Aspects of the Zel'dovich-Sunyaev mechanism

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(Received 14 August 1989)

In this paper we treat aspects of the Compton distortion of the cosmic background radiation (CBR)—what is known as the Zel'dovich-Sunyaev mechanism—which do not appear to have been fully explored in the literature. These include a novel solution to the so-called Kompaneets equation, a treatment of the simultaneous heating and cooling of free electrons, a discussion of electron-proton "recombination," and a demonstration of the ingredients necessary for efficient energy transfer. These matters are illustrated in a scenario in which the excess CBR energy observed by Matsumoto *et al.* is supplied by a radiatively decaying heavy lepton, so we also provide a careful analytic treatment of the photon spectrum produced by a heavy particle decaying in the early Universe.

I. INTRODUCTION

Interest in the so-called Zel'dovich-Sunyaev mechanism,[1] which exploits inverse Compton scattering of energetic electrons by the relatively cool cosmic background radiation (CBR) to produce distortions in the CBR, has been rekindled by observations of what may be such distortions. These observations are of two kinds: on the one hand, electrons in the hot intergalactic gas circulating within dense clusters of galaxies can inverse Compton scatter from the universally present CBR. In 1972, Sunyaev and Zel'dovich[2] noted that if the distorted CBR at low frequency, the Rayleigh-Jeans regime, were observed, its effective temperature would be *lowered* as compared to the CBR in other parts of the sky. If a radio telescope is swept across such a galactic cluster, the Rayleigh-Jeans temperature of the CBR will vary as a function of angle, with the maximum drop in temperature occurring when the telescope is pointed at the center of the cluster. This effect has apparently been seen.[3] The maximum temperature distortions are of the order of

$$\Delta T_0 \simeq -10^{-3}\ \mathrm{K}\ . \tag{1.1}$$

It has been noted that the observation of these distortions proves that the CBR is an extra galactic phenomenon, if such proof is needed, and that these observations, in conjunction with other information about these clusters, provide a novel way of measuring the Hubble parameter H_0 (Ref. 4).

On the other hand, a recent measurement[5] using a rocket-borne telescope has found significant distortions in the high-frequency, Wein, end of the spectrum. Within the limitations of this experiment these distortions appear to be isotropic,[6] as they must be if they are to be identified with the CBR. One possible explanation[7] of these results is that some early Universe phenomenon heats the electrons in the primeval electron-proton plasma. These electrons then, by inverse Compton scattering, produce distortions in the CBR. Photons are scattered to higher frequencies. Before equilibrium can be restored the Universe cools, lowering the reaction rates.

Before launching into the formal discussion of the Kompaneets equation, which determines the Zel'dovich-Sunyaev distortions, we close this introduction with a characterization of the regimes in which this analysis is to be done. On the one hand, the intergalactic gas is characterized by two temperatures—the CBR temperature today T_0 and the electron temperature T_e—and by the free electron number density n_e^G:

$$\begin{aligned} &T_0 \simeq 2.7\ \mathrm{K} \simeq 2.3\times 10^{-4}\ \mathrm{eV}\ ,\\ &T_e \simeq 7.7\ \mathrm{keV}\ ,\\ &n_e^G \simeq 3\times 10^{-3}/\mathrm{cm}^3\ . \end{aligned} \tag{1.2}$$

On the other hand, the relevant early-Universe regime for our work will be around the epoch of electron-proton recombination: i.e.,

$$T \simeq \tfrac{1}{3}\ \mathrm{eV}\ ; \tag{1.3}$$

much earlier and any distortions will be erased by inelastic scattering processes such as

$$\gamma + e \rightarrow e + \gamma + \gamma\ ;$$

much later and the Compton reaction rate will be too small to allow for significant distortions. The electron temperature will need to be considerably larger than the CBR temperature in order to produce significant distortions; however, the exact value is model dependent. We can estimate the electron density at the epoch of recombination by assuming that whatever mechanism excites the electrons keeps most of them ionized. We shall show that this is the case for the decaying-particle scenario. Thus, by charge conservation, neglecting helium and other light elements,

$$n_e \simeq n_p \simeq n_B \equiv \eta n_\gamma \ . \tag{1.4}$$

Here n_p is the proton number density, n_B the baryon density, η the baryon-to-photon ratio, and n_γ the photon density. If we take $\eta = 10^{-9}$ then

$$n_e \simeq 10^{-9} \frac{398}{\text{cm}^3} \left[\frac{\frac{1}{3}\ \text{eV}}{2.3\times 10^{-4}\ \text{eV}} \right]^3 , \tag{1.5}$$

where again we have taken the present CBR temperature to be 2.7 K$=2.3\times10^{-4}$ eV. Thus

$$n_e \simeq 10^3/\text{cm}^3 \ . \tag{1.6}$$

To analyze the electron-photon interactions, we will make use of the Boltzmann equations. In order for these transport equations to be valid, the thermal energy of the particles must be greater than their potential energy:

$$T_e \gg \frac{\alpha}{\bar r} \simeq \alpha n_e^{1/3} , \tag{1.7}$$

where $\bar r$ is the mean distance between particles. We readily verify that this condition is satisfied in our regimes by many orders of magnitude. A more sophisticated condition can be derived in terms of the Debye length and it leads to the same result.

II. GENERAL CONSIDERATIONS

All of the work in this subject begins from the so-called Kompaneets equation.[8] This is the equation that determines the distortion in the photon occupation-number distribution per polarization, $f(k,t)$, due to Compton scattering of photons by electrons. We write the equation (in natural units in which $\hbar = c = k_B = 1$) and then describe its origin. Thus,

$$\left[\frac{\partial}{\partial t} - \frac{\dot R}{R} k \frac{\partial}{\partial k}\right] f(k,t) = \frac{n_e \sigma_T}{m_e} \frac{1}{k^2} \frac{\partial}{\partial k} \left[k^4 \left[T_e \frac{\partial}{\partial k} f(k,t) + [1+f(k,t)] f(k,t) \right] \right] . \tag{2.1}$$

The notation is as follows: $R(t)$ is the Robertson-Walker scale factor, so that $\dot R/R$ is the Hubble expansion rate, σ_T is the Thomson cross section,

$$\sigma_T = \frac{8\pi\alpha^2}{3m_e^2} = 0.665\times 10^{-24}\ \text{cm}^2 \ ; \tag{2.2}$$

m_e is the mass of the electron, n_e the free electron number density, and T_e the electron temperature. The most straightforward way to derive this equation[9] is to start from the Boltzmann equation

$$\left[\frac{\partial}{\partial t} - \frac{\dot R}{R} k \frac{\partial}{\partial k}\right] f(k,t) = C_I(k) + \frac{1}{k} \int \frac{d^3k'}{(2\pi)^3} \frac{1}{2k'} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E(p)} \int \frac{d^3p'}{(2\pi)^3} \frac{1}{2E(p')} |M|^2 (2\pi)^4 \delta^4(k+p-k'-p') \times \{ f(k',t) g_e(p',t)[1+f(k,t)] - f(k,t) g_e(p,t)[1+f(k',t)] \} \ . \tag{2.3}$$

Here $|M|^2$ is the matrix element for electron Compton scattering and $g_e(p,t)$ is the electron occupation-number distribution per spin degree of freedom. The collision term $C_I(k)$ represents processes such as photon-proton Compton scattering and inelastic processes such as bremsstrahlung which we can neglect in the regimes of interest. The second term on the left arises from the influence of the cosmic gravitational field and leads to the redshift. This term is irrelevant in the case of intergalactic clusters; however, we will see that both cases can ultimately be treated by the same method.

