Lovelock gravitational field equations in cosmology

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We present a systematic study of cosmological solutions in the Lovelock theory of gravitation, including maximally symmetric space-times, Robertson-Walker universes, and product manifolds of symmetric subspaces.

I. INTRODUCTION

In 1917 Vermeil,[1] then Weyl[2] and Cartan[3] showed that Einstein's gravitational tensor (together with a "cosmological" term) was, in any dimension, the only symmetric and conserved tensor depending only on the metric and its first and second derivatives, with a linear dependence on the latter. In 1971 Lovelock[4,5] dropped the condition of linear dependence on the second derivatives of the metric and obtained the most general tensor satisfying the other conditions. He also found the Lagrangian, which generalizes that of Einstein and Hilbert, from which that tensor derives via Euler variation with respect to the metric. Important features of this "Lovelock Lagrangian" (see below for an explicit expression) are that it is nonlinear in the Riemann tensor and differs from the Einstein tensor only if space-time has more than four dimensions. Therefore it yields a most natural generalization of general relativity in higher-dimensional space-times.

The fact that space-time may have more than four dimensions is a recurrent idea in unified field theories since the original proposal by Kaluza[6] and Klein[7] (see, e.g., Ref. 8 for a review of the Kaluza-Klein program); it may even be compulsory in string theory.[9] As for modifying the Einstein-Hilbert Lagrangian by the inclusion of terms that are nonlinear in the curvature, it is an idea that dates back to Weyl[2] and Eddington[10] and was exploited in view of renormalizing the quantized theory of linearized general relativity (see, e.g., Ref. 11 for a review). The convenience, or even necessity, of considering such nonlinear terms is also apparent in quantum field theory in curved space, where they are required to renormalize the energy-momentum tensor of the quantized matter fields[12]—even in the semiclassical approach where gravity is not quantized. Their presence seems also necessary in string theory, where they appear from the requirement of conformal invariance on the world sheet of the string on a curved background.[9]

The Lovelock Lagrangian differs however from a generic nonlinear correction to the Einstein-Hilbert Lagrangian not only in that its Euler variation—the Lovelock tensor—reduces to Einstein's in four-dimensional space-times, but also in that that tensor, by construction, contains derivatives of the metric of order no higher than the second. At the classical level, to which we shall confine ourselves, the main consequence of this property is to avoid singular perturbations: that is, introducing, by increasing the order of differentiation of the field equations, new classes of solutions which do not necessarily approach the unperturbed, Einsteinian solutions when the nonlinear perturbation in the Riemann tensor tends to zero. Hence the Lovelock tensor is of little use in achieving goals that ultimately rely on the possibility of singular perturbations such as the renormalization of the graviton propagator[13,14] or the type of inflation first discussed by Starobinksi[15] (see also, e.g., Refs. 16 and 17). However, the same property guarantees that the quantization of the linearized Lovelock theory is free of ghosts and it was argued that, for this reason, the Lovelock Lagrangian would appear in the low-energy limit of superstring theory.[18–21]

Gravity theories based on the Lovelock field equations have been fairly extensively studied in the last few years (this renewal started with the work of Madore[22] and Müller-Hoissen;[23] see Ref. 24 for a review). Cosmological models in particular[25–45] have been a focus of interest as the very early Universe appears to be a privileged arena where unified theories and observation may meet. Despite the wealth of references quoted the subject is far from being exhausted, especially when considering the variety of possible couplings[38–45] (inspired or not by supergravity or superstring theories) to other matter fields.

This paper intends to bridge a number of gaps in the literature. It presents a systematic approach to the cosmological solutions of the pure Lovelock field equations (the presence of other matter fields, should the occasion arise, will be dealt with in terms of some cosmic fluid). It will encompass some previously obtained results[25–37] and present new ones which could serve as a basis for further studies.

This paper, which is intended to be self-contained, is organized as follows. In Sec. II we introduce the notations that we shall use and define the Lovelock Lagrangian and tensor. Section III briefly deals with the simple case of maximally symmetric space-times and casts the solution in the context of chaotic inflation. In Sec. IV we treat the Robertson-Walker-type cosmologies and we show in this specific case how the nonlinearity in the second-order derivatives of the metric, which is specific to the Lovelock theory, may induce causality violation problems, and in what sense. Finally we turn, in Sec. V, to space-times that are the product of an "external" Robertson-Walker space-time and an "internal" maximally symmetric compact space. These models where some dimensions become unobservable should provide potentially more realistic cosmologies. Our analysis, however, will not be exhaustive. We shall conclude with some remarks for further work.

II. NOTATIONS AND DEFINITIONS

We shall use in this paper most of the notation of Ref. 32: space-time dimension $D=1+d+n$; signature $(-,+,\ldots,+)$; $A,B,\ldots=0,1,\ldots,D$; $\alpha,\beta,\ldots=0,1,\ldots,d$; $a,b,\ldots=d+1,\ldots,D$; $\bar{A},\bar{B},\ldots=1,2,\ldots,D$; $\bar{\alpha},\bar{\beta},\ldots=1,2,\ldots,d$; g is the determinant of the metric g_{AB}; g^{AB} its inverse; $R^{A}{}_{BCD}\equiv+\partial_c\Gamma^{A}{}_{BD}\cdots$, $R_{AB}\equiv R^{C}{}_{ACB}$, $R\equiv g^{AB}R_{AB}$; $\Box\equiv g^{AB}\nabla_A\nabla_B$.

Consider now the Lovelock gravitational Lagrangian[4,5] L_g:

$$\frac{8\pi G_D}{c^4}L_g=\sum_{0\le p\le D/2}\alpha_p\lambda^{2(p-1)}L_{(p)}\ . \tag{2.1}$$

G_D is Newton's constant in a D-dimensional space-time, c is the speed of light, $p\in N$, λ is a length scale (e.g., the Planck length), α_p are real dimensionless parameters which, for want of a metatheory or observations which could determine them, we shall leave unspecified; and

$$L_{(p)}=\frac{1}{2^p}\delta^{i_1\cdots i_{2p}}_{j_1\cdots j_{2p}}R^{j_1j_2}{}_{i_1i_2}\cdots R^{j_{2p-1}j_{2p}}{}_{i_{2p-1}i_{2p}}\ , \tag{2.2}$$

where $\delta^{i_1\cdots i_{2p}}_{j_1\cdots j_{2p}}$ is the Kronecker symbol of order $2p$ and $R^{A}{}_{BCD}$ is the D-dimensional Riemann tensor. $L_{(0)}=1$; $L_{(1)}=R$ is the Einstein-Hilbert Lagrangian;

$$L_{(2)}=R_{ABCD}R^{ABCD}-4R_{AB}R^{AB}+R^2 \tag{2.3}$$

is the Lanczos Lagrangian[46,47] which reduces in $D=4$ to the Gauss-Bonnet combination;[48] $L_{(3)}$ was first obtained by Müller-Hossen;[23] $L_{(4)}$ can be found in Ref. 49 and corrects a previously published expression.[28] More generally, for $D=2N$, and if the manifold is compact with positive-definite metric, $g^{1/2}L_{(N)}$ is the generator of the characteristic Euler class.[50,23]

The Euler variation of $(-g)^{1/2}L_g$ with respect to the metric g_{AB} is the Lovelock tensor[4,5] G_{AB}:

$$G_{AB}=\sum_{0\le p<D/2}\alpha_p\lambda^{2(p-1)}G_{(p)AB} \tag{2.4}$$

with

$$G_{(p)}{}^{A}{}_{B}=-\frac{1}{2^{p+1}}\delta^{Ai_1\cdots i_{2p}}_{Bj_1\cdots j_{2p}}R^{j_1j_2}{}_{i_1i_2}\cdots R^{j_{2p-1}j_{2p}}{}_{i_{2p-1}i_{2p}}\ . \tag{2.5}$$

In particular, $G_{(0)AB}=-\frac{1}{2}g_{AB}$ and

$$G_{(1)AB}=R_{AB}-\tfrac{1}{2}g_{AB}R \tag{2.6}$$

is the D-dimensional Einstein tensor. The explicit expressions for $G_{(2)AB}$ and $G_{(3)AB}$ can be found in the work of Müller-Hoissen.[23] In a D-dimensional space-time, $D\le 2N$, all tensors $G_{(p)AB}$ with $p\ge N$ vanish identically (the vanishing of $G_{(2)AB}$ in $D=4$ dimensions is known as as the Bach identity[51]). Hence in $D=9$ or 10 dimensions, for instance, the Lovelock tensor can have five terms, up to the quartic term $G_{(4)AB}$.

Using general relativity as a guideline the field equations will be postulated to be

$$G_{AB}=\frac{8\pi G_D}{c^4}T_{AB}\ , \tag{2.7}$$

where T_{AB} is a phenomenological stress-energy tensor. In purely geometrical theories, T_{AB} is zero. Should the occasion arise we shall impose it to represent a cosmic fluid:

$$T_{AB}=\rho u_A u_B+S_{AB}\ , \tag{2.8}$$

where u_A is the velocity of the fluid element ($u_A u^A=-1$), ρ its energy density, and S_{AB} its stress tensor. The field equations (2.7) and (2.8) must then be supplemented by an equation of state (see below for specific examples).

