{}_{ab}\alpha_c . \tag{2.12}$$

Notice that this covariant derivative commutes with itself when acting on elements of the algebra although the ordinary derivative does not. That is, for f in $\mathcal{A}$,

$$D_{[a}D_{b]}f=0 .$$

The equations we have given above are with respect to an arbitrary basis λ^a but they are all tensorial in character with respect to a change of basis

$$\lambda^a\mapsto\lambda'^a=A^a_b\lambda^b, \quad (A^a_b)\in GL(n^2-1) . \tag{2.13}$$

In particular the covariant derivative (2.12) transforms as it should. The unusual fact here is that the connection transforms also as a tensor and each of the terms on the right-hand side of (2.12) is tensorial in character separately. This is related to the fact that on the factor M_n of our algebra there is no notion of a point and no analog of local variation. Each θ^a corresponds to an arbitrary globally defined moving frame in the commutative case and the transformations of θ^a correspond to the set of all local transformations. We speak of a linear connection even though the corresponding moving frames do not vary. The change of basis (2.13) is the equivalent in M_n of a coordinate transformation in $\mathcal{C}$. We can suppose that a basis has been chosen so that the Killing metric g_{ab} is equal to the Euclidean metric δ_{ab}.

One can also consider the automorphisms of M_n, given by

$$\lambda^a\mapsto\lambda'^a=g^{-1}\lambda^a g, \quad g\in GL(n) . \tag{2.14}$$

To simplify we mention only the infinitesimal transformations

$$\lambda^a\mapsto\lambda'^a=\lambda^a-[f,\lambda^a], \quad g\simeq 1+f .$$

We see that

$$\theta'^a=\theta^a-L_X\theta^a, \quad X=\mathrm{ad}(f) , \tag{2.15}$$

and in general, for any n-form α,

$$\alpha'=\alpha-L_X\alpha .$$

III. METRICS AND CONNECTIONS

We shall introduce the quadratic form, of signature n^2+1, given by

$$ds^2=g_{ij}\theta^i\otimes\theta^j=\eta_{\alpha\beta}\theta^\alpha\otimes\theta^\beta+g_{ab}\theta^a\otimes\theta^b . \tag{3.1}$$

The $\eta_{\alpha\beta}$ is the Minkowski metric. We shall refer to this quadratic form as a metric although it contains two terms of a slightly different nature. To within a rescaling the g_{ab} are the components of the unique metric g_V for M_n with respect to which all the derivations e_a are Killing derivations. Let $X=X^ae_a$ be an arbitrary derivation. It is easy to see that

$$L_Xg_V=0$$

and that conversely any metric for M_n which satisfies this equation must be a real multiple of g_V.

From the generators θ^a we can construct a one-form θ in Ω^1_V,

$$\theta=-m\lambda_a\theta^a , \tag{3.2}$$

which from (2.5) and (2.6) satisfies the zero-curvature condition

$$d\theta+\theta^2=0 . \tag{3.3}$$

We shall see below that θ is gauge invariant. It satisfies with respect to the algebraic exterior derivative d_V similar conditions to those which the Maurer-Cartan form satisfies with respect to ordinary exterior derivation on the group SU(n). We have a map of the trace-free elements of M_n onto the derivations of M_n given by $f \mapsto X = \mathrm{ad}(f)$. The one-form θ can be defined, without any reference to the θ^a, as the inverse map: $\theta(X) = -f$. Using it, the vertical component of the metric can be given by the equation

$$g_V(X,Y) = -\kappa^2 \mathrm{Tr}[\theta(X)\theta(Y)] ,$$

where X, Y are derivations of M_n. Set $X = \mathrm{ad}(f)$. Then using (3.3) we find

$$L_X\theta = i(X)d\theta + di(X)\theta = -(df + [\theta, f]) = 0 .$$

So the one-form θ is invariant under all the derivations of M_n. To within multiplication by a complex number it is the only such one-form.

In the commutative case a connection ω on the trivial principal U(1) bundle equipped with the associated canonical flat connection is an anti-Hermitian one-form, which can be split as the sum of a horizontal part, a one-form on the base manifold, and a vertical part, the Maurer-Cartan form $d\alpha$ on U(1):

$$\omega = A + d\alpha . \tag{3.4}$$

The gauge potential A is an element of $\Omega^1(\mathcal{C})$ and using it we can construct a covariant derivative on an associated vector bundle. The notion of a vector bundle can be generalized to the noncommutative case as an $\mathcal{A}$ module which in its simplest form, a free module of rank 1, can be identified with $\mathcal{A}$ itself. This is in fact the natural generalization to the algebra we are considering of a trivial U(1) bundle since M_n has replaced $\mathbb{C}$ in our models. So the U(n) gauge symmetry we shall use below comes not from the rank of the vector bundle, which we shall always choose to be equal to 1, but rather from the factor M_n in our algebra $\mathcal{A}$. The noncommutative generalization of A is an anti-Hermitian element of $\Omega^1(\mathcal{A})$, which we saw in the previous section in turn can be split as the sum of two parts, called also horizontal and vertical. We shall here designate by ω such an element of $\Omega^1(\mathcal{A})$ since we wish to reserve the letter A and the name gauge potential for the horizontal part in this latter sense. We write then

$$\omega = A + \theta + \phi , \tag{3.5}$$

where A is an element of Ω^1_H and ϕ is an element of Ω^1_V. The field ϕ is the Higgs-boson field. We have noted that θ is in many respect like a Maurer-Cartan form. Formula (3.5) with $\phi = 0$ and formula (3.4) are formally similar but the meaning of the words horizontal and vertical in the two cases is not the same. We have then a bundle over a space which itself resembles a bundle. This double-bundle structure, which is what gives rise to a quartic Higgs-boson potential as we shall see below, has been investigated in previous papers.[9,10]

Let $\mathcal{U}$ be the unitary elements of $\mathcal{A}$. With $M = M_n$ this is the group $\mathcal{U}(n)$ of smooth functions on $\mathbb{R}^4$ with values in the unitary group U(n). We shall choose it to be the group of local gauge transformations.