One then assumes (a) the electrons are nonrelativistic and (b) the photons themselves have energies much less than the electron mass. The energy transfer in any one collision is then constrained by the kinematics to be small. The initial center-of-mass energy squared is given by

$$s_I = [E(p)+k]^2 - (\mathbf{p}+\mathbf{k})^2 \simeq m_e^2 \left[1 + \frac{2k}{m_e} - \frac{2\mathbf{p}\cdot\mathbf{k}}{m_e^2} \right] . \tag{2.4}$$

This must be equal to the final center-of-mass energy squared so that

$$\frac{k}{m_e} - \frac{\mathbf{p}\cdot\mathbf{k}}{m_e^2} \simeq \frac{k'}{m_e} - \frac{\mathbf{p}'\cdot\mathbf{k}'}{m_e^2} \tag{2.5}$$

or

$$k - k' = O(pk/m_e) \ll k \ . \tag{2.6}$$

So the fractional energy transfer is small and one can expand the δ function on the right-hand side of the Boltzmann equation in powers of the small energy transfers. In both cases we will be treating the distributions as isotropic: (i) in the early Universe everything is homogeneous and isotropic and (ii) the intergalactic medium where electron-photon scatterings take place is assumed isotropic. Because of the isotropy of the f's and g's the angular averages in the matrix element can be performed. (A departure from isotropy would lead to additional terms in the Boltzmann equation.[10]) This series of steps leads to Eq. (2.1). Two general remarks are in order about this equation. If we integrate both sides of the equation over all momenta k, we have

$$\frac{1}{R^3}\frac{d}{dt}\left[R^3\int\frac{d^3k}{(2\pi)^3}f(k,t)\right] = \int\frac{d^3k}{(2\pi)^3}\frac{n_e\sigma_T}{m_e}\frac{1}{k^2}\frac{\partial}{\partial k} \times\left[k^4\left[T_e\frac{\partial}{\partial k}f(k,t)+[1+f(k,t)]f(k,t)\right]\right] =0 \tag{2.7}$$

assuming that $f(k,t)$ vanishes rapidly enough as k goes to infinity. This equation expresses the fact that the elastic electron-photon collisions conserve the number of photons.

The second remark concerns the equilibrium limit of Eq. (2.1). The Bose-Einstein distribution

$$f_0=\frac{1}{e^{k/T+\alpha}-1} \tag{2.8}$$

satisfies the equation

$$\left[\frac{\partial}{\partial t}-\frac{\dot{R}}{R}k\frac{\partial}{\partial k}\right]f_0=0 \tag{2.9}$$

provided that (i) $T(t)=\text{const}/R(t)$ and (ii) the chemical potential α is a constant. The distribution f_0 obeys the identity

$$\frac{\partial}{\partial k}f_0=-\frac{1}{T}f_0(1+f_0) . \tag{2.10}$$

If we use this identity on the right-hand side of Eq. (2.1) and set T equal to T_e this side of the equation vanishes also and we have a consistent equilibrium solution.

We are interested in *nonequilibrium* solutions of (2.1), but the departures from f_0 will not be large. Hence we will use the identity (2.10) to simplify (2.1) and write the approximate equation

$$\left[\frac{\partial}{\partial t}-\frac{\dot{R}}{R}k\frac{\partial}{\partial k}\right]f(k,t)=\frac{n_e\sigma_T(T_e-T)}{m_e}\frac{1}{k^2}\frac{\partial}{\partial k} \times\left[k^4\frac{\partial}{\partial k}f(k,t)\right] . \tag{2.11}$$

Some discussion of the meaning of the temperature T is in order. T is roughly the temperature of the CBR, but since the spectrum is not exactly blackbody, it is more accurate to simply say that T is a parameter which characterizes the photon distribution. In the applications we will make, f differs from f_0 by only a few percent. Hence, we can use Eq. (2.10) identifying T with the CBR temperature. In any event $T_e \gg T$ and small errors in treating the nonlinear terms in Eq. (2.11) will not substantially affect the final results.

We can simplify the left-hand side of Eq. (2.11) by making the substitution $\tilde{k}\equiv kR$. Then,

$$\left[\frac{\partial f}{\partial t}\right]_{\tilde{k}}=\left[\frac{\partial f}{\partial t}\right]_k\left[\frac{\partial t}{\partial t}\right]_{\tilde{k}}+\left[\frac{\partial f}{\partial k}\right]_t\left[\frac{\partial k}{\partial t}\right]_{\tilde{k}} =\left[\frac{\partial f}{\partial t}\right]_k-\left[\frac{\partial f}{\partial k}\right]_t\frac{\dot{R}}{R}\frac{\tilde{k}}{R} , \tag{2.12}$$

where the last line here is the left-hand side of (2.11). The right-hand side of (2.11) is unaffected by this substitution. Therefore, rewriting $\tilde{k}$ as k, we may write Eq. (2.11) in the form

$$\frac{\partial}{\partial t}f(k,t)=\frac{n_e\sigma_T(T_e-T)}{m_e}\frac{1}{k^2}\frac{\partial}{\partial k}\left[k^4\frac{\partial}{\partial k}f(k,t)\right] . \tag{2.13}$$

The $\dot{R}/R$ term has scaled out of this equation so that its solution is applicable to both the intergalactic and the recombination-era distortions.

To solve Eq. (2.13), we follow Zel'dovich and Sunyaev[1] and introduce the dimensionless variable y by the relation

$$dy=\frac{n_e\sigma_T(T_e-T)}{m_e}dt . \tag{2.14}$$

In the cosmological example we determine y at any time t by integrating (2.14) from some arbitrary initial time which we can call zero: that is,

$$y(t)=\int_0^t\frac{n_e\sigma_T(T_e-T)}{m_e}dt' . \tag{2.15}$$

Later in the paper we shall study the physics of $y(t)$. Here we shall simply take it as a parameter that characterizes the solution to Eq. (2.13). In the galactic gas example the integration over time is replaced by an integration along a given line of sight through the galactic cluster. The time of expansion is replaced by the distance traversed through the medium. Thus, in both cases, Eq. (2.13) can be written

$$\frac{\partial}{\partial y}f(x,y)=\frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\frac{\partial f(x,y)}{\partial x}\right] , \tag{2.16}$$

where we have introduced the dimensionless variable

$$x\equiv\frac{k}{T_0} . \tag{2.17}$$

Here T_0 is roughly the temperature of the CBR today, keeping in mind that the photon distribution is to be fit with two free parameters: T_0 and y. Equation (2.16) has the formal solution

$$f(x,y)=\exp\left[\frac{y}{x^2}\frac{\partial}{\partial x}x^4\frac{\partial}{\partial x}\right]f(x,0) . \tag{2.18}$$

In our applications we can take

$$f(x,0)=\frac{1}{e^x-1} \tag{2.19}$$

since in the absence of distortions ($y=0$), the cosmic photon spectrum is pure blackbody.

If $x^2y<1$ we can expand the exponential in (2.18) (Ref. 11): that is,

$$f(x,y)\simeq f(x,0)+\frac{y}{x^2}\frac{\partial}{\partial x}\left[x^4\frac{\partial}{\partial x}f(x,0)\right] . \tag{2.20}$$

To reduce (2.20) to a more useful form we can employ the differential identity

$$\frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\frac{\partial}{\partial x}\frac{1}{e^x-1}\right]=\frac{xe^x}{(e^x-1)^2}\left[\frac{x}{\tanh(x/2)}-4\right] . \tag{2.21}$$

Thus,[12] for $x^2y<1$,

$$f(x,y)\simeq\frac{1}{e^x-1}\left[1+\frac{xye^x}{e^x-1}\left[\frac{x}{\tanh(x/2)}-4\right]\right] . \tag{2.22}$$

When one compares to experiment it is often useful to use $f(x,y)$ to compute the so-called "effective blackbody temperature." This is a frequency-dependent temperature $T(x)$ defined so that

$$\frac{T(x)_y}{T_0}=\frac{x}{\ln[1+f(x,y)^{-1}]} . \tag{2.23}$$

Note that for $f(x,0)$ given by Eq. (2.19), the blackbody,

$$\frac{T(x)_0}{T_0}=1 . \tag{2.24}$$

Hence Eq. (2.23) is a useful way of characterizing departures from the blackbody spectrum. For the approximate $f(x,y)$ given by Eq. (2.22),

$$\frac{T(x)_y}{T_0}\simeq 1+y\left[\frac{x}{\tanh(x/2)}-4\right] . \tag{2.25}$$

We shall shortly explore the limits on the validity of this expression. However, it is surely valid as $x\to 0$, the Rayleigh-Jeans limit. In this limit

$$\frac{T(x)_y}{T_0}\to 1-2y \tag{2.26}$$

so that

$$\frac{\Delta T}{T_0}\simeq -2y . \tag{2.27}$$

This is the effect apparently observed in the galactic clusters. Indeed if we use the value of $\Delta T/T_0$ given by Eq. (1.1) we see that it corresponds to a y of about 5×10^{-4}. This y for the cluster is usually written[3] in the form

$$y=\frac{T_e\tau}{m_e} , \tag{2.28}$$

where τ is the optical depth

$$\tau=\int n_e\sigma_T dl \tag{2.29}$$

and the integral is taken along the line of sight. If we use Eq. (1.1) we find an optical depth τ of about 0.003.