III. MAXIMALLY SYMMETRIC SPACE-TIMES

Let us first consider D-dimensional maximally symmetric space-times, such that

$$R_{ABCD}=\frac{\kappa}{\lambda^2}(g_{AC}g_{BD}-g_{AD}g_{BC})\ , \tag{3.1}$$

where κ is a real number and λ is the characteristic length of the theory (e.g., the Planck length). The Lovelock Lagrangians and tensors (2.1) and (2.2) and (2.4) and (2.5) then reduce to[22]

$$L_{(p)}=\left[\frac{\kappa}{\lambda^2}\right]^p\frac{D!}{(D-2p)!}\ ,\qquad G_{(p)AB}=-\frac{1}{2}\left[\frac{\kappa}{\lambda^2}\right]^p\frac{(D-1)!}{(D-2p-1)!}g_{AB}\ , \tag{3.2}$$

so that the vacuum ($T_{AB}=0$) field equations (2.7) become

$$\sum_{0\le p<D/2}\beta_p\kappa^p=0 \tag{3.3}$$

with

$$\beta_p=\frac{(D-1)!}{(D-2p-1)!}\alpha_p\ . \tag{3.4}$$

[Matter can be added to the model and is compatible with the prescribed geometry (3.1) if it is a perfect fluid

$$T_{AB}=(\rho+p)u_A u_B+pg_{AB} \tag{3.5}$$

with equation of state $p=-\rho$, where p is the pressure. Hence the addition of matter only amounts to changing the bare cosmological constant α_0 into $\bar{\alpha}_0=\alpha_0-16\pi G_D\lambda^2\rho/c^4$.]

Equation (3.3) is of degree $[(D-1)/2]$ in κ (where $[r]$ denotes the integer part of r). Only in $D=4$ is the unique root $\kappa=-\bar{\alpha}_0/6\alpha_1$ guaranteed to be real and one then recovers the standard (anti-)de Sitter universe. In higher dimensions the condition of reality of at least one root imposes constraints on the α_p's when D is even. (When the equation is of very high degree and the parameters α_p are normally distributed, the probability that it has a real root can in fact be calculated.[17,52]) In any case if (3.3) has several roots the theory does not provide any criterion for choosing one rather than the other. Giving up the idea of maximal symmetry but keeping that of constant curvature one can envision a solution, akin to some inflationary scenarios (see, e.g., Ref. 53), consisting of patches covering space-time, each of constant curvature, the value of which is given by one of the solutions of (3.3). However, contrary to what happens in inflationary scenarios, this picture of a "bubbly" universe does not arise from say α_0 effectively varying (from ~ 1 to $\sim 10^{-120}$) from patch to patch because of some phase transition process; it arises from the possible multiplicity of the real roots of (3.3), with the α_p's being given and constant. Just as in inflationary models, though, the distribution of patches is not determined by the field equations and what constraints to impose on the geometry of the domain walls is unclear.[54,55]

IV. ROBERTSON-WALKER SPACE-TIMES

Let us now consider the case, touched upon in Ref. 28, of a D-dimensional Robertson-Walker space-time with line element

$$ds^2=-dt^2+a^2(t)d\sigma^2 , \tag{4.1}$$

where $d\sigma^2$ describes a $(D-1)$-dimensional space of constant curvature. The nonvanishing components of the Riemann tensor are

$$\begin{aligned} R_{\bar{A}0\bar{B}0}&=-g_{\bar{A}\bar{B}}\ddot{a}/a , \\ R_{\bar{A}\bar{B}\bar{C}\bar{D}}&=\frac{A}{\lambda^2}(g_{\bar{A}\bar{C}}g_{\bar{B}\bar{D}}-g_{\bar{A}\bar{D}}g_{\bar{B}\bar{C}}) , \end{aligned} \tag{4.2}$$

with

$$A\equiv\frac{\lambda^2(k+\dot{a}^2)}{a^2} , \tag{4.3}$$

where an overdot denotes differentiation with respect to t, and k characterizes the curvature of the spatial sections and can take the values 1, 0, or -1 upon appropriate rescaling of the coordinates.

From the form of the Lovelock tensor (2.4) and (2.5) in this metric up to cubic order[29,56] one infers the general expression to be

$$\lambda^2 G_{00}=\frac{1}{2}\sum_{0\le p<D/2}\beta_p A^p , \tag{4.4a}$$

$$\lambda^2 G_{\bar{A}\bar{B}}=-\frac{1}{2}\frac{g_{\bar{A}\bar{B}}}{2(D-1)}\times\sum_{0\le p<D/2}\beta_p A^{p-1}\left[2p\lambda^2\frac{\ddot{a}}{a}+(D-2p-1)A\right] \tag{4.4b}$$

with β_p given by (3.4).

In the presence of a perfect fluid described by the stress-energy tensor (3.5) together with the equation of state

$$p=\mu_D\rho , \tag{4.5}$$

μ_D being a constant, the field equations (2.7) then read

$$\sum_{0\le p<D/2}\beta_p A^p=\tilde{\rho} . \tag{4.6a}$$

$$\sum_{0\le p<D/2}\beta_p A^{p-1}\left[2p\lambda^2\frac{\ddot{a}}{a}+[(D-1)(\mu_D+1)-2p]A\right]=0 , \tag{4.6b}$$

where $\tilde{\rho}\equiv 16\pi G_D\lambda^2\rho/c^4$. [N.B. The p appearing in (4.6b) represents integers, not the pressure.]

A. The vacuum solutions

In the vacuum case the field equation (4.6a), $\tilde{\rho}=0$, is an algebraic equation of degree $[(D-1)/2]$ for A, which therefore must be equal to one of its roots κ: $A=\kappa$. The equation is then identical to (3.3) so that the same comments, in particular about the reality of κ, can be carried over. Now from $A=\kappa$ and if $\dot{a}\neq 0$ it follows that $\lambda^2\ddot{a}/a=A$ so that (4.6b) reduces to (4.6a); hence the other field equations $G_{\bar{A}\bar{B}}=0$ are identically satisfied. For generic values of the α_p's, the empty Robertson-Walker universes of the Lovelock theory are then necessarily locally isometric to maximally symmetric (de Sitter) space-times. "Einstein static" universes ($\dot{a}=0$) can also be solutions of the vacuum field equations, but at the cost of fine-tuning, say, the bare cosmological constant α_0 in order that

$$y(A)=\sum_{0\le p<D/2}\beta_p A^p=0$$

and

$$\sum_{0\le p<D/2}\beta_p(D-2p-1)A^p=0 \tag{4.7}$$

have one (or several) common root(s) $A=\kappa$. [The conditions (4.7) are in fact equivalent to $y(A)=0$ and $dy/dA=0$.]

B. The spatially flat ($k=0$) solutions

In that case, setting

$$x=\lambda\dot{a}/a \tag{4.8}$$

we have, from (4.3), $A=x^2$ and $\lambda^2\ddot{a}/a=\dot{x}\lambda+x^2$, so that the field equations (4.6) reduce to a quadrature

$$\tilde{\rho}=\sum_{0\le p<D/2}\beta_p x^{2p} , \tag{4.9}$$

$$\frac{1}{\lambda\alpha}\frac{dt}{dx}=-\frac{\sum p\beta_p x^{2(p-1)}}{\sum\beta_p x^{2p}} , \tag{4.10}$$

where $\alpha\equiv 2/[(D-1)(\mu_D+1)]$. Using (4.9), Eq. (4.10) reads

$$\frac{1}{\lambda\alpha}\frac{dt}{dx}=-\frac{1}{2x}\frac{d\tilde{\rho}/dx}{\tilde{\rho}} \tag{4.11}$$

which yields the first integral

$$a\tilde{\rho}^{\alpha/2}=\text{const} . \tag{4.12}$$

It must be noted however that (4.10) is a consequence of (4.6b) under the condition that the coefficient of $\ddot{a}$ in that equation, that is, the numerator of (4.10) or, equivalently, $x^{-1}d\tilde{\rho}/dx$, does not vanish. As for the denominator of (4.10), which is simply $\tilde{\rho}$, we shall impose it to be always positive, although such a requirement is fairly arbitrary—in this context where all local physics is ignored.

In the particularly simple case of Einstein theory in D dimensions (4.10) reads

$$\frac{t}{\alpha\lambda}=-\int\frac{\beta_1}{\beta_0+\beta_1x^2}dx \tag{4.13}$$

and yields the standard results (a_0 and t_0 are integration constants):

(a) $\beta_0=0$,

$$a(t)=a_0(t-t_0)^{\alpha}, \quad \tilde{\rho}=\lambda^2\beta_1\alpha^2(t-t_0)^{-2} ; \tag{4.14a}$$

(b) $\beta_0/\beta_1>0$,

$$a(t)=a_0\left\{\cos\left[\left[\frac{\beta_0}{\beta_1}\right]^{1/2}\frac{t-t_0}{\lambda\alpha}\right]\right\}^{\alpha},$$
$$\tilde{\rho}=\beta_0\left\{\cos\left[\left[\frac{\beta_0}{\beta_1}\right]^{1/2}\frac{t-t_0}{\lambda\alpha}\right]\right\}^{-2}; \tag{4.14b}$$

(c) $\beta_0/\beta_1<0$,

$$a(t)=a_0\left\{\cosh\left[\left[\frac{-\beta_0}{\beta_1}\right]^{1/2}\frac{t-t_0}{\lambda\alpha}\right]\right\}^{\alpha},$$
$$\tilde{\rho}=\beta_0\left\{\cosh\left[\left[\frac{-\beta_0}{\beta_1}\right]^{1/2}\frac{t-t_0}{\lambda\alpha}\right]\right\}^{-2}. \tag{4.14c}$$

The solutions (a) and (b) correspond to universes starting from a singularity at $t=t_0$ which either expand forever or recollapse. The case (c) is to be rejected if, as demanded by local physics, ρ is to be positive (i.e., $\beta_0>0$) and gravity attractive ($\beta_1>0$).