A gauge transformation defines a mapping of $\Omega^1(\mathcal{A})$ into itself of the form

$$\omega' = g^{-1}\omega g + g^{-1}dg . \tag{3.6}$$

We define

$$\theta' = g^{-1}\theta g + g^{-1}d_V g, \quad A' = g^{-1}Ag + g^{-1}d_H g , \tag{3.7}$$

and so ϕ transforms under the adjoint action of $\mathcal{U}$:

$$\phi' = g^{-1}\phi g .$$

It can be readily seen that in fact θ is invariant,

$$\theta' = \theta , \tag{3.8}$$

and so the transformed potential ω' is again of the form (3.5).

The generalization of the globally defined maps of a manifold onto itself is the set of automorphisms of the algebra $\mathcal{A}$. This consists of automorphisms of $\mathcal{C}$ and of the automorphisms (2.14) of M_n. Gauge transformations can be identified then with a subset of generalized coordinate transformations. The relation between λ^a and e_a is similar to that which exists between the coordinate x^μ and the partial derivative ∂_μ. The difference lies in the fact that e_a is a linear expression in λ_a. When e_a acts on an element f of $\mathcal{A}$ it yields the components of a one-form. In the expression (3.2) for θ, λ_a transforms as the derivative of a scalar but because of (3.3) there is no f such that $\theta = df$.

The fact that θ is invariant under a gauge transformation means in particular that it cannot be made to vanish by a choice of gauge. We have then a potential with vanishing curvature but which is not gauge equivalent to zero. If M_n were an algebra of functions over a compact manifold, the existence of such a one-form would be due to the nontrivial topology of the manifold.

We define the curvature two-form Ω and the field strength F as usual:

$$\Omega = d\omega + \omega^2, \quad F = d_H A + A^2 .$$

They satisfy the relation

$$\Omega = F + D_H\phi + (D_V\phi - \phi^2) . \tag{3.9}$$

We have defined here the covariant exterior derivative

$$D\phi = d\phi + \omega\phi + \phi\omega$$

and decomposed it into horizontal and vertical parts. In terms of components, with $\phi = \phi_a\theta^a$ and $A = A_\alpha\theta^\alpha$ and with

$$\Omega = \tfrac{1}{2}\Omega_{ij}\theta^i \wedge \theta^j, \quad F = \tfrac{1}{2}F_{\alpha\beta}\theta^\alpha \wedge \theta^\beta , \tag{3.10}$$

we find

$$\begin{aligned} &\Omega_{\alpha\beta} = F_{\alpha\beta}, \quad \Omega_{\alpha a} = D_\alpha\phi_a , \\ &\Omega_{ab} = [\phi_a, \phi_b] - mC^c{}_{ab}\phi_c . \end{aligned} \tag{3.11}$$

The Kaluza-Klein construction involves essentially three steps, the first of which was the identification of the internal structure. Using the one-forms θ^i we can consider the algebra $\mathcal{A}$ in certain aspects as the algebra of functions over a formal manifold of dimension $4+n^2-1$: a product of an ordinary manifold of dimension 4 and an algebraic structure of dimension n^2-1. We shall use this analogy to construct a linear connection on $\Omega^1(\mathcal{A})$ using ω and a linear connection on $\Omega^1(\mathcal{C})$. Formally, the invariance group of the complete structure is $SO(3+n^2-1,1)$. If we restrict the local rotations to those which do not mix the ordinary θ^α with the algebraic θ^a, this group reduces to $SO(3,1)\times SO(n^2-1)$. The group U(n) acts on the algebraic structure through the adjoint representation

$$U(n)\to SO(n^2-1) .$$

Only the group $SU(n)/\mathbb{Z}_n$ acts nontrivially and we have an embedding

$$SU(n)/\mathbb{Z}_n \hookrightarrow SO(n^2-1) .$$

We shall be forced then to suppose that

$$\omega_a^0=0, \quad A_\alpha^0=0 , \tag{3.12}$$

and that $g\in\mathcal{SU}(n)$, the local $SU(n)$ gauge transformations.

With the condition (3.12) the connection ω can be written out explicitly as

$$\omega=(A_\alpha^a\theta^\alpha-m\theta^a+\phi_b^a\theta^b)\lambda_a . \tag{3.13}$$

Consider the derivations

$$\tilde{e}_\alpha=e_\alpha+\kappa A_\alpha^a e_a, \quad \tilde{e}_a=e_a+\kappa\omega_a^b e_b=\kappa\phi_a^b e_b , \tag{3.14}$$

of the algebra $\mathcal{A}$. These constitute part of the covariant derivative which we shall later apply to spinor fields. Dual to $\tilde{e}_i$ are the one-forms $\tilde{\theta}^i$ given by

$$\tilde{\theta}^\alpha=\theta^\alpha, \quad \tilde{\theta}^a=m\chi_b^a(\theta^b-\kappa A_\alpha^b\theta^\alpha) . \tag{3.15}$$

We have here used the inverse χ_b^a to the matrix ϕ_b^a:

$$\chi_b^a\phi_c^b=\delta_c^a . \tag{3.16}$$

First we restrict our considerations to that special class of connections for which the internal curvature vanishes:

$$\Omega_{ab}=0 . \tag{3.17}$$

This is the case which most resembles ordinary Kaluza-Klein theory. From (3.11) either ϕ_b^a vanishes or ϕ_b^a belongs to the gauge orbit of $m\delta_b^a$. We shall see below that these values correspond to the two stable vacua of the theory. The first gives rise to a singular set of one-forms $\tilde{\theta}^a$. Consider the second value, which corresponds to the physical vacuum. It yields a frame $\tilde{\theta}^i$ which is formally very similar to the usual moving frame constructed on a principal $SU(n)$ bundle. On the other hand if we compare with (3.13) we see that the vertical component of the connection vanishes. It is in the other vacuum $\phi=0$ that the connection ω most resembles a connection in a trivial $SU(n)$ bundle.