When the condition $x^2y<1$ is not satisfied we must solve Eq. (2.16) exactly. Several solutions have been presented in the literature.[13] We wish to present a novel solution worked out by one of us (J.B.) in collaboration with Brown.[14] To this end, we call D the differential operator:

$$D\equiv x\frac{\partial}{\partial x} . \tag{2.30}$$

We can write

$$f(x,y)=e^{y(D^2+3D)}f(x,0) . \tag{2.31}$$

We notice that

$$\sqrt{\pi}=\int_{-\infty}^{\infty}ds\, e^{-(s+\sqrt{y}D)^2} . \tag{2.32}$$

Thus

$$e^{yD^2}=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}ds\, e^{-s^2-2D\sqrt{y}s} . \tag{2.33}$$

Therefore,

$$\begin{aligned}f(x,y)&=\frac{e^{3yD}}{\sqrt{\pi}}\int_{-\infty}^{\infty}ds\, e^{-s^2-2D\sqrt{y}s}f(x,0)\\&=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}ds\, e^{-s^2}e^{(3y-2\sqrt{y}s)D}f(x,0) .\end{aligned} \tag{2.34}$$

We may now use the well-known operator identity,[15] with λ a number,

$$e^{\lambda D}f(x)e^{-\lambda D}=f(e^{\lambda}x) . \tag{2.35}$$

The $e^{-\lambda D}$ factor goes to 1 since the derivative has nothing to act on. Hence,

$$f(x,y)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}ds\, e^{-s^2}f(e^{-2s\sqrt{y}+3y}x,0) , \tag{2.36}$$

where, to be explicit about the notation,

$$f(e^{-2s\sqrt{y}+3y}x,0)=\frac{1}{\exp(xe^{-2s\sqrt{y}+3y})-1} .$$

It is clear that for $y=0$, Eq. (2.36) reduces to the Planck distribution. Before exploiting this solution it is instructive to relate it to others in the literature.[13] To relate Eq. (2.36) to the *correct* version of the expression given in Zel'dovich and Novikov[1] we make the substitution

$$z=xe^{-2s\sqrt{y}+3y}$$

in Eq. (2.36). This leads at once to

$$f(x,y)=\frac{1}{\sqrt{4\pi y}}\int_0^{\infty}\frac{dz}{z}\exp\left[-\frac{[3y+\ln(x/z)]^2}{4y}\right]\times f(z,0) . \tag{2.37}$$

This expression should replace Eq. (8.7.3) in Zel'dovich and Novikov.[1] To derive the expression of Chan and Jones[13] make the substitution

$$z=x\frac{T_0}{T}$$

in Eq. (2.37). Thus

$$f(x,y)=\frac{1}{\sqrt{4\pi y}}\int_0^\infty \frac{dT}{T_0}\exp\left[4y-\frac{1}{4y}[\ln(T/T_0)+5y]^2\right] \times f(xT_0/T,0)\ , \tag{2.38}$$

which is given in Chan and Jones.

Any of these expressions can be used for numerical work. However, Eq. (2.36) is especially well suited to compute the moments of the distribution $f(x,y)$. Hence consider[16]

$$P_n(y)\equiv\int_0^\infty dx\, x^n f(x,y) = \int_0^\infty dx\, x^n \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} ds\, e^{-s^2} f(e^{-2s\sqrt{y}+3y}x,0)\ . \tag{2.39}$$

A short calculation then shows that

$$P_n(y)=e^{(n+1)(n-2)y}P_n(0)\ . \tag{2.40}$$

We see that, for $n=2$,

$$P_2(y)=P_2(0)\ . \tag{2.41}$$

This is the statement that the elastic Compton scatterings preserve the number of photons. For $n=3$ we have the statement

$$P_3(y)=e^{4y}P_3(0)\ . \tag{2.42}$$

This is the well-known[17] connection between the modified energy density $\rho(y)$ and the blackbody density $\rho(0)$. To make this more explicit, suppose that $\rho(0)$ is defined in terms of the temperature T_0: that is,

$$\rho(0)=\sigma T_0^4\ . \tag{2.43}$$

Thus,

$$\rho(y)=e^{4y}\sigma T_0^4\ . \tag{2.44}$$

In principle, if the observed spectrum is described by Eq. (2.36) we can go to the limit where x goes to zero to measure the Rayleigh-Jeans temperature $T_{\rm RJ}$. From Eq. (2.38) we see that

$$\lim_{x\to 0} f(x,y)=\frac{e^{-2y}}{x} \tag{2.45}$$

so that

$$\lim_{x\to 0}\frac{T(x)_y}{T_0}\equiv\frac{T_{\rm RJ}}{T_0}=e^{-2y}\ . \tag{2.46}$$

That is, the effective blackbody temperature in the Rayleigh-Jeans regime is frequency independent. Hence if we express $\rho(y)$ in terms of $T_{\rm RJ}$, we find

$$\rho(y)=e^{12y}\sigma T_{\rm RJ}^4\ . \tag{2.47}$$

We shall end this section by giving our best fits to the data in terms of a plot of $T(x)_y$. Before doing this we want to indicate that the data[5] support an interpretation in terms of a Zel'dovich-Sunyaev mechanism with $y \ll 1$. To this end we note that with a base temperature T_0 given by 2.74 K,

$$\rho(y=0)\simeq 0.28\frac{\rm eV}{\rm cm^3}\ . \tag{2.48}$$

The new data points[5] indicate an excess over $\rho(0)$ of about $\Delta\rho\simeq 3\times 10^{-2}$ eV/cm^3. Thus,

$$\frac{\rho(y)}{\rho(0)}\simeq 1+4y\simeq 1.1\ . \tag{2.49}$$

Solving for y we have $y\simeq 0.025$ which means that our approximations, specifically the use of the identity (2.10) on the right-hand side of the Boltzmann equation, are self-consistent.

Below we present two curves for $T(x)_y$ (Figs. 1 and 2). The purpose of the first curve is pedagogical. It is to show that using the approximate form given by Eq. (2.25) is seriously wrong for $x\sim 10$, the submillimeter regime. This form should not be used to fit $T(x)_y$. The second curve is our best fit to $T(x)_y$ using Eq. (2.36). We find that the best fit is given by $y=0.021$, $T_0=2.81$ K. It must be emphasized that this fit involves only the Zel'dovich-Sunyaev mechanism and ignores inelastic processes such as bremsstrahlung which could modify the curve for $x<1$. In any event, finding a y and a T_0 that fits the data does not solve the real physics problem: namely, to find a plausible mechanism for generating such y's. In the following sections we explore the ingredients that must go into such a mechanism.

III. ELECTRON TEMPERATURE

We have seen that the parameter y determines the shape of the cosmic-background spectrum. To calculate

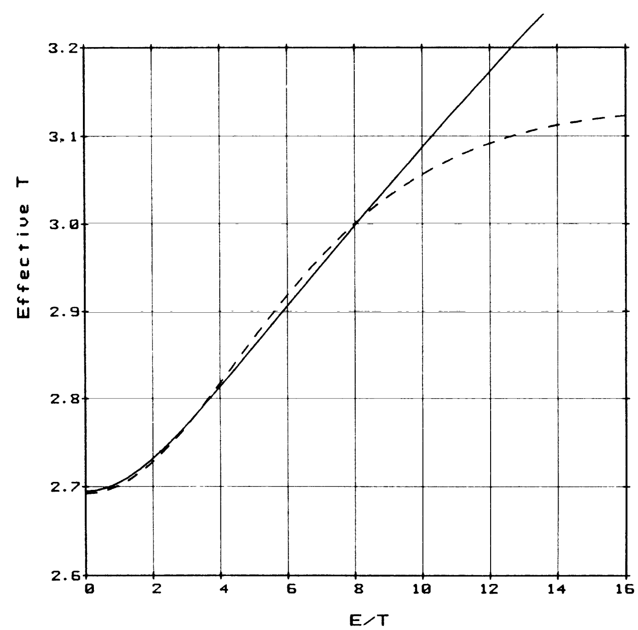

FIG. 1. Effective temperature arising from the Zel'dovich-Sunyaev mechanism with $y=0.02$ and $T_0=2.81$. The solid line is the exact solution, Eq. (2.36) and the dashed line is the approximate solution (2.25). The x axis is the frequency of the photons, normalized to T_0 so that, for example, the new submillimeter points are at $x=4.4$, 7.2, and 10.6 (frequencies $\nu=259$, 423, and 624 GHz).