In the next simplest case, that of the quadratic (or Lanczos) theory, (4.10) reads

$$\frac{t}{\alpha\lambda}=-\int\frac{\beta_1+2\beta_2x^2}{\beta_0+\beta_1x^2+\beta_2x^4}dx . \tag{4.15}$$

This integral can be explicitly calculated. More enlightening however is a qualitative analysis. The parameter plane (γ_0,γ_1), with $\gamma_0\equiv\beta_0/\beta_2$ and $\gamma_1\equiv\beta_1/\beta_2$, splits into five regions according to the signs of the numerator and the denominator of (4.15):

$$N=\gamma_1+2x^2=\frac{1}{2\beta_2x}\frac{d\tilde{\rho}}{dx}$$

and

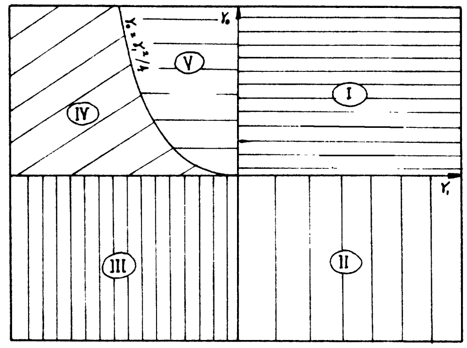

FIG. 1. Setting $x_{\pm}^2=\frac{1}{2}(-\gamma_1\pm\sqrt{\gamma_1^2-4\gamma_0})$, we have (I) $D>0$ for all x^2, $N>0$ for all x^2, (II) $D>0$ for all $x^2>x_+^2$, $N>0$ for all x^2, (III) $D>0$ for all $x^2>x_+^2$, $N>0$ for all $x^2>-\gamma_1/2$ ($x_+^2>-\gamma_1/2$), (IV) $D>0$ for all $x^2>x_+^2$ and all $x^2<x_-^2$, $N>0$ for all $x^2>-\gamma_1/2$, (V) $D>0$ for all x^2, $N>0$ for all $x^2>-\gamma_1/2$.

$$D=\gamma_0+\gamma_1x^2+x^4=\tilde{\rho}/\beta_2$$

(see Fig. 1).

In region I, with $\beta_2>0$ so that $\tilde{\rho}$ is positive, the solution is essentially the same as in Einstein theory with a positive cosmological constant [cf. Eq. (4.14b)]: the Universe starts from a singularity, expands to a maximum, and then recollapses. In regions II and III ($\gamma_0<0$) when $\beta_2>0$, the energy density is positive if $x^2>x_+^2$, so again the solution starts from a singularity, but then in general approaches a de Sitter-type solution: $a(t)\sim a_0\exp(x_+t/\lambda)$ as $t\to+\infty$; when $x_+=0$, that is $\beta_0=0$ the expansion approaches a power law: $a(t)\sim a_0t^{\alpha}$. In region II, when $\beta_2<0$, $\tilde{\rho}>0$ if $x^2<x_+^2$, so the solution here is regular: it bounces at the origin and asymptotically tends to de Sitter behavior (unless $\beta_0=0$, in which case the expansion follows a power law). Just as in Einstein theory, however [see Eq. (4.14c)], this solution may have to be rejected when one restricts the β_p's to yield a positive effective gravitational coupling constant in four di-

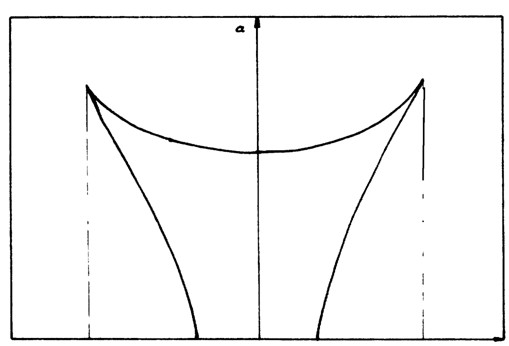

FIG. 2. An example of a pathological behavior of the scale factor of the Robertson-Walker cosmological models in the Lovelock theory, corresponding to $\gamma_0=\beta_0/\beta_2$ and $\gamma_1=\beta_1/\beta_2$ lying in region V of Fig. 1, $\beta_2>0$.

mensions (see, e.g., Ref. 32 for examples of such a requirement).

More interesting, however, for our purpose of illustrating some of the specific properties of the Lovelock theory as compared to Einstein's are the regions IV and V of Fig. 1, for all β_2, and region III when $\beta_2<0$, where the solutions are pathological. An example (corresponding to γ_0 and γ_1 in region V, and $\beta_2>0$) is depicted in Fig. 2. It describes a universe which comes into being at some $t=t_I$, but differs from a standard big-bang model in two essential respects: $t=t_I$ is not a curvature singularity [$\rho(t_I)$ is finite] and the solution is multivalued, so that the solution can either follow the lower branch and end up into a curvature singularity at $t=-t_0$ or follow the upper branch and end up into nothingness at $t=t_F$, without any divergence in the curvature invariants signaling the approach of that event. The origin of these pathologies lies in that the numerator of (4.15), that is, the coefficient of $\ddot{a}$ of the field equation (4.6b) which determines the evolution of the scale factor, vanishes at $t=t_I,t_F$. Hence starting from some initial data, at $t=0$, for example, (4.15) for $a(t)$ cannot predict its evolution beyond $t=t_F$.

This occasional failure of the field equations to predict the time evolution of the geometry, of which the case treated above is a particularly simple example, can ultimately be traced back to the fact that the Lovelock field equations, contrary to Einstein's, are nonlinear in the second-order derivatives of the metric. In this respect they are to be contrasted with the field equations derived from a generic, nonlinear in the curvature, Lagrangian which, just as the Einstein-Hilbert Lagrangian, yields equations that are linear in the higher-order derivatives of the metric (fourth in the generic case). Other examples of pathologies arising from this property of the Lovelock theory have been given in Refs. 28(b) and 35. The closely connected problem of the wave propagation in this theory has been examined in Refs. 57 and 58, as well as the resulting difficulties in setting up the Hamiltonian formalism.[59] As for the Cauchy problem of the theory and the equation of the characteristic surfaces, they have been studied in Refs. 60 and 49.

This thorough analysis of the spatially flat Robertson-Walker universes in the Lanczos (quadratic) theory gives the pattern of the various possible types of behavior in the generic case. We first note [see Eqs. (4.9) and (4.10)] that when $\beta_q>0$, with q the degree of the numerator and denominator of (4.10), the solution necessarily exhibits a curvature singularity. Indeed the asymptotic region $x^2\to+\infty$ is then allowed since there $\rho>0$. From (4.10) one has

$$\frac{1}{\lambda\alpha}\frac{dt}{dx}\sim-\frac{q}{x^2}\quad \text{as } x^2\to\infty \tag{4.16}$$

so that $a\propto(t-t_I)^{\alpha q}$ and $\tilde{\rho}\propto(t-t_I)^{-2q}$ as $t\to t_I$, and $\tilde{\rho}$ diverges. The subsequent evolution of the nonpathological solutions [i.e., those cases where the numerator of (4.10) never vanishes] depends on whether the denominator is positive for all x^2, in which case, by time symmetry, the model will recollapse to a big-crunch singularity, or whether it goes to zero at $x^2=x_+^2$, in which case the model approaches a de Sitter stage (or a power-law expansion if $\beta_0=0$). If $\beta_q<0$ the asymptotic region $x^2\to\infty$ is forbidden, and the nonpathological solutions are singularity free: they bounce and tend asymptotically to de Sitter or power-law stages.

C. The static solutions and their linear stability

The Lovelock Robertson-Walker static universes generalize in the present context the Einstein static universe. They are the solutions of the field equations (4.6a) and (4.6b) such that $\dot{a}=\ddot{a}=0$, $a(t)=\bar{a}$, $A=\bar{A}=\lambda^2k/\bar{a}^2$. Introducing the polynomial

$$z(A)=\sum_{0\le p<D/2}\beta_p A^p(\kappa-2p) \tag{4.17a}$$

with $\kappa\equiv(D-1)(\mu_D+1)$, it is easily seen that the static solutions are the zeros of $z(A)$,

$$z(\bar{A})=0\ , \tag{4.17b}$$

and that the corresponding density is

$$\tilde{\rho}(\bar{A})\equiv\bar{\rho}=\frac{2\bar{A}}{\kappa}\sum_{0\le p<D/2}p\beta_p\bar{A}^{p-1}\ . \tag{4.18}$$

Equations (4.17a) and (4.17b) may or may not have real solutions, depending on the values taken by the β_p's, μ_D, and D. If it has, then the corresponding $\bar{\rho}$, given by (4.18), may or may not be positive. If it is, one may inquire about the linear stability of the solution. Setting $a=\bar{a}(1+\epsilon_0e^{i\omega t})$ and linearizing the field equation (4.6b) one finds

$$\omega^2=-\frac{2\bar{A}^2}{\kappa\bar{\rho}}\frac{dz}{dA}\bigg|_{A=\bar{A}}\ . \tag{4.19}$$

Plotting the curve $z(A)$ we see that generically if the zero $A=\bar{A}_k$ corresponds to a stable (unstable) solution, its neighbors $A=\bar{A}_{k-1}$ and $A=\bar{A}_{k+1}$ are unstable (stable).