Let $\omega^\alpha{}_\beta$ now be a linear connection in $\Omega^1(\mathcal{C})$, an so(3,1)-valued one-form satisfying the structure equations

$$d\theta^\alpha+\omega^\alpha{}_\beta\wedge\theta^\beta=0, \quad d\omega^\alpha{}_\beta+\omega^\alpha{}_\gamma\wedge\omega^\gamma{}_\beta=\Omega^\alpha{}_\beta . \tag{3.18}$$

We must construct an $so(3+n^2-1,1)$-valued one-form $\tilde{\omega}^i{}_j$ on $\Omega^1(\mathcal{A})$ satisfying the first structure equation

$$d\tilde{\theta}^i+\tilde{\omega}^i{}_j\wedge\tilde{\theta}^j=0 . \tag{3.19}$$

Under the condition (3.17) the solution to these equations is given by

$$\tilde{\omega}^\alpha{}_\beta=\omega^\alpha{}_\beta+\tfrac{1}{2}\kappa F_a{}^\alpha{}_\beta\tilde{\theta}^a ,$$

$$\tilde{\omega}^\alpha{}_a=\tfrac{1}{2}\kappa F_a{}^\alpha{}_\beta\theta^\beta , \tag{3.20}$$

$$\tilde{\omega}^a{}_b=-\tfrac{1}{2}mC^a{}_{bc}\tilde{\theta}^c+\kappa C^a{}_{cb}A_\alpha^c\theta^\alpha .$$

Except for an additional term on the right-hand side of the equation for $\tilde{\omega}^a{}_b$, this connection is formally the same as the usual one which one constructs on an SU(n) bundle. The extra term, we shall see below, is what remains of the covariant derivative of the Higgs-boson fields. In a previous paper,[5] in which the Higgs-boson field was interpreted as torsion, the extra term was absorbed into a redefinition of the vertical part of the exterior derivative.

Consider now a general SU(n) connection with a general Higgs-boson field. The matrix $m\chi_b^a$ can be considered in (3.15) as a transformation of the frame $\tilde{\theta}^a$ away from its value in the physical vacuum. Set

$$\Omega_{ij}=\Omega^a{}_{ij}\lambda_a , \tag{3.21}$$

and define

$$\Omega'^a{}_{ij}=\chi_b^a\Omega^b{}_{ij} . \tag{3.22}$$

Then the solution to (3.19) is given by

$$\tilde{\omega}^\alpha{}_\beta=\omega^\alpha{}_\beta+\tfrac{1}{2}\Omega'_a{}^\alpha{}_\beta\tilde{\theta}^a ,$$

$$\tilde{\omega}^\alpha{}_a=\tfrac{1}{2}\Omega'_b{}^\alpha{}_a\theta^b+\tfrac{1}{2}\Omega'_a{}^\alpha{}_i\tilde{\theta}^i , \tag{3.23}$$

$$\tilde{\omega}^a{}_b=-\tfrac{1}{2}mC^a{}_{bc}\tilde{\theta}^c+\tfrac{1}{2}\Omega'_c{}^a{}_b\tilde{\theta}^c+\tfrac{1}{2}(\Omega'^a{}_{ib}-\Omega'_{bi}{}^a)\tilde{\theta}^i .$$

As in the normal Kaluza-Klein construction, the connection contains structure constants and terms which vanish with the curvature.

To complete the Kaluza-Klein construction it is necessary to consider the second structure equations

$$d\tilde{\omega}^i{}_j+\tilde{\omega}^i{}_k\wedge\tilde{\omega}^k{}_j=\tilde{\Omega}^i{}_j \tag{3.24}$$

and the equations of motion which follow from a suitable action. That this be the most general invariant which yields second-order field equations is the only condition one can impose in the absence of any criterion of renormalizability. Invariance under local $SO(3,1)\times SU(n)$ transformations permits an infinite formal sum of terms involving arbitrary powers of the components $\Omega_{\alpha\beta}$, $\Omega_{\alpha a}$, Ω_{ab} of Ω as well as of the components ϕ_a of the Higgs-boson field. If on the other hand one imposes a rather formal $SO(3+n^2-1,1)$ invariance which mixes the space-time θ^α with the algebraic θ^a, then the most general invariant can be constructed using only the components of the curvature form $\tilde{\Omega}^i{}_j$. It is a finite sum,

$$\mathcal{L}=\sum_{i=0}^{2+N}\alpha_i\mathcal{L}_i, \quad N=\left[\frac{n^2-1}{2}\right], \tag{3.25}$$

with arbitrary coefficients α_i, of the generators of the Euler classes in even dimensions up to and including $4+n^2-1$ (Ref. 11). Although it has been shown[12] in the usual Kaluza-Klein case with a manifold as internal structure that a consistent classical theory with a reasonable stable vacuum can be only based on a Lagrangian which includes some of the higher-order terms in the above expansion, we shall restrict our attention here to the Einstein-Hilbert term $\mathcal{L}_1=\tilde{R}$.