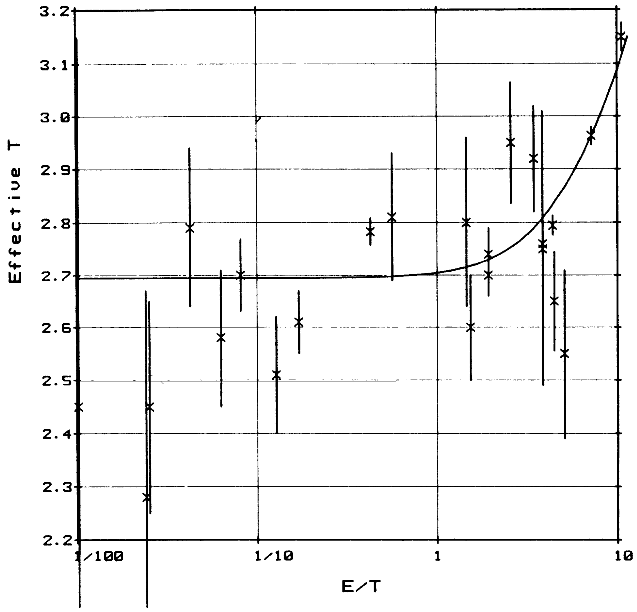

FIG. 2. The Zel'dovich-Sunyaev best fit to 21 data points (in Ref. 18) from recent measurements. Here $T_0=2.81$ K and $y=0.021$; the χ^2 for the 21 data points is 44.

y, we need to know the temperature difference between electrons and photons, T_e-T, at all times. In this section, we will write down and solve the evolution equation which governs the electron temperature.

Free electrons interact with other free electrons and with free protons via Coulomb scattering; they also inverse Compton scatter from photons. The full evolution equation contains collision terms representing each of these interactions:

$$\left[\frac{\partial}{\partial t}-\frac{\dot{R}}{R}p\frac{\partial}{\partial p}\right]g_e(p,t)=C_{e\gamma}(p,t)+C_{ee}(p,t)+C_{eP}(p,t)\ , \tag{3.1}$$

where $g_e(p,t)$ is the electron number distribution per spin degree of freedom. In the limit of small velocities, the rate for Coulomb scattering is very large. The only solutions to (3.1) therefore are those distributions which make the Coulomb collision terms vanish: the electrons and protons are both constrained to have Maxwellian distributions at a common temperature T_e. If we now multiply (3.1) by the kinetic energy $(p^2/2m_e)$ and integrate over all momenta, since the Coulomb collision terms vanish with Maxwellian distributions, we are left with

$$R^{-5}\frac{d}{dt}(\tfrac{3}{2}n_eT_eR^5)=2\int\frac{d^3p}{(2\pi)^3}C_{e\gamma}(p,t)\frac{p^2}{2m_e}\ . \tag{3.2}$$

The factor of 2 on the right-hand side of (3.2) reflects the two spins in n_e. As long as most electrons are free (an issue we will return to later), $n_e\simeq n_B$, the baryon-number density. Since baryon number is strictly conserved, n_B, and therefore n_e, falls off strictly as R^{-3}. Therefore,

$$R^{-2}\frac{d}{dt}(T_eR^2)=\frac{4}{3n_e}\int\frac{d^3p}{(2\pi)^3}C_{e\gamma}(p,t)\frac{p^2}{2m_e}\ . \tag{3.3}$$

The collision term $C_{e\gamma}(p,t)$ depends upon the photon distribution. Under the same set of assumptions that were used to derive the Kompaneets equation, $C_{e\gamma}(p,t)$ can be simplified and expressed in terms of the moments of the photon distribution. Specifically, defining

$$\rho_\gamma(t)\equiv2\int\frac{d^3k}{(2\pi)^3}kf_\gamma(k,t) \tag{3.4}$$

and

$$T_\gamma(t)\equiv\frac{1}{2\rho_\gamma}\int\frac{d^3k}{(2\pi)^3}k^2f_\gamma(k,t)[1+f_\gamma(k,t)]\ , \tag{3.5}$$

it can be shown[19] that

$$C_{e\gamma}(p,t)=\frac{2\rho_\gamma\sigma_T}{3m_ep^2}\frac{\partial}{\partial p}\left[p^3\left[g_e(p,t)+T_\gamma\frac{m_e}{p}\frac{\partial}{\partial p}g_e(p,t)\right]\right]\ . \tag{3.6}$$

Note that ρ_γ and T_γ are defined for any photon distribution. If the photons happen to be in equilibrium with a temperature T, then it is straightforward to check that T_γ as defined is equal to T. Now we integrate the right-hand side of Eq. (3.3) by parts and write the evolution equation for the electron temperature as

$$\dot{T}_e+2\frac{\dot{R}}{R}T_e=\frac{4}{3}\frac{\rho_\gamma\sigma_T}{m_e}(T_\gamma-T_e)\ . \tag{3.7}$$

Again we emphasize that, although Eq. (3.7) resembles the equation for electron cooling that is often used, it is more general as it determines the electron temperature for electrons in contact with *any* photon distribution, not just an equilibrium one.

As an example, let us consider the standard picture of the early Universe, wherein the only photons are those in equilibrium at a temperature T. Then the solution to Eq. (3.7) is

$$T_e(t)=T_e(t_i)\left[\frac{R(t_i)}{R(t)}\right]^2\exp\left[-\int_{t_i}^{t}\frac{dt'}{\tau_C(t')}\right]+\int_{t_i}^{t}\frac{dt'}{\tau_C(t')}\left[\frac{R(t')}{R(t)}\right]^2\exp\left[-\int_{t'}^{t}\frac{dt''}{\tau_C(t'')}\right]T(t')\ , \tag{3.8}$$

where t_i is some early time and the Compton time is defined as

$$\frac{1}{\tau_C}\equiv\frac{4}{3}\frac{\rho_\gamma\sigma_T}{m_e}\ . \tag{3.9}$$

There are two limits to consider: (i) the Compton time $\tau_C\gg(\dot{R}/R)^{-1}$, so there is very little energy transfer between the electrons and the photons; in this limit the first term on the right side of Eq. (3.8) dominates and we retrieve the familiar result that the electron temperature falls off as R^{-2}; (ii) if $\tau_C\ll(\dot{R}/R)^{-1}$ then the photons can transfer some of their energy to the electrons. Integrating by parts, we find

$$R^2(t)T_e(t)\simeq R^2(t)T(t)-\tau_C\frac{\partial}{\partial t}[R^2(t)T(t)]\ . \tag{3.10}$$

The second term on the right here is smaller than the first by a factor of order $\dot{R}/R\tau_C$ and the additional ones we have not written are smaller still, so we find

$$T_e \simeq T \quad \text{if } \tau_C \frac{\dot{R}}{R} \ll 1 \ . \tag{3.11}$$

When the ambient temperature is above 4×10^{-3} eV, this condition is satisfied. Hence throughout this epoch, free electrons have the same temperature as the blackbody photons unless some additional energy source is available to heat them. This means that in the standard picture of the early Universe, with such sources absent, the electrons and photons have the same temperature and therefore $y=0$.