If all the parameters of the theory (the β_p's) are of order unity, the solutions of (4.17a) and (4.17b) and ω^2 are also of the same order. For one solution, say $\bar{A}=\bar{A}_m$, to be very small compared to the others, $\bar{A}_m\sim10^{-2N}$, so that $\bar{a}_m\sim10^N\lambda$ be of cosmological size, it is sufficient that $\beta_0\sim10^{-2N}$ while the others remain of order unity. That solution is then approximately the Einstein static universe: $\bar{A}_m\simeq A_E$, $\bar{\rho}_m\simeq\rho_E$, $\omega_m^2\simeq\omega_E^2$, with

$$A_E=-\frac{\kappa\beta_0}{\beta_1(\kappa-2)}\ ,$$

$$\rho_E=-\frac{2\beta_0}{\kappa-2}\ ,$$

$$\omega_E^2=\frac{\kappa\beta_0}{\beta_1}\sim10^{-2N}\ . \tag{4.20}$$

We then recover Eddington's result that this cosmological solution, for all $\mu_D>-(D-3)/(D-1)$, is unstable in a cosmological time scale if ρ_E is to be positive (which implies $\beta_0<0$) and gravity attractive ($\beta_1>0$, that is,

$k=+1$ and $\omega_E^2<0$). As for the other static, "Planckian" $[\bar{a}=O(\lambda)]$, solutions, they evolve on a Planck time scale λ.

D. The phase-space diagram of the $k=+1$ solutions: a qualitative analysis

The previous stability analysis suffices to obtain the qualitative features of the general solution of the Robertson-Walker field equations (4.6a) and (4.6b) by means of phase-space diagrams. Figure 3 is an example of such a diagram where Eq. (4.17) for the static case is assumed to have three real positive solutions; this implies that the Lovelock theory under consideration includes terms up to cubic order at least. It is further assumed that β_0 is of order 10^{-2N} and negative, $\beta_1>0$ and $k=+1$, so that the cosmological solution $a=\bar{a}_m$, of order $10^N\lambda$, is approximately the Einstein static solution, and is unstable on a cosmological time scale $10^N\lambda$.

The curves labeled 1 and 2 in Fig. 3 describe universes which start from an initial singularity, expand to a maximum of Planckian size and recollapse. The curve 3 represents a universe which oscillates about a Planckian mean value. The curve 4 is a model which starts from a singularity and, for generic initial values, blows up in a Planckian time scale. Therefore, for a solution to evolve from a big bang to a universe of cosmological size with scale factor of order $\bar{a}_m$ (curve 4′), it must belong to the family 4 and the initial velocity $\dot{a}_0$ must be carefully fine-tuned to avoid blowing up in a Planckian time scale.

We know from the analysis of the $k=0$ case (Sec. IV B) that the coefficient of $\ddot{a}$ in the field equation (4.6b) may vanish in some cases. This phenomenon would correspond to the existence, in the phase-space diagram, of curves which intersect. Such curves are shown in Fig. 3 with discontinuous lines; the dashed one corresponds to a universe qualitatively exhibiting the evolution of Fig. 2.

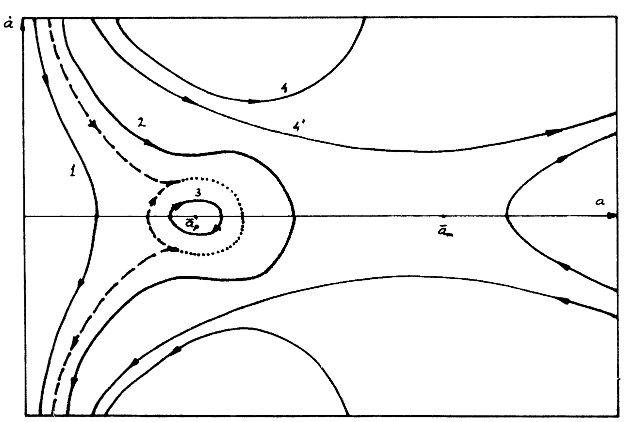

FIG. 3. A qualitative picture of the phase space of a closed Robertson-Walker space-time. The diagram is compressed along the horizontal axis to allow its representation, but $\bar{a}_m$ is many orders of magnitude larger than $\bar{a}_P$. For explanations see Sec. IV D in the main text.

V. PRODUCT SPACES

Let us finally consider the case when the manifold is a product of an "external" $(d+1)$-dimensional Robertson-Walker space-time and an n-dimensional compact "internal" space of constant curvature: $\mathcal{V}_D=\mathcal{V}^{(e)}_{(d+1)}\times\mathcal{V}^{(i)}_n$. When $d=3$ and the size of the internal space is small enough such a geometry is a potentially realistic candidate for describing today's Universe. In an appropriate coordinate system the line element can be written as

$$ds^2=-dt^2+g_{\bar\alpha\bar\beta}dx^{\bar\alpha}dx^{\bar\beta}+g_{ab}dx^a dx^b$$
$$=-dt^2+a_d^2(t)d\sigma_d^2+a_n^2(t)d\sigma_n^2\ , \qquad (5.1)$$

where $d\sigma_d^2$ and $d\sigma_n^2$, respectively, describe a d-dimensional maximally symmetric space (not necessarily compact) and an n-dimensional compact space of constant curvature. The nonvanishing components of the Riemann tensor are

$$R_{\bar\alpha 0\bar\beta 0}=-\frac{\ddot a_d}{a_d}g_{\bar\alpha\bar\beta},\quad R_{\bar\alpha\bar\beta\bar\gamma\bar\delta}=\frac{A_d}{\lambda^2}(g_{\bar\alpha\bar\gamma}g_{\bar\beta\bar\delta}-g_{\bar\alpha\bar\delta}g_{\bar\beta\bar\gamma})\ ,$$
$$R_{a0b0}=-\frac{\ddot a_n}{a_n}g_{ab},\quad R_{a\bar\alpha b\bar\beta}=-\frac{x_n x_d}{\lambda^2}g_{ab}g_{\bar\alpha\bar\beta}\ , \qquad (5.2)$$
$$R_{abcd}=\frac{A_n}{\lambda^2}(g_{ac}g_{bd}-g_{ad}g_{bc})\ ,$$

where $x_d=\lambda\dot a_d/a_d$, $A_d=\lambda^2k_d/a_d^2+x_d^2$, $x_n=\lambda\dot a_n/a_n$, $A_n=\lambda^2k_n/a_n^2+x_n^2$, k_d takes the values $+1$, 0, or -1 but k_n is restricted to the values $+1$ or 0 if $\mathcal{V}_n$ is to be compact. It is then a matter of a straightforward calculation to obtain the Lovelock tensor (2.5) for the metric (5.1). We shall write it as

$$G_{00}=\tfrac{1}{2}F\ ,$$
$$G_{\bar\alpha\bar\beta}=-\tfrac{1}{2}g_{\bar\alpha\bar\beta}\left[f_d+\frac{\ddot a_d}{a_d}g_d+\frac{\ddot a_n}{a_n}h_d\right]\ , \qquad (5.3)$$
$$G_{ab}=-\tfrac{1}{2}g_{ab}\left[f_n+\frac{\ddot a_d}{a_d}h_n+\frac{\ddot a_n}{a_n}g_n\right]\ .$$

The explicit expressions of $F,f_d,g_d,h_d,f_n,g_n,h_n$ are given in Appendix A up to cubic order and agree with the result of Müller-Hoissen.[29]

There are two particular cases where they can easily be generalized to any order. These are (a) when the spaces $\mathcal{V}^{(e)}_d$ and $\mathcal{V}^{(i)}_n$ are static (a_d=const, a_n=const, $k_d=\pm1$, $k_n=+1$) and (b) when they are both flat ($k_d=k_n=0$ and $\mathcal{V}^{(i)}_n$ compactified on a torus). See Appendix A, Eqs. (A4)–(A6), respectively.

Knowing the explicit expression for the Lovelock tensor as a function of a_d and a_n we can write the field equations (2.7) as

$$\tilde\rho=F\ ,$$
$$\frac{\ddot a_d}{a_d}g_d+\frac{\ddot a_n}{a_n}h_d=-(f_d+\mu_dF)\ , \qquad (5.4)$$
$$\frac{\ddot a_d}{a_d}h_n+\frac{\ddot a_n}{a_n}g_n=-(f_n+\mu_nF)\ ,$$

when the components of the stress tensor are taken to be

$$\frac{8\pi G_D}{c^4}T_{00}=-\tilde{\rho}\ ,$$
$$\frac{8\pi G_D}{c^4}T_{\bar{\alpha}\bar{\beta}}=\mu_d\tilde{\rho}g_{\bar{\alpha}\bar{\beta}}\ , \quad (5.5)$$
$$\frac{8\pi G_D}{c^4}T_{ab}=\mu_n\tilde{\rho}g_{ab}\ .$$

The constants μ_d and μ_n characterize the equations of state of the cosmic fluid in the external and internal spaces, respectively. One would expect them to be different when the scales of both subspaces differ substantially, since "long"-wavelength fields would be excluded from a Planck-size internal space.