A straightforward calculation yields that to within a divergence we have

$$\begin{aligned}\tilde{R}=R&+\tfrac{1}{4}\Omega'_{aij}\Omega'^{aij}-\tfrac{1}{4}m^2C_{abc}C^{abc}\\&+\tfrac{1}{2}\Omega'^a{}_{ab}\Omega'^{ba}{}_a+\Omega'^a{}_{aa}\Omega'^{ba}{}_b\\&+\tfrac{1}{2}(\Omega'^a{}_{bc}+mC^a{}_{bc})\Omega'^{cb}{}_a\ .\end{aligned} \tag{3.26}$$

All of the terms on the right-hand side are gauge invariant. In fact, under a gauge transformation, the components ϕ^a_b transform only with respect to the upper index:

$$\phi^a_b\to\phi'^a_b=\Lambda^a_r\phi^r_b\ ,$$
$$\lambda'_a=g^{-1}\lambda_a g=\Lambda^b_a\lambda_b, \quad \Lambda\in SO(n^2-1)\ ,$$

and therefore χ^a_b transforms only with respect to the lower index. The quantities $\Omega'^a{}_{ij}$ defined then in (3.22) are gauge invariants.

The second term in (3.26) is a modified version of the gauge-boson Lagrangian which we shall use in the next section. It includes not only (4.2) given below but also an infinite sum of terms with insertions of Higgs-boson fields which come from the inverse χ^a_b of ϕ^a_b near the physical vacuum. The third term is an effective cosmological constant. The three last terms do not appear in usual Kaluza-Klein theories. They modify in an essential way the Higgs Lagrangian which we shall use below. For example, the scalar potential which comes from (3.26) is given by

$$V(\phi)=m^2\tilde{\omega}_{bac}\tilde{\omega}^{cab}\ , \tag{3.27}$$

where

$$\tilde{\omega}_{abc}=\tfrac{1}{2}(\tilde{C}_{abc}+\tilde{C}_{bac}-\tilde{C}_{cab})$$

and

$$\tilde{C}^a{}_{bc}=\chi^a_r C^r{}_{pq}\phi^p_b\phi^q_c\ .$$

This potential has a stable minimum only near $\phi=0$ where the curvature is singular. Here also higher-order terms in the expansion (3.25) will have to be included to stabilize the physical vacuum.

If we assume an internal structure given by a compact manifold F, then it is possible to obtain a Kaluza-Klein theory with a finite number of modes simply by expanding the elements of $\mathcal{C}(F)$ in terms of a basis and truncating after a finite number of terms. We have then a finite-dimensional vector space V. It is sometimes possible to introduce a product on this space and make it into an associative algebra. Sufficient conditions for this are given by the Gelfand-Naimark-Segal construction[13] which states that, subject to certain topological restrictions, an algebra M can be embedded as a vector space into the continuous functions on the set F of its pure states. The product in M defines then a product in the image V in $\mathcal{C}(F)$.

IV. GAUGE BOSONS

The straightforward generalization of the Maxwell action to the geometry defined by $\mathcal{A}$ is given by

$$S_{\text{YM}}=\frac{1}{4g^2}\text{Tr}\int(\Omega_{ij}\Omega^{ij})\ . \tag{4.1}$$

The integration is over space-time and the trace is what replaces this integration on the factor M of $\mathcal{A}$. Together the two define an integration in the algebra $\mathcal{A}$. The constant g is the Yang-Mills coupling constant. We shall set it equal to 1. Written out explicitly in the case $l=1$ the Lagrangian becomes

$$\mathcal{L}_{\text{YM}}=\tfrac{1}{4}\text{Tr}(F_{\alpha\beta}F^{\alpha\beta})+\tfrac{1}{2}\text{Tr}(D_\alpha\phi_a D^\alpha\phi^a)-V(\phi)\ , \tag{4.2}$$

where the Higgs-boson potential $V(\phi)$ is given by

$$V(\phi)=-\tfrac{1}{4}\text{Tr}(\Omega_{ab}\Omega^{ab})\ . \tag{4.3}$$

It is a quartic polynomial in ϕ with the mass scale m as free parameter.

From (3.11) we see that $V(\phi)$ vanishes for the values

$$\phi_a=0, \quad \phi_a=m\lambda_a\ . \tag{4.4}$$

Referring back to the expansion (3.5) of the connection we see that the first value corresponds to $\omega_V=\theta$ and the second value to $\omega_V=0$. The orbit of the second value under the action of the gauge group can be identified with SU(n). The Higgs-boson potential has therefore two absolute minima, a point at the origin and a submanifold of dimension n^2-1, which are separated by a potential barrier. There are therefore two stable phases.

We shall consider first the symmetric phase $\phi_a=0$ and then the broken phase $\phi_a=m\lambda_a$. We expand the Higgs-boson field and the gauge potential

$$\phi_a=\lambda_b\phi^b_a+\frac{i}{\sqrt{n}}\phi^0_a, \quad A_\alpha=\lambda_a A^a_\alpha+\frac{i}{\sqrt{n}}A^0_\alpha\ ,$$

so that we can identify the basic modes, the elementary scalars (ϕ^b_a,ϕ^0_a) and the elementary vector bosons (A^a_α, A^0_α). In the symmetric phase the masses of all the Higgs-boson modes are equal and they are real since the corresponding value of the potential is a stable minimum. Using the expression given above for Ω_{ab}, we find that the mass is given by

$$m_H^2=nm^2\ . \tag{4.5}$$

The gauge bosons of course have a vanishing mass in the symmetric phase.