The Zel'dovich-Sunyaev mechanism is a two-step process in which a source of energetic photons energizes free electrons which, in turn, Compton scatter from the CBR photons. Hence we may break up the photon distribution into two parts:

$$f_\gamma(k)=f_{\rm CBR}(k)+f_H(k) \ , \tag{3.12}$$

where $f_{\rm CBR}(k)$ is the blackbody distribution with temperature T and $f_H(k)$ is the distribution of heated photons. The arguments that lead to the equilibration of the photon and electron temperatures apply here if we understand that the photon "temperature" T_γ is given by Eq. (3.5) where we use the sum of the two photon distributions to evaluate the integral. Thus,

$$T_\gamma=\frac{1}{2\rho_\gamma}\int\frac{d^3k}{(2\pi)^3}k^2[f_{\rm CBR}(k)+f_H(k)] \times[1+f_{\rm CBR}(k)+f_H(k)] \ . \tag{3.13}$$

Since the total photon energy density ρ_γ is the sum of the energy in the two distributions: $\rho_\gamma=\rho_{\rm CBR}+\rho_H$, we can write

$$\begin{aligned} T_\gamma&=\frac{1}{2(\rho_{\rm CBR}+\rho_H)}\int\frac{d^3k}{(2\pi)^3}k^2\{f_{\rm CBR}(k)[1+f_{\rm CBR}(k)]+f_H(k)[1+f_H(k)]+2f_{\rm CBR}(k)f_H(k)\} \ . \\ &=\frac{1}{\rho_{\rm CBR}+\rho_H}(\rho_{\rm CBR}T+\rho_H T_H+I) \ . \end{aligned} \tag{3.14}$$

Here I is the interference term

$$I\equiv\int\frac{d^3k}{(2\pi)^3}k^2 f_{\rm CBR}(k)f_H(k) \ . \tag{3.14a}$$

We will see that this interference term is much smaller than the other terms in equation (3.14) so we shall neglect it. Since

$$T_e\simeq T_\gamma$$

we can write, neglecting I,

$$T_e-T\simeq T_\gamma-T=\frac{\rho_H}{\rho_{\rm CBR}+\rho_H}(T_H-T) \ . \tag{3.15}$$

Equation (3.15) is exactly what we need to compute y: an expression for the temperature difference between electrons and CBR photons. We note that if the additional photons all have energies less than T, so that $T_H<T$, then the electron temperature is *not* raised above the blackbody photon temperature. Therefore, dumping many low-energy photons into the Universe will not create the sort of distortions needed to account for observations.[20]

It is convenient to switch from time to temperature variables. To this end we use the approximate entropy conservation condition, RT=const. Photon production in neutrino decays causes only a small amount of entropy production. Hence, we now have an expression for y solely in terms of the photon distributions

$$y=\int_{T_0}^{\infty}\frac{dT}{T}\frac{\rho_H}{\rho_{\rm CBR}}\epsilon \ . \tag{3.16}$$

Thus y is proportional to the efficiency[21] ϵ of converting energy from the hot to the cold photons and to the available excess energy ρ_H. The efficiency ϵ is defined to be

$$\begin{aligned} \epsilon&\equiv\frac{n_e\sigma_T(T_H-T)}{m_e\frac{\dot{R}}{R}}\frac{\rho_{\rm CBR}}{\rho_{\rm CBR}+\rho_H} \\ &\simeq\frac{n_e\sigma_T}{m_e\frac{\dot{R}}{R}}T_H \ , \end{aligned} \tag{3.17}$$

where n_e, T_H, $\dot{R}/R$, $\rho_{\rm CBR}$, and ρ_H are functions of T. To get the approximate equality in Eq. (3.17), we assume that the energy density of the hot photons is less than the background energy density and that the hot photons are much more energetic than those in the background.

Before going on to discuss the hot photon spectrum, we mention that the efficiency decreases as time goes on. The electron number density falls off as R^{-3}, while the Hubble rate drops (at most) as R^{-2}. Therefore, to get a given value of y from electron heating at late times requires more energy (relative to the background energy) than at early times.[22]

IV. RECOMBINATION

In order to calculate y, we need to know the free electron number density n_e. This is a trivial problem at high temperatures ($T>\epsilon_0\equiv13.6$ eV), since there are enough high-energy photons around to ensure that the electron-proton plasma is fully ionized. The problem becomes more difficult as the temperature drops and fewer photons with energy greater than ϵ_0 exist. In the problem we are analyzing there is a new wrinkle: there are additional photons besides those in equilibrium at a temperature T. Therefore, in this section we will analyze the "recombination" problem for an arbitrary photon distribution.

The evolution equation which determines the abundance of hydrogen atoms is

$$R^{-3}\frac{d}{dt}(n_H R^3) = 4\int \frac{d^3p}{(2\pi)^3} g_e(p) \int \frac{d^3p'}{(2\pi)^3} g_P(p')(\sigma_{eP\to H\gamma} v) - 4\int \frac{d^3k}{(2\pi)^3} f_\gamma(k) \int \frac{d^3k'}{(2\pi)^3} g_H(k')\sigma_{H\gamma\to eP} , \quad (4.1)$$

where n_H is the hydrogen number density and the distribution functions $g_e(p)$, $g_P(p')$, $f_\gamma(k)$, and $g_H(k')$ are for free electrons, free protons, photons, and hydrogen atoms, respectively. The factors of 4 here arise since the hydrogen atom has four spin states. Strictly speaking, there should be a $[1+f_\gamma(k)]$ factor in the term representing hydrogen production above. This factor can be set to one here though, since $f_\gamma(k)$ is always much less than one in the regions of interest.

The cross sections are independent of the momenta of the free protons and the hydrogen atoms, so the k' and p' integrals can be evaluated leading to a right-hand side

$$2n_P \int \frac{d^3p}{(2\pi)^3} g_e(p)(\sigma_{eP\to H\gamma} v) - n_H \int \frac{d^3k}{(2\pi)^3} f_\gamma(k)\sigma_{H\gamma\to eP} . \quad (4.2)$$

We are considering only transitions between the ground state of hydrogen and the continuum, since capture to other states is slower, and even if such captures occur the excited states quickly decay to the ground state.

To reduce this equation further, we use the following pieces of information.

(i) If we neglect the small fraction of helium[23] atoms, the number of free electrons is equal to the number of free protons. The sum of either of these with the number of hydrogen atoms is just the baryon number of the Universe:

$$n_e \simeq n_P \simeq n_B - n_H . \quad (4.3)$$

Within an order of magnitude, the baryon-number density is known today to be about $n_B \simeq 10^{-9} n_\gamma$. If we define the ratio r as

$$r \equiv \frac{n_H}{n_B} \quad (4.4)$$

then Eq. (4.3) says that

$$n_e \simeq n_P \simeq n_B(1-r) . \quad (4.5)$$

(ii) Total baryon number is conserved, so

$$n_B R^3 = \text{const} . \quad (4.6)$$

(iii) Electrons at temperature T_e and number density n_e have the Maxwellian distribution function

$$g_e(p) = \frac{n_e}{2}\left[\frac{2\pi}{m_e T_e}\right]^{3/2} e^{-p^2/2m_e T_e} . \quad (4.7)$$

(iv) Each of the cross sections is well known:[24]

$$\sigma_{H\gamma\to eP} = \frac{64\pi}{3\sqrt{3}}\alpha a_0^2 \left[\frac{\epsilon_0}{k}\right]^3 g(k) \equiv \sigma_I \left[\frac{\epsilon_0}{k}\right]^3 g(k) , \quad (4.8)$$

where a_0 is the Bohr radius and $g(k)$ is the gaunt factor, not to be confused, by virtue of its lack of subscript, with any of the distribution functions. The gaunt factor is a slowly varying function of order unity which contains some of the energy dependence in the cross section.[24] Here we have defined the ionization cross section σ_I; it is 7.9×10^{-18} cm^2. To get the inverse cross section, we use detailed balance

$$\sigma_{eP\to H\gamma} = \sigma_{H\gamma\to eP}\left[\frac{k}{p}\right]^2 = \sigma_{H\gamma\to eP}\left[\frac{\epsilon_0 + p^2/2m_e}{p}\right]^2 , \quad (4.9)$$

where the second equality holds by conservation of energy. Since the electron velocity v is p/m_e, we find

$$(\sigma_{eP\to H\gamma} v) = \sigma_I g\left[\epsilon_0 + \frac{p^2}{2m_e}\right]\frac{\epsilon_0^2}{pm_e}\frac{1}{1+\dfrac{p^2}{2m_e\epsilon_0}} . \quad (4.10)$$