A. Maximally symmetric external space-time with no matter

When the external space-time is imposed to be de Sitter (that is, x_d=const and $k_d=0$), the computation of the Lovelock tensor (5.3) at a point where $\dot{a}_n=0$ shows that the vacuum field equations [Eqs. (5.4) with $\mu_d=\mu_n=\rho=0$] are satisfied if (1) $\ddot{a}_n=0$ and (2) the Hubble constant x_d and the curvature $A_n=1/a_n^2$ of the internal space are solutions of the coupled algebraic equations:

$$\sum_{0\le p<D/2}\alpha_p\sum_{k=0}^{p}C_k^p\frac{d!}{(d-2p+2k)!}\frac{n!}{(n-2k)!}x_d^{2(p-k)}A_n^k=0\ ,$$
$$\sum_{0\le p<D/2}\alpha_p\sum_{k=0}^{p}C_k^p\frac{(d+1)!}{(d+1-2p+2k)!}\frac{(n-1)!}{(n-1-2k)!}x_d^{2(p-k)}A_n^k=0\ . \quad (5.6)$$

The first condition, $\ddot{a}_n=0$, shows that when the external manifold is de Sitter space-time and $T_{AB}=0$ the internal space is necessarily static. When the dimension of the external space is taken to be $d=3$, Eq. (5.6) simplifies to

$$6x_d^2\left[\sum_{0\le p<D/2}p\alpha_p\frac{n!}{(n+2-2p)!}A_n^{p-1}\right]+\sum_{0\le p<D/2}\alpha_p\frac{n!}{(n-2p)!}A_n^p=0\ , \quad (5.7a)$$

$$12x_d^4\left[\sum_{0\le p<D/2}p(p-1)\alpha_p\frac{(n-1)!}{(n+3-2p)!}A_n^{p-2}\right]+12x_d^2\left[\sum_{0\le p<D/2}p\alpha_p\frac{(n-1)!}{(n+1-2p)!}A_n^{p-1}\right]+\sum_{0\le p<D/2}\alpha_p\frac{(n-1)!}{(n-1-2p)!}A_n^p=0\ , \quad (5.7b)$$

which in the case $n=4$, for example, reads

$$6x_d^2(\alpha_1+24\alpha_2A_n+72\alpha_3A_n^2)+\alpha_0+12\alpha_1A_n+24\alpha_2A_n^2=0\ , \quad (5.8a)$$

$$24x_d^4(\alpha_2+18\alpha_3A_n)+12x_d^2(\alpha_1+12\alpha_2A_n)+\alpha_0+6\alpha_1A_n=0\ . \quad (5.8b)$$

The existence and number of solutions of (5.6)–(5.8) depend on the values taken by the various parameters (the α_p's, d, and n) and must be examined case by case.

If all the α_p are of order unity the external manifold can be Minkowski space-time ($x_d=0$) if

$$\sum_{0\le p<D/2}\alpha_p\frac{n!}{(n-2p)!}A_n^p=0\ ,$$
$$\sum_{0\le p<D/2}\alpha_p\frac{(n-1)!}{(n-1-2p)!}A_n^p=0\ . \quad (5.9)$$

These conditions have already been obtained by many authors (see, e.g., Refs. 21–27 and 32). The first equation (5.9) gives the radius of the internal manifold as a function of its dimension n and the parameters α_p; the second is a condition on one of the α_p, α_0, say, which is determined as a function of the others. When $n=4$, for example, the conditions (5.9) are $A_n=-\alpha_0/6\alpha_1$ and $\alpha_0=3\alpha_1^2/2\alpha_2$.

The linear stability of this candidate ground state for an effective quantum field theory in four-dimensional Minkowski space-time was analyzed in a number of papers[27,32,61] (the analysis of Ref. 61 being the most complete) and will not be repeated here.

Concentrating on the example (5.8) we shall only remark that when α_0 is fine-tuned so that $x_d=x_n=0$ is a solution of (5.8) (that is, $\alpha_0=3\alpha_1^2/2\alpha_2$ and $A_n^{(1)}=-\alpha_1/4\alpha_2$) then there exists another solution which is symmetrical to that one: $A_n^{(2)}=0$, $x_d^{(2)2}=-\alpha_1/4\alpha_2$. As for the remaining three solutions they are such that $A_n=x_d^2$ and of order unity. The question as to whether these manifolds, where both the external and internal spaces are of Planckian size, can serve as an initial (nonsingular) state for the Universe which, because of some instability, would evolve toward the present state (5.9), requires a numerical integration of the field equations (5.4) that will not be presented here.

Another way for the external manifold to be Minkowski space-time, or at least to tend to it, is that the coefficient $\alpha_{[D/2-1]}$ of the leading term in the Lovelock tensor [α_3 if $D=8$, see Eq. (5.8)] be infinitely large as compared to the others: $\alpha_{[D/2-1]}\sim10^{2N}$, $\alpha_p\sim1$, $N\to\infty$. In such theories (the values $D=8$, $d=3$, $n=4$ can serve again as an example) it is easy to see that the solutions scale as follows: one *ad futurum* solution such that $A_n\sim1$, $x_d^2\sim10^{-2N}$; one *ad initium* solution $A_n\sim10^{-2N}$,

$x_d^2 \sim 1$ obtained (when $d=3$, $n=4$) by an interchange of x_d^2 and A_n; and up to three "big-bang" solutions $A_n = x_d^2 \sim 10^{-2N/3}$, which have the interesting property of being highly curved but not to the point that quantum gravity corrections must be taken into account. The stability analysis of these ground states (in the spirit of Ref. 61) has not been done, as far as we know, but should be straightforward (although painstaking) and will not be undertaken here. Nor shall we study the possibility of dynamical evolution from one of these ground states to another. We shall only remark that if the sign of the determinant

$$\text{Det} = g_d g_n - h_n h_d \tag{5.10}$$

of the system (5.4) is positive, say, in the vicinity of the "cosmological" solutions and negative near the "Planckian" or "intermediate" ones then no dynamical evolution from one solution to another is possible since the system (5.4) exhibits a breakdown of predictability when the determinant goes through zero.

B. On the static solutions and their linear stability

When the coefficient $\alpha_{[D/2-1]}$ of the term of highest degree in the Lovelock tensor is $\sim 10^{2N}$, with N large ($N \sim 60$, to be more specific), and the others are of order 1 then the Lovelock field equations (5.4) admit a static solution ($\dot{a}_n = \dot{a}_d = 0$) such that the curvature of the external manifold is much smaller than that of the internal space ($A_d \sim 10^{-2N}$, $A_n \sim 1$). In Ref. 34 it was shown that this solution can be linearly stable against perturbations of the radii of the two subspaces if the α_p belong to an appropriate range, in contrast with Einstein's static solutions to the field equations of the four-dimensional general relativity, which was shown by Eddington[10] to be always unstable. We shall present in this section a more thorough, although not exhaustive, analysis of these static solutions.

When a_d and a_n, the radii of the external and internal spaces, respectively, are imposed to be time independent, the field equations (5.4) reduce to

$$\bar{Y}_d \equiv \bar{f}_d + \mu_d \bar{F} = 0, \quad \bar{Y}_n \equiv \bar{f}_n + \mu_n \bar{F} = 0 \,, \tag{5.11}$$

where the functions $\bar{f}_d$, $\bar{F}$, and $\bar{f}_n$ are given in Appendix A [Eq. (A4)]. In the following we shall restrict ourselves to the case $d=3$ of a four-dimensional external space-time [Eq. (A5) of Appendix A]. For the sake of an example, when $d=3$ and $n=4$, Eq. (5.11) reads

$$\bar{A}_d = -\frac{1+\mu_d}{2(1+3\mu_d)} \frac{\alpha_0 + 12\alpha_1 \bar{A}_n + 24\alpha_2 \bar{A}_n^2}{\alpha_1 + 24\alpha_2 \bar{A}_n + 72\alpha_3 \bar{A}_n^2} \,, \tag{5.12a}$$

$$\frac{2\mu_n}{1+3\mu_d}(\alpha_0 + 12\alpha_1 \bar{A}_n + 24\alpha_2 \bar{A}_n^2) = \alpha_0 + 6\alpha_1 A_n - \frac{3(1+\mu_d)}{1+3\mu_d} \frac{(\alpha_0 + 12\alpha_1 \bar{A}_n + 24\alpha_2 \bar{A}_n^2)(\alpha_1 + 12\alpha_2 \bar{A}_n)}{\alpha_1 + 24\alpha_2 \bar{A}_n + 72\alpha_3 \bar{A}_n^2} \,. \tag{5.12b}$$

When $\mu_n \neq 0$ (the case $\mu_n = 0$ yields unstable solutions as shown in Ref. 34, and will not be considered any further) the equation for $\bar{A}_n$, (5.12b), is of degree n (4 in the example considered) and has therefore up to n real solutions, to each of which corresponds a value for $\bar{A}_d$, through Eq. (5.12a).

We shall now consider two different types of theories.

(I) The coefficient $\alpha_{[D/2-1]} = \alpha_{[n/2+1]}$ is of order 10^{2N} while all others are of order 1 (this is the case considered in Ref. 34).

As one can see from its structure, the equation for A_n [see Eq. (5.12b), for example] has then $n/2$ "cosmological solutions" ($A_n \sim 1$ with corresponding A_d of order 10^{-2N}) which cause the coefficient of $\alpha_{[n/2+1]}$ to vanish. The other $n/2$ "big-bang" solutions are of order $10^{-4N/n}$ (and not 10^{-2N}) and very close to each other (in the case $n=4$, for example [see Eq. (5.12b)] these two solutions are of order 10^{-N} but their difference is of order 10^{-2N}); as for the corresponding A_d, they are of order 1.