In the broken phase, to calculate the Higgs-boson masses we use the vertical part $\omega_a\theta^a$ of the connection as a new Higgs-boson field and we expand it in terms of the basic field modes:

$$\omega_a=\lambda_b\omega_a^b+\frac{i}{\sqrt{n}}\omega_a^0 .$$

The analysis here is rather messy and we give it only in the case $n=2$. We decompose $\omega_{ab}=g_{bc}\omega_a^c$ into its irreducible parts, a trace, a trace-free symmetric, and an antisymmetric part:

$$\omega_{ab}=\frac{\tau}{\sqrt{3}}g_{ab}+\sigma_{ab}+\alpha_{ab} .$$

The masses are, respectively, given by

$$m_\tau^2=2m^2,\quad m_\sigma^2=8m^2,\quad m_\alpha^2=0,\quad m_0^2=2m^2 , \tag{4.6}$$

where m_0 is the mass of the modes ω_a^0. The 3 degrees of freedom in the α are the 3 gauge degrees of freedom. They correspond to the modes which have been absorbed as the longitudinal part of the now massive gauge bosons. In the broken phase the mass of the A^0 remains equal to zero. The mass term for the remaining vector bosons is

$$\mathcal{L}_{m_A}=\tfrac{1}{2}m^2\mathrm{Tr}([A_\alpha,\lambda_a][A^\alpha,\lambda^a]) . \tag{4.7}$$

The mass of these bosons is given therefore by the equation

$$m_A^2=2nm^2 . \tag{4.8}$$

We have two classical vacua separated by a potential barrier. There is however no instanton solution which tunnels between them since there is not even a smooth field configuration with finite action which could take as values $\omega=0$ at $t\to-i\infty$ and $\omega=\theta$ at $t\to+i\infty$.

The potential $V(\phi)$ can be identified with the (Euclidean) action of the vertical part ω of the connection. The trace replaces in the algebra M_n the integration over the points of the manifold in the usual case. If we define an instanton to be a finite-action field configuration, solution to the Yang-Mills field equations, then we have found all instantons. They are given by

$$\phi=0,\quad \phi=-\tfrac{1}{2}\theta,\quad \phi=-\theta . \tag{4.9}$$

The first and last solutions have zero curvature and zero action. The second has positive action but is unstable.

V. DIRAC FERMIONS

With the frame θ^i which was introduced above, the geometry of the algebra $\mathcal{A}$ resembles in some aspects ordinary commutative geometry in dimension $4+n^2-1$. Let g_{ij} be the Minkowski metric in this dimension and γ^i the associated Dirac matrices. The space $\mathcal{H}$ of spinors must be a left module with respect to the Clifford algebra. It is therefore of the form

$$\mathcal{H}=\mathbb{C}^{2^{(2+N)}}\otimes\mathcal{H}',\quad N=\left[\frac{n^2-1}{2}\right] .$$

We choose also $\mathcal{H}$ to be an $\mathcal{A}$ module, a right module for convenience. The unique irreducible finite projective module is given by

$$\mathcal{H}'=\mathcal{C}\otimes\mathbb{C}^n .$$

We can write then

$$\mathcal{H}=\mathbb{C}^4\otimes\mathbb{C}^{2^N}\otimes\mathcal{C}\otimes\mathbb{C}^n .$$

The first factor describes a Dirac spinor, the second factor combined with the last factor describe isospin. With the above decomposition, the Dirac matrices can be written

$$\gamma^i=(\gamma^\alpha\otimes 1\quad \gamma^5\otimes\gamma^a) ,$$

where the γ matrices on the right-hand side are with respect to the first and second factors in the decomposition of $\mathcal{H}$. This is a particular case of a general dimensional-reduction formula for spinors. We refer to Ref. 14 for a more thorough discussion. The definition of the derivative which we shall give is similar to that which is used on spinors defined in higher-dimensional Kaluza-Klein-type theories. The essential difference lies in the fact that the usual derivative in the hidden interior directions is here replaced by an abstract algebraic one.

The Dirac operator is a linear first-order operator of the form

$$\not{D}=\gamma^k D_k , \tag{5.1}$$

where D_k is the appropriate covariant derivative which we must now define. The space-time components are the usual ones if we take into account the right action of the gauge group on $\mathcal{H}$:

$$D_\alpha\psi=e_\alpha\psi-\psi A_\alpha+\tfrac{1}{4}\omega_\alpha{}^\beta{}_\gamma\gamma_\beta\gamma^\gamma\psi . \tag{5.2}$$

By analogy we would like to write

$$D_a\psi=e_a\psi-\psi\omega_a-\tfrac{1}{8}mC^b{}_{ca}\gamma_b\gamma^c\psi , \tag{5.3}$$

but we have first to define the derivative $e_a\psi$. Since $\mathcal{H}$ is a right $\mathcal{A}$ module, for any f in $\mathcal{A}$ we wish to have the relation

$$e_i(\psi f)=(e_i\psi)f+\psi e_i(f) . \tag{5.4}$$

We must set therefore[6,7]

$$e_a\psi=-m\psi\lambda_a . \tag{5.5}$$

The first two terms on the right-hand side of (5.3) simplify then and we have

$$D_a\psi=-\psi\phi_a-\tfrac{1}{8}mC^b{}_{ca}\gamma_b\gamma^c\psi . \tag{5.6}$$

Strictly speaking, this does not define a connection on the right $\mathcal{A}$-module structure of $\mathcal{H}$ unless ϕ_a takes one of its ground-state values given below. A connection on $\mathcal{H}$ considered as a right $\mathcal{A}$ module must satisfy the relation

$$D_i(\psi f)=(D_i\psi)f+\psi e_i f , \tag{5.7}$$

where f is an element of $\mathcal{A}$. This is the covariant generalization of (5.4). A covariant derivative on the other hand must satisfy

$$D_i' \psi' = (D_i \psi) g, \quad \psi' = \psi g \ , \tag{5.8}$$

where g is an element of the gauge group $\mathcal{U} = \mathcal{U}(n)$. We have identified $\mathcal{U}(n)$ as a subset of $\mathcal{A}$ so both of the above relations cannot be satisfied simultaneously. With the given definition of $D_i\psi$ the relation (5.7) is satisfied only for those f which are in the center Z of $\mathcal{A}$. If g is in $Z \cap \mathcal{U}(n) = \mathcal{U}(1)$ then the relations (5.7) and (5.8) are identical.