The first fact we use is (ii). Armed with the definition of the hydrogen ratio, Eq. (4.4), we use (ii) to write the left-hand side of Eq. (4.1) as $n_B dr/dt$. Then using (i) and (iii), we get

$$\frac{dr}{dt} = n_B(1-r)^2\left[\frac{2\pi}{m_e T_e}\right]^{3/2} \times \int \frac{d^3p}{(2\pi)^3} e^{-p^2/2m_e T_e}(\sigma_{eP\to H\gamma} v) - r\int \frac{d^3k}{(2\pi)^3} f_\gamma(k)\sigma_{H\gamma\to eP} . \quad (4.11)$$

Performing the angular integrations and inserting the expressions for the cross sections (iv) leaves

$$\frac{dr}{dt} = \frac{n_B\sigma_I(1-r)^2}{\sqrt{\pi m_e/2}\,T_e^{3/2}}\left[\frac{\epsilon_0}{m_e}\right]^2 \times \int_0^\infty \frac{dp\, pg(\epsilon_0 + p^2/2m_e)e^{-p^2/2m_e T_e}}{1+p^2/2m_e\epsilon_0} - \frac{r\sigma_I\epsilon_0^3}{2\pi^2}\int \frac{dk}{k} f_\gamma(k) g(k) . \quad (4.12)$$

This equation can be written more compactly if we make two further definitions. Define the average gaunt factor as

$$\bar{g} \equiv \int_0^\infty \frac{dx}{1+T_e x/\epsilon_0} e^{-x} g(\epsilon_0 + T_e x) \tag{4.13}$$

and define

$$\tilde{n}_\gamma \equiv \frac{\epsilon_0^3}{2\pi^2} \int_{\epsilon_0}^\infty \frac{dk}{k} f_\gamma(k) g(k) \ . \tag{4.14}$$

Finally we have

$$\frac{dr}{dt} = \tilde{n}_\gamma \sigma_I \left[\frac{n_B}{\tilde{n}_\gamma} \frac{\epsilon_0^2 \bar{g}}{\sqrt{\pi T_e / 2 m_e^{3/2}}} (1-r)^2 - r \right] . \tag{4.15}$$

We cannot solve Eq. (4.15) without knowing the time dependence of $\tilde{n}_\gamma$, T_e, and $\bar{g}$. However, we can make the qualitative statement that as long as the ionization rate $\tilde{n}_\gamma \sigma_I$ is greater than the expansion rate, the system is near equilibrium so that the terms in brackets cancel each other, thereby determining r via

$$r \simeq \frac{n_B}{\tilde{n}_\gamma} \frac{\epsilon_0^2 \bar{g}}{\sqrt{\pi T_e / 2 m_e^{3/2}}} (1-r)^2 \ \text{ if } \tilde{n}_\gamma \sigma_I > \frac{\dot{R}}{R} \ . \tag{4.16}$$

To close this section, we use Eq. (4.16) to analyze recombination in the standard cosmological model, where the only photons are those with temperature T. Then for $T < \epsilon_0$, the $f_\gamma(k)$ which determines $\tilde{n}_\gamma$ is simply $e^{-k/T}$. If we change variables in the integral equation (4.14) to $x \equiv (\epsilon_0 - k)/T$, then—recalling that in the standard model the electron temperature T_e is equal to T—we find that $\tilde{n}_\gamma$ can be simply expressed in terms of $\bar{g}$:

$$\tilde{n}_\gamma \simeq \frac{\epsilon_0^2 T}{2\pi^2} \bar{g} e^{-\epsilon_0/T} \ . \tag{4.17}$$

We assume that the Universe is flat and matter dominated now so that $\dot{R}/R = (\dot{R}/R)_0 (T/T_0)^{3/2}$. Then

$$\frac{\tilde{n}_\gamma \sigma_I}{\dfrac{\dot{R}}{R}} \simeq 5 \times 10^{20} \left[\frac{1\ \text{eV}}{T} \right]^{1/2} e^{-\epsilon_0/T} \tag{4.18}$$

so that the two rates are equal when $T = 0.28$ eV; hence Eq. (4.16) holds above this temperature. Substituting for r in Eq. (4.16) then leads to

$$n_H = n_e n_P e^{\epsilon_0/T} \left[\frac{2\pi}{m_e T} \right]^{3/2} . \tag{4.19}$$

This is the familiar Saha equation. The significance of Eq. (4.16) is now clearer: it is the generalized Saha equation which can be applied to any photon distribution, not just the standard blackbody. Further, the condition $\tilde{n}_\gamma \sigma_I > \dot{R}/R$ tells us when the generalized Saha equation is valid. We can use the Saha equation to determine the temperature at which the number of free protons is equal to the number of bound hydrogen atoms. Using $n_B = 10^{-9} n_\gamma$ leads to

$$n_H \simeq \frac{n_B}{2} \simeq n_e \simeq n_P \ \text{ at } T = 0.34 \text{ eV} \tag{4.20}$$

in the standard model. We will see that the recombination temperature, as might be defined by Eq. (4.20), is often much lower when there are additional photons in the early Universe.

V. DECAY PHOTON SPECTRUM

The most interesting feature of the microwave distortions is the new physics they may teach us. In the standard cosmology there is no other source of photons that might heat up the electrons. The search for this source inevitably leads to new physics. Here we will quantitatively examine one proposal: massive radiatively decaying neutrinos. The neutrino is perhaps the most natural candidate for providing the extra energy. First, it exists. Second, standard cosmology predicts that there are about as many neutrinos in the Universe as there are photons. If neutrinos have mass—still an open question experimentally—and if they decay at least partially into photons, they will indeed deposit a large amount of energy into the electron-photon plasma of the early Universe.

The scenario, initially proposed by Fukugita[25] for example, then is the following: one species of neutrino, ν_H, has mass m_ν and decays with a lifetime τ into

$$\nu_H \to \bar{\nu}_L \nu_L \nu_L \tag{5.1}$$

or

$$\nu_H \to \nu_L X \ , \tag{5.1a}$$

where X is a weakly interacting particle and with all the light species effectively massless. There is also a small branching ratio B into photons

$$\nu_H \to \nu_L \gamma \ . \tag{5.2}$$

If the neutrinos decay at rest, the photon has energy $m_\nu/2$ when produced and then loses energy due to the cosmological redshift. Our task in this section is to calculate the spectrum of photons produced.[26] We will make two approximations to simplify the calculations. First, we will assume that the heavy neutrino is at rest when it decays. The momentum of a neutrino with a mass smaller than the decoupling temperature (~ 1 MeV) is roughly equal to the neutrino temperature T_ν. But within a factor of 2 the neutrino temperature is equal to the cosmic temperature T. Therefore, if the temperature at the time of decay, $T(\tau)$, is much smaller than m_ν, then the heavy neutrino velocities will be of order[27] $p/m_\nu \sim T(\tau)/m_\nu$; they will be small. We want the decay photons to heat the electrons, so $m_\nu/2$, the energy of a decay photon, must be greater than the typical thermal energies $\sim T(\tau)$. We therefore expect this first approximation to be valid. The second approximation we will make here is to neglect the way the decay photon spectrum is distorted by interactions with electrons. In the next section, however, these interactions will be considered.