(II) $\mu_d = -1 + \kappa$, with $\kappa \sim 10^{-2N}$ and all the other parameters of order 1. As we are dealing with product manifolds this condition does not reduce to merely renormalizing the "bare cosmological constant" α_0. Then the equation for A_n has n solutions all of order 1. It turns out that $n/2$ of them are the zeros of the denominator of the equation for A_d [see e.g., Eqs. (5.12a) and (5.12b)]. This means that the corresponding solutions for A_d are of order 10^{-2N} while the other $n/2$ are of order 1. These results are summarized in Fig. 4.

The analysis of the linear stability of the static solutions (5.11) [or (5.12) when $d=3$, $n=4$] is greatly simplified by the fact that the field equations (5.4) do not

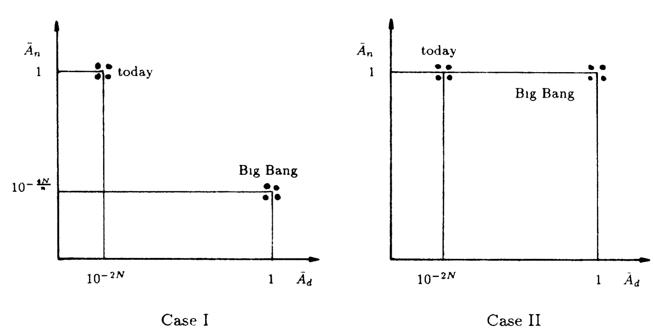

FIG. 4. Schematic representation of the orders of magnitude of the static solutions of the Lovelock field equations when space-time is imposed to be the product $R \times S_3 \times S_n$, S_3 and S_n being spaces of constant curvature of 3 and n dimensions, respectively ($n=8$ in the figure). Case I corresponds to theories such that the coefficient of the leading term in the Lovelock tensor (α_5 when $n=8$) is very large (of order 10^{2N}, $N \sim 60$) with respect to all the other parameters of the theory. Case II corresponds to theories where all α_p are of order unity but where the equation of state for the (nonbaryonic) fluid has an index $\mu_d = -1 + \kappa$, $\kappa \sim 10^{-2N}$.

contain any linear term in $x_d \equiv \dot{a}_d/a_d$ or x_n. At the linear approximation therefore we have, when $a_d = \bar{a}_d(1+\epsilon_d)$, $a_n = \bar{a}_n(1+\epsilon_n)$,

$$F \sim \bar{F} - 2\left[\frac{\partial \bar{F}}{\partial \bar{A}_d}\bar{A}_d\epsilon_d + \frac{\partial \bar{F}}{\partial \bar{A}_n}\bar{A}_n\epsilon_n\right] \quad (5.13)$$

and similar expressions for the other quantities f_d, h_d, etc., appearing in (5.4). The field equations hence read, when linearized about the static solution,

$$\ddot{\epsilon}_d\bar{g}_d + \ddot{\epsilon}_n\bar{h}_d = 2\left[\frac{\partial \bar{Y}_d}{\partial \bar{A}_d}\bar{A}_d\epsilon_d + \frac{\partial \bar{Y}_d}{\partial \bar{A}_n}\bar{A}_n\epsilon_n\right],$$
$$\ddot{\epsilon}_d\bar{h}_n + \ddot{\epsilon}_n\bar{g}_n = 2\left[\frac{\partial \bar{Y}_n}{\partial \bar{A}_d}\bar{A}_d\epsilon_d + \frac{\partial \bar{Y}_n}{\partial \bar{A}_n}\bar{A}_n\epsilon_n\right], \quad (5.14)$$

where $\bar{Y}_d$ and $\bar{Y}_n$ are given by Eq. (5.11) and the quantities $\bar{f}_d$, $\bar{h}_d$, etc., are given in Appendix A [Eqs. (A4) and (A5)]. Setting $\epsilon_d = \epsilon_{0d}e^{i\omega t}$, $\epsilon_n = \epsilon_{0n}e^{i\omega t}$, the linear stability of the static solutions is determined by the sign of the roots of the equation for ω^2:

$$0 = 4\bar{A}_d\bar{A}_n\left[\frac{\partial \bar{Y}_d}{\partial \bar{A}_d}\frac{\partial \bar{Y}_n}{\partial \bar{A}_n} - \frac{\partial \bar{Y}_d}{\partial \bar{A}_n}\frac{\partial \bar{Y}_n}{\partial \bar{A}_d}\right] + 2\omega^2\left[\bar{g}_d\bar{A}_n\frac{\partial \bar{Y}_n}{\partial \bar{A}_n} + \bar{g}_n\bar{A}_d\frac{\partial \bar{Y}_d}{\partial \bar{A}_d} - \bar{h}_d\bar{A}_d\frac{\partial \bar{Y}_n}{\partial \bar{A}_d} - \bar{h}_n\bar{A}_n\frac{\partial \bar{Y}_d}{\partial \bar{A}_n}\right] + \overline{\mathrm{Det}}\,\omega^4, \quad (5.15)$$

where Det is given by Eq. (5.10).

Let us explore a few simple cases and start with ordinary Einstein theory (all α_p zero but for α_0 and α_1). Then there exists one static solution:

$$A_d = -\frac{\alpha_0}{\alpha_1}\frac{1+\mu_d}{d-1}\frac{1}{D-3+d\mu_d+n\mu_n},$$
$$A_n = A_d\frac{d-1}{n-1}\frac{1+\mu_n}{1+\mu_d}. \quad (5.16)$$

In order for the external space-time to be vastly less curved that the internal space ($A_d \sim 10^{-2N}$, $A_n \sim 1$) it must be filled with a cosmic fluid with equation of state $\mu_d = -1+\kappa$, $\kappa \sim 10^{-2N}$. The linear perturbations away from this static solution (5.16) are characterized by two eigenfrequencies, solutions of (5.14), that in this particularly simple case read

$$\omega^2 = \omega_P^2 \simeq -\frac{\alpha_0}{\alpha_1}\frac{(d-1)(1+\mu_n)}{D-2},$$
$$\omega^2 = \omega_H^2 \simeq -2\left[\frac{\alpha_0}{\alpha_1}\right]\frac{\kappa}{(d-1)(n-2-n\mu_n)}. \quad (5.17)$$

The static solution is then linearly stable against perturbations of Planck frequency ($\omega^2 = \omega_p^2 > 0$) if $\alpha_0/\alpha_1 < 0$ (assuming $1+\mu_n > 0$). This implies [cf. Eq. (5.16)] that $A_n > 0$ which is compatible with the requirement that the internal space be closed. This static solution can also be stable against perturbations that evolve on cosmological time scales ($\omega_H^2 > 0$) if $k_d > 0$ in which case the external space is closed. Finally the energy density of the solution

$$\bar{\rho} = -\frac{2\alpha_0}{D-3+d\mu_d+n\mu_n} \quad (5.18)$$

is positive if $\alpha_0 < 0$ which implies $\alpha_1 > 0$, that is, attractive gravity. However, before concluding that Einstein could have saved his static model from the accusation of instability of Eddington, had he more firmly believed in the extra dimensions proposed by Kaluza, one should study, in the spirit of Refs. 61–64, the stability of this ground state against general perturbations of the metric and matter fields of which those considered here are a very special subclass. Such an analysis, however, is beyond the scope of this paper and is left to further work.

Let us finally turn to the case next to simplest of the Lanczos theory in $D=6$ dimensions ($d=3$, $n=2$) and start with theories previously called type I, which are such that $\alpha_2 \sim 10^{2N}$, all other parameters being of order unity. Solving Eqs. (5.11) and (5.14) and (5.15) in this case we find that there exist two static solutions. A "cosmic" static solution given by

$$\bar{A}_d^{(c)} \simeq -\frac{(1+\mu_d)\alpha_1}{4\alpha_2(1-2\mu_n+3\mu_d)} \sim 10^{-2N},$$
$$\bar{A}_n^{(c)} \simeq -\frac{(1+3\mu_d-2\mu_n)\alpha_0}{4\mu_n\alpha_1} \sim 1, \quad (5.19a)$$

corresponding to a density $\rho^c \simeq -\alpha_0/\mu_n$. The linearized solution near this ground state oscillates with the frequencies

$$\bar{\omega}_P^{(c)2} \simeq -\frac{\alpha_0(1+3\mu_d-2\mu_n)}{2\alpha_1} \sim 1,$$
$$\bar{\omega}_H^{(c)2} \simeq \frac{\alpha_1}{4\alpha_2}\frac{1+\mu_d}{1+\mu_n} \sim 10^{-2N}. \quad (5.19b)$$

The constraints are $\omega_p^{(c)2} > 0$ (stability on the Planck scale), $\bar{A}_n^c > 0$ (compact internal space), $\rho^c > 0$. This implies

$$\mu_n < 0, \quad \alpha_0 > 0, \quad \alpha_1 < 0. \quad (5.20)$$

And the solution is linearly stable on a cosmic time scale if $\alpha_2 < 0$, in which case the external space is noncompact ($\bar{A}_d^c < 0$). As for the other "Planckian" solution it is given by

$$\bar{A}_d^{(P)} \simeq -\frac{(1-2\mu_n+3\mu_d)\alpha_0}{6\alpha_1(1+3\mu_d)} \sim 1,$$
$$\bar{A}_n^{(P)} \simeq \frac{(1+\mu_n)\alpha_1}{2\alpha_2(1-2\mu_n+3\mu_d)} \sim 10^{-2N}, \quad (5.21)$$

corresponding to a density $\rho^P \simeq -2\alpha_0/(1+3\mu_d)$. Since $A_n^P \sim 10^{-2N}$ this solution could hardly serve as an initial state of the universe. Moreover it appears that it can evolve on a cosmic time scale only, as the two eigenfrequencies of its linear perturbations happen to be both of order 10^{-N}.