The covariant derivative which we have defined here coincides with the one which is given by the connection (3.23), to within terms linear in the curvature Ω. These extra nonminimal terms do not influence the conclusions which we wish to draw here concerning the classical mass spectrum. However, this spectrum depends in an essential way on the equilibrium value of ϕ, a value which we shall suppose in the next section to be given by the minimum of the classical potential (4.3). The effective potential, with the inclusion of quantum corrections, could very well have different minima. In fact we mentioned in Sec. III that, by analogy with usual Kaluza-Klein theory, one could expect even a classical theory based on the Lagrangian (3.25) to have equilibrium values which depend on the parameters α_i.

The straightforward generalization of the Dirac action to the geometry defined by $\mathcal{A}$ is given by

$$S_D = \mathrm{Tr} \int (\bar{\psi} D\!\!\!/ \psi) \ . \tag{5.9}$$

To this would have to be added an explicit mass term

$$S_m = m\zeta \mathrm{Tr} \int (\bar{\psi}\psi) \ ,$$

with ζ a real parameter, unless a parity operation could be defined which would force it to vanish. We shall not consider this term here.

VI. THE SIMPLEST MODEL

The generalization of the Maxwell-Dirac action is given by

$$S = S_{\mathrm{YM}} + S_D \ .$$

We shall discuss here the particle spectrum this action implies in the case $l = 1$, $n = 2$. In the case $n = 2$ we can write $\mathcal{H}$ in the form

$$\mathcal{H} = \mathbb{C}^4 \otimes \mathcal{A} \ .$$

The module $\mathcal{H}$ is a free $\mathcal{A}$ module of rank 1. We showed above that in this case there are two stable phases, a symmetric phase with massless gauge bosons and a broken phase with three massive gauge bosons of mass $m_A = 2m$.

The Dirac matrices can be identified with the seven matrices $\gamma^i = (\gamma^\alpha \otimes 1, \gamma^5 \otimes \sigma^a)$, where the σ^a are the Pauli matrices: $\lambda^a = (i/\sqrt{2})\sigma^a$. The algebraic components of the covariant derivative become

$$D_a \psi = -\psi\phi_a + \tfrac{1}{2} m \lambda_a \psi \ .$$

To find the mass spectrum we must solve the eigenvalue equation

$$i\sigma^a D_a \psi = m_A \mu \psi \tag{6.1}$$

for ψ in $\mathcal{A}$. Because of the simplicity of the supplementary algebraic structure this can be done completely and explicitly, as in the bosonic case.

In the symmetric phase, ϕ_a vanishes and Eq. (6.1) reduces to

$$\mu = -\frac{3}{4\sqrt{2}} \ .$$

All of the eigenvalues are equal. In the broken phase $\phi_a = m\lambda_a$ and Eq. (6.1) becomes

$$\sigma_a \psi \sigma^a = (2\sqrt{2}\mu + \tfrac{3}{2})\psi \ . \tag{6.2}$$

There is one mode ψ_0, proportional to the unit matrix in M_2, of mass

$$m_0 = \frac{3}{4\sqrt{2}} m_A \ ,$$

and there are three modes ψ_a proportional to the Pauli matrices, of mass

$$m_a = \frac{5}{4\sqrt{2}} m_A \ .$$

The existance of these two mass values is a reflection of the broken U(2) symmetry.

The bosonic sector consists of four gauge bosons, of which one is massless and can be identified with the photon. There are as well nine massive scalar Higgs bosons. The photon is the component of A_α proportional to the unit matrix in M_2 so all of the fermions have charge given by

$$e = \frac{1}{\sqrt{2}} \ . \tag{6.3}$$

The bosons are all neutral. There are in all $4 \times 4 = 16$ fermionic degrees of freedom and $2 + 3 \times 3 + 9 = 20$ bosonic degrees of freedom in this simplest model. This is to be compared with the infinite number of modes in usual Kaluza-Klein theories.

VII. SEMISIMPLE EXTENSIONS

We shall now show that the formalism presented in the previous sections can be extended in a straightforward way to the general semisimple algebra M introduced in Sec. II. One of the many drawbacks which characterizes the Dirac operator of the preceding section is the lack of zero modes. The mass spectrum, including the zero modes, is given by the eigenvalues of the algebraic part of the Dirac operator, that which corresponds to the operator on the internal space in the Kaluza-Klein description. Here this operator is essentially a finite matrix and it cannot have a nonzero index. We are thus forced to adopt the point of view that a correct description of nature is given by a left-right-symmetric model in which there are an equal number of zero modes of helicity plus and minus. Even with this allowance we shall see that, at least for the simple $\mathcal{A}$-module structure which we assume, the Dirac operator has no zero modes for finite $l > 1$.