The starting point is the Boltzmann equation for the distribution of decay photons. Recalling Eq. (3.12), we call this distribution function $f_H(k,t)$. The Boltzmann equation is

$$\left[\frac{\partial}{\partial t}-\frac{\dot{R}}{R}k\frac{\partial}{\partial k}\right]f_H(k,t)$$
$$=C_D(k,t)$$
$$=\frac{1}{2k}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2p}\int\frac{d^3q}{(2\pi)^3}\frac{F(q,t)}{2E(q)}(2\pi)^4$$
$$\times\delta^4(q-p-k)|M|^2\ , \quad (5.3)$$

where $F(q,t)$ is the distribution of massive neutrinos and M is the amplitude for decay into (ν_L,γ). $|M|^2$ is a relativistic invariant; as such it depends on the only invariant that can be formed from the three momenta, $(p+k)^2=q^2=m_\nu^2$. Therefore, it is a constant and may be taken outside the integrals. After performing the p integration, we find the decay term to be

$$C_D(k,t)=\frac{\pi|M|^2}{k}\int\frac{d^3q}{(2\pi)^3}\frac{F(q,t)}{2E(q)}\frac{\delta(E(q)-|\mathbf{q}-\mathbf{k}|-k)}{2|\mathbf{q}-\mathbf{k}|}\ . \quad (5.4)$$

We know that $F(q)$ constrains q to be of order T, so if $m_\nu \gg T$, we have the following simplifications:

$$\delta[E(q)-|\mathbf{q}-\mathbf{k}|-k]\rightarrow\tfrac{1}{2}\delta(k-m_\nu/2)\ ,$$
$$2E(q)2|\mathbf{q}-\mathbf{k}|\rightarrow 4m_\nu k\ . \quad (5.5)$$

These lead to

$$C_D(k,t)=\frac{B\pi^2}{\tau k^2}\delta(k-m_\nu/2)\left[2\int\frac{d^3q}{(2\pi)^3}F(q,t)\right] \quad (5.6)$$

with the identification of the radiative lifetime as

$$\tau_\gamma\equiv\frac{\tau}{B}\equiv\frac{16\pi m_\nu}{|M|^2}\ . \quad (5.7)$$

The term in large parentheses in Eq. (5.6) is the heavy neutrino number density, with the factor of 2 accounting for antineutrinos. Ordinarily we could just replace this with[28] $n_\nu=(3/11)n_\gamma$. Here, because of decays there is an additional exponential factor, so we write

$$2\int\frac{d^3q}{(2\pi)^3}F(q,t)=n_\nu(t)e^{-t/\tau}$$

with the understanding that n_ν represents the familar neutrino number density in the absence of decays. Since RT is a constant, if we assume that the Universe is radiation dominated,[29] T falls off as $t^{-1/2}$; therefore, $n_\nu=n_\nu(\tau)(\tau/t)^{3/2}$. The Boltzmann equation becomes

$$\left[\frac{\partial}{\partial t}-\frac{\dot{R}}{R}k\frac{\partial}{\partial k}\right]f_H(k,t)$$
$$=\frac{Bn_\nu(\tau)\pi^2}{\tau k^2}\left[\frac{\tau}{t}\right]^{3/2}e^{-t/\tau}\delta(k-m_\nu/2)\ . \quad (5.8)$$

To solve this partial differential equation we *cannot* make the simple substitution used to solve Eq. (2.11)—the fundamental difference being that the right-hand side here is not invariant under such a substitution. Therefore, we employ a more general method which will be useful in solving the more intricate equation we encounter in Sec. VI. Consider a family of curves in the (k,t) plane, each satisfying $k=(m_\nu/2)(t_I/t)^{1/2}$ with each curve distinguished by a different value of t_I. We can parametrize each curve with the variable $\lambda\equiv t-t_I$. Along each curve f_H is a function of λ only and the partial differential equation (5.8) becomes an ordinary differential equation

$$\frac{df_H}{d\lambda}=\frac{Bn_\nu(\tau)\pi^2}{\tau(m_\nu/2)^3}\left[\frac{\tau}{t_I+\lambda}\right]^{3/2}e^{-(t_I+\lambda)/\tau}\delta(\lambda)2(t_I+\lambda)\ . \quad (5.9)$$

What boundary condition is needed here? Note that $\lambda<0$ corresponds to $k>m_\nu/2$. We know though that there are no photons with $k>m_\nu/2$. So we choose as the boundary condition

$$f_H(\lambda=0-\epsilon)=0\ , \quad (5.9a)$$

where ϵ is some small positive number. We now integrate Eq. (5.9) from $\lambda'=0-\epsilon$ up to $\lambda'=\lambda$. With the delta function, the integral is trivial, and the boundary condition allows us to set $f_H=0$ at the lower limit. So we have

$$f_H(\lambda)=\frac{Bn_\nu(\tau)2\pi^2}{\tau(m_\nu/2)^3}\left[\frac{\tau}{t_I}\right]^{3/2}e^{-t_I/\tau}t_I\Theta(\lambda)\ . \quad (5.10)$$

The step function comes in here because if λ is negative the interval of integration does not go through $\lambda'=0$ and therefore the integral over the right-hand side of Eq. (5.9) vanishes. To turn f_H back into a function of t and k, we replace t_I by $t[k/(m_\nu/2)]^2$. Then

$$f_H(k,t)=\frac{Bn_\nu(\tau)2\pi^2}{k(m_\nu/2)^2}\left[\frac{\tau}{t}\right]^{1/2}\exp\left[-\frac{t}{\tau}\left[\frac{k}{m_\nu/2}\right]^2\right]$$
$$\times\Theta\left[\frac{m_\nu}{2}-k\right]\ , \quad (5.11)$$

where the step function reflects the fact that no photons are produced with energy greater than $m_\nu/2$. One can of course verify that Eq. (5.11) is the solution to Eq. (5.8) by direct substitution. In Appendix B we review the above derivation in the framework of a matter-dominated universe and compare our expression with others in the literature. Now we extend the work of this section to include the environment into which the photons are injected.

VI. PHOTON SPECTRUM WITH COMPTON SCATTERING

In the previous section, we computed the spectrum of photons emanating from massive decaying particles assuming that the photons do not scatter off of the background electrons. Here we include the effect of scattering. Again the starting point is the Boltzmann equation, but now we can write down the equation immediately based on the work of the previous sections:

$$\left[\frac{\partial}{\partial t}-\frac{\dot{R}}{R}k\frac{\partial}{\partial k}\right]f_H(k,t)$$
$$=\frac{B}{\tau}\frac{\pi^2}{k^2}n_\nu e^{-t/\tau}\delta(k-m_\nu/2)$$
$$+\frac{n_e\sigma_T}{m_e}\frac{1}{k^2}\frac{\partial}{\partial k}\left[k^4\left[T_e\frac{\partial}{\partial k}f_H(k,t)+f_H(k,t)[1+f_H(k,t)]\right]\right] . \quad (6.1)$$

The first term on the right is the source term derived in Sec. V; the second term is the Compton-scattering term whose derivation was sketched in Sec. II. We stress that this expression is valid only in the nonrelativistic limit, $k<m_e$, so we will restrict ourselves to that regime, i.e., to neutrinos with mass less than the electron mass.

The Compton-scattering term depends on the electron temperature, so Eq. (6.1) should be solved in conjunction with the equation for T_e, Eq. (3.7). However, we expect the first term in the large parentheses on the right-hand side to be of order $(T_e/k)f_H$. So if we restrict ourselves to photons with energy $k\gg T_e$, then the second term in these large parentheses, $f_H(1+f_H)$, is larger than the first by a factor of order (k/T_e). In fact, this high-energy regime ($k\gg T_e$) is exactly the one we are interested in; for, we want the spectrum of decay photons and these typically have energies much greater than the electron temperature. The fact that the electron temperature dependence drops out corresponds to the physical fact that the electrons are essentially at rest compared with the hot photons produced; their thermal energy is irrelevant. There is one more approximation we can make to simplify Eq. (6.1). We will see that $f_H(k,t)$ is typically much less than 1 for decay photons. Therefore, we let

$$f_H(k,t)[1+f_H(k,t)]\to f_H(k,t) . \quad (6.2)$$

With these approximations, the Boltzmann equation becomes

$$\left[\frac{\partial}{\partial t}-\left[\frac{\dot{R}}{R}k+\frac{n_e\sigma_T}{m_e}k^2\right]\frac{\partial}{\partial k}\right]f_H(k,t)$$
$$=\frac{B}{\tau}\frac{\pi^2}{k^2}n_\nu e^{-t/\tau}\delta(k-m_\nu/2)+\frac{4n_e\sigma_T}{m_e}kf_H(k,t) . \quad (6.3)$$

Before plunging ahead and solving this equation, let us consider the underlying physics. A photon is produced with energy $m_\nu/2$. It scatters off the background electrons at a rate $n_e\sigma_T$. Since the scattering is in the nonrelativistic regime, in each collision a photon loses only a fraction (k/m_e) of its energy, so that the rate at which a photon loses energy is $n_e\sigma_T k/m_e$. This is to be compared with the expansion rate $\dot{R}/R$; if the energy-loss rate is larger than the expansion rate, then we expect the photons to transfer an appreciable amount of its energy to the electrons. Therefore, we can define a critical energy

$$k_C\equiv\frac{m_e(\dot{R}/R)(\tau)}{n_e(\tau)\sigma_T} , \quad (6.4)$$

where we evaluate $\dot{R}/R$ and n_e at cosmic time $t=\tau$ so that k_C is a constant. Roughly, photons produced with $k>k_C$ scatter down, while those with lower energies travel freely. The previous section, in which we assumed that the decay photon spectrum is not affected by collisions with electrons, is then valid only when $m_\nu/2\ll k_C$. Here we treat the general case.