Consider now the type-II theories where $\mu_d = -1+\kappa$, $\kappa \sim 10^{-2N}$, and all the other parameters of the theory (that is, α_0, α_1, and α_2) are of order unity. Then the "cosmic" static solution is given by

$$\overline{A}_d^{(c)} \simeq -\frac{\alpha_0\alpha_1\kappa}{4[\mu_n\alpha_1^2 - 2\alpha_0\alpha_2(1+\mu_n)]} \sim 10^{-2N} ,$$
$$\overline{A}_n^{(c)} \simeq -\frac{1+\mu_n}{2\mu_n}\frac{\alpha_0}{\alpha_1} \sim 1 , \tag{5.22a}$$

corresponding to a density $\rho^c \simeq -\alpha_0/\mu_n$ and the frequencies of its linear oscillations are given by

$$\omega_P^{(c)2} \simeq -\frac{\alpha_0(1+\mu_n)}{2\alpha_1}\frac{\mu_n\alpha_1^2 - 2\alpha_0\alpha_2(1+\mu_n)}{\mu_n\alpha_1^2 + \alpha_0\alpha_2(1+\mu_n)} \sim 1 ,$$
$$\overline{\omega}_H^{(c)2} \simeq 2\overline{A}_d^c \sim 10^{-2N} . \tag{5.22b}$$

The constraints are $(\omega_p^c)^2 > 0$, $\overline{A}_n^c > 0$, $\rho^c > 0$, as before, and can be satisfied. Moreover with α_2 belonging to an appropriate range we can have

$$\mu_n > 0, \quad \alpha_0 < 0, \quad \alpha_1 > 0 , \tag{5.23}$$

and stability on a cosmic time scale for compact external spaces. As for the "Planckian" static solution,

$$\overline{A}_d^{(P)} \simeq -\frac{\mu_n\alpha_1^2 - 2\alpha_0\alpha_2(1+\mu_n)}{12\alpha_1\alpha_2} \sim 1 ,$$
$$\overline{A}_n^{(P)} \simeq -\frac{\alpha_1}{4\alpha_2} \sim 1 , \tag{5.24a}$$

corresponding to a density $\rho^c \simeq (2\alpha_0\alpha_2 - \alpha_1^2)/2\alpha_2$; it oscillates with frequencies which are both of order 1:

$$\omega_P^{(1)2} + \omega_P^{(2)2} \simeq -\frac{8\alpha_1\overline{A}_d^{(P)}}{\alpha_1 + 4\alpha_2\overline{A}_d^{(P)}} ,$$
$$\omega_P^{(1)2}\omega_P^{(2)2} \simeq -\frac{16\alpha_1\alpha_2\overline{A}_d^{(P)2}\overline{A}_n^{(P)}}{(\alpha_1 + 4\alpha_2\overline{A}_d^{(P)})^2} . \tag{5.24b}$$

We can therefore conclude this analysis of the static product manifolds that solve the Lovelock field equations thus: there are essentially two ways to impose that one of these solutions—the "cosmic" solution—be "realistic" (by this we mean that the internal space is of Planckian size and the external space is almost flat); the first is to fine-tune one of the parameters of the Lovelock Lagrangian (the "bare cosmological constant" α_0, say); the other is to impose the presence of a dilute nonbaryonic cosmic fluid with an index fine-tuned to be $\mu \simeq -1+\kappa$, κ very small. Usually the first condition is adopted (see, e.g., Ref. 32). However, we showed here (inasmuch as our conclusions, pertaining to a few particular cases, can be extended to the general case) that the second condition might be preferable since the other static solutions seem more satisfactory: they are such that both the external and internal spaces are of Planckian size and oscillate on a Planckian time scale; hence they could serve as an initial nonsingular state for the Universe. An analysis of whether a dynamical solution could indeed evolve from such an initial state to the "cosmic" solution requires a numerical integration of the field equations that will not be presented here. Also, a more thorough analysis of the stability of the "cosmic" solution (see Refs. 61–64) should be undertaken, to check in particular whether gravity remains attractive at the local level.

C. The case of flat subspaces without matter

When the two subspaces are flat ($k_d = k_n = 0$, $\mathcal{V}^{(i)}$ compactified on a torus) the field equations (5.4) reduce, in absence of matter, to

$$F_0 = 0 , \tag{5.25a}$$
$$\mathring{x}_d\mathring{g}_d + \mathring{x}_n\mathring{g}_d = -(\mathring{f}_d + x_d^2\mathring{g}_d + x_n^2\mathring{h}_d) , \tag{5.25b}$$
$$\mathring{x}_d\mathring{h}_n + \mathring{x}_n\mathring{g}_n = -(\mathring{f}_n + x_d^2\mathring{h}_n + x_n^2\mathring{g}_n) , \tag{5.25c}$$

where the quantities $F_0(x_d, x_n)$, $\mathring{g}_d, \mathring{g}_n, \mathring{h}_d, \mathring{h}_n, \mathring{f}_d, \mathring{f}_n$ are given in the Appendix [Eq. (A6)]. In the particular case of the Lanczos theory without a bare cosmological constant (only α_1 and α_2 nonzero) with moreover $\alpha_1 = \alpha_2$, these equations were studied by Ishihara.[33] We shall not attempt here to redo or generalize his work but only remark that there are in fact two curves of interest in the plane (x_d, x_n): first the curve $F_0 = 0$ which constraints the solution (see Ref. 32) but also the curve

$$\mathrm{Det}^{(0)} \equiv \mathring{g}_d\mathring{g}_n - \mathring{h}_d\mathring{h}_n \tag{5.26}$$

which divides the plane into causally disconnected regions.

VI. CONCLUSIONS

We have analyzed systematically the simplest solutions to the Lovelock field equations, with an increasingly less symmetric and more realistic metric, which can serve as ground states or cosmological initial states.

We have given the equations that characterize the maximally symmetric space-time solutions, and noted that, when maximal symmetry is abandoned in favor of several regions with constant curvature, there arises a scenario akin to some inflationary models.

With a higher-dimensional Robertson-Walker symmetry we have studied three cases. In a vacuum we have found that the solution is, in general, of de Sitter type; only for particular values of the Lagrangian coefficients can there also be a static Einstein solution. The nonvacuum spatially flat solutions show five qualitatively different kinds of behavior. Four of these arise in Einstein theory already: "big-bang" recollapsing universes; collapse from infinity that stops at some minimum value of the scale factor and bounces back to approach asymptotically a de Sitter phase; and unlimited expansion from a singularity, either following a power-law (Friedmann universes) or an asymptotically exponential behavior (Lemaître universes) (and their corresponding time-reversal ones). It should be noted, however, that those solutions correspond to quite different situations than their analogs in Einstein theory; for instance, the recollapsing universe is here spatially flat. The fifth family of solutions reveals a peculiarity of the Lovelock theory. It consists of pathological solutions which stem from the nonlinearity of the field equations in the highest derivatives: this allows the determinant of the system to go through zero at some points, preventing the prediction of a unique evolution from then onward. We have finally characterized the static solutions with Robertson-Walker symmetry and found that, when more than one is present, consecutive solutions have opposite stability properties against linear perturbations. The analysis allows one to draw some conclusions about the general behavior of the positively curved solutions, summarized in the qualitative phase plane portrait in Fig. 3.

The more realistic case studied is that of product spaces. A maximally symmetric external space allows only a static internal companion. We have reencountered in this case already known conditions for the existence of a four-dimensional Minkowski space-time with an internal static space, and presented new ones for an asymptotic behavior of that kind. An analysis of the static solutions has rendered specific examples of the two types of theories in which it is possible to find an external subspace of "cosmological" size accompanied by an internal one of a Planckian size. These two types of theories are (I) those where the coefficient α_p of the leading order in the Lovelock tensor is $\sim 10^{2N}(N\sim 60)$, and all the others ~ 1 and (II) those for which the external space is filled with some nonbaryonic cosmic fluid with equation of state characterized by $\mu_d=-1+\kappa$, with $\kappa\sim 10^{-2N}$ and all the other parameters of order unity. The stability properties of the various static solutions found seem to depend on the features peculiar to each model. Among these solutions, the more promising candidate for a nonsingular initial state that might have given rise to our Universe, via some kind of instability, corresponds to case (II) and consists of two Planckian subspaces capable of oscillating on a Planckian time scale.

The study of less symmetric and potentially more realistic situations is of great interest. Therefore the search for and study of possible cosmological initial states in the Lovelock theory has been extended to the case of Kasner solutions, in work that will be presented in a separate paper.[65] Also, as it has been pointed out in previous sections, a numerical integration of the field equations is required to study the dynamical departure of the Universe from static or very highly symmetric solutions that might serve as initial nonsingular states, and this work is currently in progress.