Consider also the algebra $\mathcal{D}$ of derivations. We have supposed that this algebra is the complete algebra $D(\mathcal{A})$ of derivations of $\mathcal{A}$. If M is taken to be equal to M_n this means that a basis can be chosen for $\mathcal{D}$ with e_a of the form (2.1) with λ_a which lie in the fundamental representation of the Lie algebra of the special unitary group SU(n). It is also possible to suppose that the λ_a lie in an arbitrary representation. This is equivalent to supposing that the Lie algebra of derivations is a subalgebra of $\mathcal{D}(\mathcal{A})$ for some $\mathcal{A}$:

$$\mathcal{D}\subseteq D(\mathcal{A}) . \tag{7.1}$$

The representation determines how an element of $\mathcal{A}$ behaves under derivation by an element of $\mathcal{D}$. To ensure that the λ_a generate all of M as an associative algebra we shall suppose that the representation is irreducible. As an example consider the three derivations of M_2 acting on M_n. The n^2 degrees of freedom in M_n can be compared with the n^2 states of the hydrogen atom up to quantum number n. So the theory we are discussing resembles quantum mechanics on S^2 with a truncation in the energy at the level n.

We would like to consider the semisimple case with an M which is the direct sum of l copies of M_n. To be specific we suppose that the derivations are those of a direct sum of l copies of M_2. We have then as bases of the derivations the set

$$e_a^{(i)}=m^{(i)}\text{ad}(\lambda_a^{(i)}), \quad 1\le i\le l, \quad 1\le a\le 3 , \tag{7.2}$$

where the $\lambda_a^{(i)}$, for each i, form the irreducible representation of dimension n of the Lie algebra of SU(2):

$$[\lambda_a^{(i)},\lambda_b^{(j)}]=C^c{}_{ab}\lambda_c^{(i)}\delta^{ij} . \tag{7.3}$$

Each element of $\mathcal{A}$ is a polynomial in the $\lambda_a^{(i)}$ with coefficients in $\mathcal{C}$. If $n=2$ this polynomial is of degree 1.

If we choose $\mathcal{U}$, the set of unitary elements of $\mathcal{A}$, as gauge group, then $\mathcal{U}$ can be identified with the set of smooth functions on $\mathbb{R}^4$ with values in the product of l copies of U(n):

$$g:\mathbb{R}^4\rightarrow\prod_i^l \text{U}(n), \quad g\in\mathcal{U} . \tag{7.4}$$

The generalizations of the first few formulas of Sec. II can be readily written down. The derivations satisfy the commutation relations

$$[e_a^{(i)},e_b^{(j)}]=m^{(i)}C^c{}_{ab}e_c^{(i)}\delta^{ij} . \tag{7.5}$$

To define the one-forms dual to $e_a^{(i)}$ we fix for a moment the index i and consider the basis e_r, $1\le r\le n^2-1$, of the Lie algebra of all derivations of $M_n^{(i)}$. The $e_a^{(i)}$ can be chosen as a subset of three elements of the set of e_r. We define θ^r as dual to e_s according to (2.4). The θ^r satisfy Eqs. (2.5) with structure constants those of SU(n). Choose as $\theta^{(i)a}$ the three elements of the set of θ^r which satisfy the equation

$$\theta^{(i)a}(e_b^{(j)})=\delta_b^a\delta^{ij} . \tag{7.6}$$

We define the differential of an element in $M_n^{(i)}$ in a way similar to formula (2.3) but using only derivatives in the three directions $e_a^{(i)}$:

$$df=e_a^{(i)}(f)\theta^{(i)a}, \quad f\in M_n^{(i)} . \tag{7.7}$$

Equation (2.6) follows from (2.2) and (2.4) and so it remains valid:

$$d\theta^{(i)a}=-\tfrac{1}{2}m^{(i)}C^a{}_{bc}\theta^{(i)b}\wedge\theta^{(i)c} . \tag{7.8}$$

The algebra $\Omega^{(i)*}$ generated by the $\theta^{(i)a}$ is a differential subalgebra of $\Omega^*(M_n^{(i)})$. By definition the first of Eqs. (2.5) remains valid,

$$d\lambda^{(i)a}=m^{(i)}C^a{}_{bc}\lambda^{(i)b}\theta^{(i)c} , \tag{7.9}$$

but the second equation would have to be modified. Here, in Eqs. (7.8) and (7.9), the structure functions are those of SU(2).

From the generators $\theta^{(i)a}$ we can construct, for each i, a one-form $\theta^{(i)}$:

$$\theta^{(i)}=-m^{(i)}\lambda_a^{(i)}\theta^{(i)a} , \tag{7.10}$$

which satisfies as before the zero-curvature condition (3.3). If we define

$$\theta=\bigoplus_{i=1}^{l}\theta^{(i)}$$

then it too satisfies the condition (3.3).

The connection form and the curvature are defined as before, the main difference being in the more complicated structure of the vertical part Ω_V of the latter:

$$\begin{aligned}\Omega_V&=\tfrac{1}{2}\bigoplus_{i=1}^{l}\Omega_{ab}^{(i)}\theta^{(i)a}\wedge\theta^{(i)b} ,\\ \Omega_{ab}^{(i)}&=[\phi_a^{(i)},\phi_b^{(i)}]-m^{(i)}C^c{}_{ab}\phi_c^{(i)} .\end{aligned} \tag{7.11}$$

Here we have expanded the Higgs field ϕ,

$$\phi=\bigoplus_{i=1}^{l}\phi_a^{(i)}\theta^{(i)a} , \tag{7.12}$$

and we have used the fact that

$$[\phi_a^{(i)},\phi_b^{(j)}]=0, \quad i\neq j .$$

For each value of i, the $\phi_a^{(i)}$ take their values in a different copy of the Lie algebra of U(n).