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APPENDIX A

Consider the line element

$$ds^2=-dt^2+a_d^2d\sigma_d^2+a_n^2d\sigma_n^2 , \tag{A1}$$

where $d\sigma_d^2$ and $d\sigma_n^2$ describe d- and n-dimensional spaces of constant curvature, respectively. Then the Lovelock tensor [Eq. (2.5) in text] is

$$\begin{aligned} G_{00}&=\tfrac{1}{2}F , \\ G_{\bar\alpha\bar\beta}&=-\frac{1}{2}g_{\bar\alpha\bar\beta}\left[f_d+\frac{\ddot a_d}{a_d}g_d+\frac{\ddot a_n}{a_n}h_d\right] , \\ G_{ab}&=-\frac{1}{2}g_{ab}\left[f_n+\frac{\ddot a_d}{a_d}h_n+\frac{\ddot a_n}{a_n}g_n\right] , \end{aligned} \tag{A2}$$

where the functions $F,f_d,g_d,h_d,f_n,g_n,h_n$ read, up to cubic order,

$$\begin{aligned} F=&\alpha_0+\alpha_1(d_1A_d+2dnx_dx_n+n_1A_n)+\alpha_2[d_3A_d^2+4d_2nA_dx_dx_n+2d_1n_1(A_dA_n+2x_d^2x_n^2)+4dn_2A_nx_nx_d+n_3A_n^2)] \\ &+\alpha_3[d_5A_d^3+6d_4nA_d^2x_dx_n+3d_3n_1(A_d^2A_n+4A_dx_d^2x_n^2)+4d_2n_2(2x_d^3x_n^3+3A_dA_nx_dx_n) \\ &\quad+3d_1n_3(A_dA_n^2+4A_nx_d^2x_n^2)+6dn_4A_n^2x_dx_n+n_5A_n^3] , \\ f_d=&\alpha_0+\alpha_1[(d-1)_2A_d+2(d-1)nx_dx_n+n_1A_n] \\ &+\alpha_2[(d-1)_4A_d^2+4(d-1)_3nA_dx_dx_n+2(d-1)_2n_1A_dA_n+4(d-1)_2n_1x_d^2x_n^2+4(d-1)n_2A_nx_nx_d+n_3A_n^2] \\ &+\alpha_3[(d-1)_6A_d^3+3(d-1)_4n_1A_d^2A_n+6(d-1)_5nA_d^2x_dx_n+12(d-1)_4n_1A_dx_d^2x_n^2 \\ &\quad+12(d-1)_3n_2A_dA_nx_dx_n+8(d-1)_3n_2x_d^3x_n^3+12(d-1)_2n_3A_nx_d^2x_n^2+6(d-1)n_4A_n^2x_dx_n \\ &\quad+3(d-1)_2n_3A_dA_n^2+n_5A_n^3] , \\ g_d=&2(d-1)\{\alpha_1+2\alpha_2[(d-2)_3A_d+2(d-2)nx_dx_n+n_1A_n] \\ &+3\alpha_3[(d-2)_5A_d^2+2(d-2)_3n_1A_dA_n+4(d-2)_4nA_dx_dx_n \\ &\quad+4(d-2)_3n_1x_d^2x_n^2+4(d-2)n_2A_nx_dx_n+n_3A_n^2]\} , \\ h_d=&2n\{\alpha_1+2\alpha_2[(n-1)_2A_n+2(n-1)(d-1)x_nx_d+(d-1)_2A_d] \\ &+3\alpha_3[(d-1)_4A_d^2+2(d-1)_2(n-1)_2A_dA_n+4(d-1)_3(n-1)A_dx_dx_n \\ &\quad+4(d-1)_2(n-1)_2(x_dx_n)^2+4(d-1)(n-1)_3A_nx_dx_n+(n-1)_4A_n^2]\} , \end{aligned} \tag{A3}$$

where $x_d=\lambda\dot{a}_d/a_d$, $A_d=\lambda^2k_d/a_d^2+x_d^2$, and $(d-k)_j\equiv(d-k)(d-k-1)\cdots(d-j)$; the expressions for x_n, A_n, f_n, g_n, h_n, n_j are obtained by interchange of n and d. We note that $h_n=(d/n)h_d$. The expressions (A2) and (A3) agree with the corresponding ones of Müller-Hoissen.[29]

In some particular cases they can be generalized to any order. These are the following.

(a) The static case. $a_n\equiv\bar{a}_n$, $x_n=0$, $A_n\equiv\bar{A}_n=k_n/\bar{a}_n^2$, and similar conditions substituting n for d. Then

$$\begin{aligned}
F=\bar{F}&=\sum_{0\le p<D/2}\alpha_p\sum_{k=0}^{p}C_k^p\frac{d!}{(d-2p+2k)!}\frac{n!}{(n-2k)!}\bar{A}_d^{p-k}\bar{A}_n^k ,\\
f_d=\bar{f}_d&=\sum_{0\le p<D/2}\alpha_p\sum_{k=0}^{p}C_k^p\frac{(d-1)!}{(d-1-2p+2k)!}\frac{n!}{(n-2k)!}\bar{A}_d^{p-k}\bar{A}_n^k ,\\
g_d=\bar{g}_d&=2(d-1)\sum_{0\le p<D/2}p\alpha_p\sum_{k=0}^{p-1}C_k^{p-1}\frac{(d-2)!\bar{A}_d^{p-1-k}}{(d-2p+2k)!}\frac{n!\bar{A}_n^k}{(n-2k)!} ,\\
h_d=\bar{h}_d&=2n\sum_{0\le p<D/2}p\alpha_p\sum_{k=0}^{p-1}C_k^{p-1}\frac{(d-1)!\bar{A}_d^{p-1-k}}{(d-2p+1+2k)!}\frac{(n-1)!\bar{A}_n^k}{(n-1-2k)!} ,
\end{aligned}\tag{A4}$$

and $\bar{f}_n,\bar{g}_n,\bar{h}_n$ are obtained by interchanging n and d. When $d=3$ these functions become linear in $\bar{A}_d$:

$$\begin{aligned}
\bar{F}(d=3)&=6\bar{A}_d\left[\sum_{0\le p<D/2}p\alpha_p\frac{n!}{(n+2-2p)!}\bar{A}_n^{p-1}\right]+\sum_{0\le p<D/2}\alpha_p\frac{n!}{(n-2p)!}\bar{A}_n^p ,\\
\bar{f}_d(d=3)&=2\bar{A}_d\left[\sum_{0\le p<D/2}p\alpha_p\frac{n!}{(n+2-2p)!}\bar{A}_n^{p-1}\right]+\sum_{0\le p<D/2}\alpha_p\frac{n!}{(n-2p)!}\bar{A}_n^p ,\\
\bar{f}_n(d=3)&=6\bar{A}_d\left[\sum_{0\le p<D/2}p\alpha_p\frac{(n-1)!}{(n+1-2p)!}\bar{A}_n^{p-1}\right]+\sum_{0\le p<D/2}\alpha_p\frac{(n-1)!}{(n-1-2p)!}\bar{A}_n^p ,\\
\bar{g}_d(d=3)&=4\sum_{0\le p<D/2}p\alpha_p\frac{n!}{(n+2-2p)!}\bar{A}_n^{p-1} ,\\
\bar{g}_n(d=3)&=12(n-1)\bar{A}_d\left[\sum_{0\le p<D/2}p(p-1)\alpha_p\frac{(n-2)!}{(n+2-2p)!}\bar{A}_n^{p-2}\right]\\
&\quad+2(n-1)\sum_{0\le p<D/2}p\alpha_p\frac{(n-2)!}{(n-2p)!}\bar{A}_n^{p-1} ,\\
\bar{h}_d(d=3)&=4n\bar{A}_d\left[\sum_{0\le p<D/2}p(p-1)\alpha_p\frac{(n-1)!}{(n+3-2p)!}\bar{A}_n^{p-2}\right]+2n\sum_{0\le p<D/2}p\alpha_p\frac{(n-1)!}{(n+1-2p)!}\bar{A}_n^{p-1} ,\\
\bar{h}_n(d=3)&=\frac{3}{n}\bar{h}_d(d=3) .
\end{aligned}\tag{A5}$$

(b) The flat subspaces case. $k_d=k_n=0$. Then

$$\begin{aligned}
F=\mathring{F}&=\sum_{0\le p<D/2}\alpha_p\sum_{k=0}^{2p}C_k^{2p}\frac{d!}{(d-2p+k)!}\frac{n!}{(n-k)!}x_d^{2p-k}x_n^k ,\\
f_d=\mathring{f}_d&=\sum_{0\le p<D/2}\alpha_p\sum_{k=0}^{2p}C_k^{2p}\frac{(d-1)!}{(d-1-2p+k)!}\frac{n!}{(n-k)!}x_d^{2p-k}x_n^k ,\\
g_d=\mathring{g}_d&=2(d-1)\sum_{0\le p<D/2}p\alpha_p\sum_{k=0}^{2p-2}C_k^{2p-2}\frac{(d-2)!x_d^{2p-2-k}}{(d-2p+k)!}\frac{n!x_n^k}{(n-k)!} ,\\
h_d=\mathring{h}_d&=2n\sum_{0\le p<D/2}p\alpha_p\sum_{k=0}^{2p-2}C_k^{2p-2}\frac{(d-1)!x_d^{2p-2-k}}{(d-2p+1+k)!}\frac{(n-1)!x_n^k}{(n-1-k)!} ,
\end{aligned}\tag{A6}$$

and $\mathring{f}_n,\mathring{g}_n,\mathring{h}_n$ are obtained by interchanging n and d. When $d=3$ these functions become quadratic in x_d.

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