The main difference with the simple situation of Sec. II lies in the more complicated structure of the vacuum.[4] A vacuum configuration is defined by a connection form ω with a vanishing curvature two-form. This means that the potential A must vanish and that the components $\phi_a^{(i)}$ of the Higgs-boson field must assume constant values $\mu_a^{(i)}$ which satisfy the equations

$$[\mu_a^{(i)},\mu_b^{(i)}]-m^{(i)}C^c{}_{ab}\mu_c^{(i)}=0, \quad 1\le i\le l . \tag{7.13}$$

The number of solutions is in general greater than 2. For each index value i, $\mu_a^{(i)}$ is an n-dimensional representation of the Lie algebra of SU(2). The number of such representations is given by the partition function $p(n)$, and is a strongly increasing function of n. Only one representation is irreducible for each value of i. There are in total then $lp(n)$ possible vacua.

We are now in a position to write down the Dirac equation and to study its zero modes. With the frame

$\theta^{(i)a}$ which was introduced above, the geometry of the algebra $\mathcal{A}$ resembles in some aspects ordinary commutative geometry in dimension $4+3l$. Let g_{ij} be the Minkowski metric in this dimension and γ^i the associated Dirac matrices. The space $\mathcal{H}$ of spinors must be a left module with respect to the Clifford algebra. It is therefore of the form

$$\mathcal{H}=\mathbb{C}^{2^{(2+N)}}\otimes\mathcal{H}', \quad N=\left[\frac{3l}{2}\right] .$$

We choose $\mathcal{H}'$ to be a right $\mathcal{A}$ module of the form

$$\mathcal{H}'=\mathcal{C}\otimes P , \tag{7.14}$$

where P is a right M module.

To investigate the spectrum of the Dirac operator we separate the space-time part and consider only the algebraic part

$$\not{D}=i\sum_{j=1}^{l}\gamma^{(j)a}D_a^{(j)} , \tag{7.15}$$

where $D_a^{(j)}$ is given by

$$D_a^{(j)}\psi=-\psi\mu_a^{(j)}-\tfrac{1}{8}m^{(j)}C^b{}_{ca}\gamma_b^{(j)}\gamma^{(j)c}\psi . \tag{7.16}$$

Here ψ is an element of the space $\mathbb{C}^{2^N}\otimes P$ and we have replaced the Higgs-boson field by its vacuum expectation value. The $3l$ Dirac matrices $\gamma^{(i)a}$ are $2^N\times 2^N$ matrices which satisfy

$$\{\gamma^{(i)a},\gamma^{(i)b}\}=2\delta^{ij}g^{ab} . \tag{7.17}$$

They can be constructed as N-fold tensor products of the Pauli matrices.

We shall consider here only the simplest possibility for the right module P:

$$P=\bigoplus_{1}^{l}\mathbb{C}^n . \tag{7.18}$$

For each i, $\phi_a^{(i)}$ acts on the ith copy of $\mathbb{C}^n$ from the right. An element of the space $\mathbb{C}^n$ is therefore considered as a row vector and ψ is a $2^N\times ln$ matrix.

We are interested in the eigenvalues of the equation

$$\not{D}\psi=\omega\psi , \tag{7.19}$$

where $\not{D}$ is the operator (7.15). This is a finite matrix equation and the solutions ω yield the finite fermion mass spectrum. This spectrum depends on the integers n and l and on the choice of vacuum for the Higgs fields.

The spectrum of Eq. (7.19) is more conveniently investigated using the associated Laplace operator

$$\Delta=\not{D}^2 .$$

We have, from (7.16),

$$\not{D}\psi=\sum_{j=1}^{l}i(-\gamma^{(j)a}\psi\mu_a^{(j)}+\tfrac{3}{4}\sqrt{2}\,m^{(j)}\gamma^{(j)1}\gamma^{(j)2}\gamma^{(j)3}\psi) . \tag{7.20}$$

So Δ splits as the sum of l terms:

$$\Delta=\sum_{1}^{l}m^{(i)2}P^{(i)} . \tag{7.21}$$

We renormalize $\mu_a^{(j)}$,

$$\mu_a^{(j)}=\frac{i}{\sqrt{2}}m^{(j)}\Sigma_a^{(j)} ,$$

so that the $\Sigma_a^{(j)}$ satisfy the commutation relations of the Pauli matrices. The operator

$$\mu_a^{(i)}\mu^{(i)a}=-\tfrac{1}{2}m^{(i)2}\Sigma_a^{(i)}\Sigma^{(i)a}$$

is the Casimir operator of SU(2) and is proportional to the identity when the representation (7.13) is irreducible. We define also $\sigma_a^{(j)}$ by

$$\sigma_a^{(j)}=\frac{i}{2}\epsilon_{abc}\gamma^{(j)b}\gamma^{(j)c} .$$

Then $P^{(i)}$ is given by

$$2P^{(i)}\psi=\tfrac{9}{4}\psi+\psi\Sigma_a^{(i)}\Sigma^{(i)a}-\sigma_a^{(i)}\psi\Sigma^{(i)a} . \tag{7.22}$$

The first term is due to the effective curvature of the algebraic structure. The second term is a mass term. It depends on the representation of the Higgs-boson fields and can be positive or zero. The last term is a sort of Pauli spin term. It can yield a contribution to the spectrum of any sign. There is no term which resembles a kinematical term. As we noticed above in (5.6), the ordinary derivative is absorbed in the Higgs-boson field. It is easily seen that for each i, $P^{(i)}$ is a positive matrix. In fact, if we define $A^{(i)a}$ by

$$A^{(i)a}\psi=\tfrac{1}{2}\sigma^{(i)a}\psi-\psi\Sigma^{(i)a}$$

then we have

$$P^{(i)}=\tfrac{3}{4}+\tfrac{1}{2}A_a^{(i)}A^{(i)a}>0 .$$

So Δ, as a sum of positive matrices, can have no zero modes.

We see then that the calculations which have been carried out in the case of a simple matrix algebra M_n can be extended to the more general case $l>1$. However, this generalization does not solve the problem of the absence of zero modes for the Dirac operator.

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