It is convenient then to rewrite Eq. (6.3) in terms of two dimensionless variables

$$\theta\equiv\frac{k}{k_C},\quad \phi\equiv\frac{R(t)}{R(\tau)}=\left[\frac{t}{\tau}\right]^{1/2} , \quad (6.5)$$

where the last equality holds in a radiation-dominated universe. The "timelike" variable ϕ starts off at 0 and becomes large at late times. Since the number densities $n_e(t),n_\nu(t)$ fall off as R^{-3} (apart from the exponential decay factor for n_ν), Eq. (6.3) becomes

$$\left[\phi^2\frac{\partial}{\partial\phi}-(\theta\phi+\theta^2)\frac{\partial}{\partial\theta}\right]f_H(\phi,\theta)$$
$$=C(m_\nu/2k_C)e^{-\phi^2}\delta(\theta-(m_\nu/2k_C))+4\theta f_H(\phi,\theta) , \quad (6.6)$$

where the constant C is

$$C\equiv\frac{2\pi^2Bn_\nu(\tau)}{(m_\nu/2)^3} . \quad (6.7)$$

To solve this equation, we take the same approach as in Sec. V. Consider a family of curves in the (ϕ,θ) plane, each of which starts at a different point $[\phi_0,\theta_0=(m_\nu/2k_C)]$. Each curve satisfies the equation

$$\frac{\theta}{m_\nu/2k_C}=\frac{\phi_0}{\phi}\frac{1}{1+(m_\nu/4k_C\phi_0)[1-(\phi_0/\phi)^2]} . \quad (6.8)$$

Again we parametrize each curve with λ defined as

$$\lambda\equiv\frac{1}{\phi_0}-\frac{1}{\phi} \quad (6.9)$$

so that $\lambda<0$ corresponds to values of k such that $f_H(k,t)=0$ always. Along any of these curves, the partial differential equation (6.6) becomes an ordinary differential equation

$$\frac{df_H(\lambda)}{d\lambda}-4\frac{(m_\nu/2k_C)(1-\phi_0\lambda)}{1+(m_\nu/2k_C)\lambda(1-\phi_0\lambda/2)}f_H(\lambda)$$
$$=C(m_\nu/2k_C)\exp\left[-\left[\frac{\phi_0}{1-\phi_0\lambda}\right]^2\right]$$
$$\times\delta\left[\frac{(m_\nu/2k_C)(1-\phi_0\lambda)}{1+(m_\nu/2k_C)\lambda(1-\phi_0\lambda/2)}-(m_\nu/2k_C)\right] . \quad (6.10)$$

There are two features of this equation which make its solution tractable. First, the "damping" coefficient in front of f_H on the left is integrable, so that the left-hand side is simply

$$[1+(m_\nu/2k_C)\lambda(1-\phi_0\lambda/2)]^4\frac{d}{d\lambda}\times\left[\frac{f_H(\lambda)}{[1+(m_\nu/2k_C)\lambda(1-\phi_0\lambda/2)]^4}\right].$$

The second simplifying feature is the presence of the delta function in the decay term. If we integrate from $\lambda'=0-\epsilon$ up to $\lambda'=\lambda$, the delta function constrains λ' to be 0. This together with the boundary condition, $f_H(\lambda'=0-\epsilon)=0$ [corresponding to $f_H(k>m_\nu/2)=0$], leaves

$$f_H(\lambda)=Ce^{-\phi_0^2}\frac{[1+(m_\nu/2k_C)\lambda(1-\phi_0\lambda/2)]^4}{(m_\nu/2k_C)+\phi_0}\times\Theta((m_\nu/2k_C)-\theta)\ . \tag{6.11}$$

We want $f_H(\phi,\theta)$, so we must substitute for λ via Eq. (6.9); this gives $f_H(\phi,\phi_0)$. To get $f_H(\phi,\theta)$, we invert Eq. (6.8) to get ϕ_0 in terms of ϕ and θ. The result is

$$f_H(\phi,\theta)=\Theta((m_\nu/2k_C)-\theta)C\left[\frac{\phi}{2\phi+\theta}\right]^4$$
$$\times\frac{\{1+[1+(m_\nu/2k_C)^2(2\phi+\theta)/(\phi^2\theta)]^{1/2}\}^4}{(m_\nu/2k_C)+(2k_C/m_\nu)[\theta\phi^2/(2\phi+\theta)]\{1+[1+(m_\nu/2k_C)^2(2\phi+\theta)/(\phi^2\theta)]^{1/2}\}}$$
$$\times\exp\left[-\frac{\theta^2\phi^4}{(2\phi+\theta)^2}(2k_C/m_\nu)^2\{1+[1+(m_\nu/2k_C)^2(2\phi+\theta)/(\phi^2\theta)]^{1/2}\}^2\right]. \tag{6.12}$$

There are three interesting limits of Eq. (6.12). First, we recover the solution of Sec. V by letting $n_e\rightarrow 0$ (or equivalently $k_C \gg m_\nu/2$). In terms of (k,t),

$$\lim_{(2k_C/m_\nu)\rightarrow\infty} f_H(k,t)=C\frac{m_\nu}{2k(t/\tau)^{1/2}}\exp[-(t/\tau)(2k/m_\nu)^2]\Theta(m_\nu/2-k) \tag{6.13}$$

in agreement with Eq. (5.11). Second, we can take the limit of very efficient scattering, $k_C \ll m_\nu/2$:

$$\lim_{(2k_C/m_\nu)\rightarrow 0} f_H(k,t)=C\left[\frac{m_\nu}{2k}\right]^3\frac{k_C}{k}e^{-t/\tau}\Theta(m_\nu/2-k)\quad\text{for } t=O(\tau)\ . \tag{6.14}$$

The limit, Eq. (6.14), only holds if ϕ (or t/τ) is not too large. This limit is initially disturbing, since it appears that the $(1/k^4)$ dependence will make the energy density and the number density diverge. Recall, though, that in going from Eq. (6.1) to Eq. (6.3) we assumed that $k>T_e$, so Eq. (6.14) holds only for photons in this energy regime. The divergences, then, reflect qualitatively the fact that most of the photons produced have scattered down. It is fortunate that to calculate y, we need only find

$$\rho_H T_H=\tfrac{1}{2}\int\frac{d^3k}{(2\pi)^3}k^2 f_H(k) \tag{6.15}$$

and, with the distribution equation (6.14), this converges. Since this integral goes simply as $\int dk$, the largest contribution comes from the highest-energy photons ($k\sim m_\nu/2$) even though there are many more photons at any given time that have already scattered down. Equivalently, the decay photons affect the electron temperature most significantly in the short time before they lose all their energy due to Compton scattering.

Finally, we will be interested in finding the spectrum of photons produced in decays that remain today. Thus, we want the $\phi\rightarrow\infty$ limit of Eq. (6.12):

$$\lim_{t\rightarrow\infty} f_H(k,t)=\Theta(m_\nu/2-k)C\left[\frac{1+\sqrt{1+m_\nu^2/[2kk_C(t/\tau)^{1/2}]}}{2}\right]^4$$
$$\times\exp(-(k/m_\nu)^2(t/\tau)\{1+\sqrt{1+m_\nu^2/[2kk_C(t/\tau)^{1/2}]}\}^2)$$
$$\times((m_\nu/2k_C)+[k(t/\tau)^{1/2}/m_\nu]\{1+\sqrt{1+m_\nu^2/[2kk_C(t/\tau)^{1/2}]}\})^{-1}\ . \tag{6.16}$$

In Sec. VIII we will compare this spectrum with the observed ultraviolet spectrum and come away with bounds on the parameters τ, B, and m_ν. But here we will use Eq. (6.16) and the results of Sec. IV to demonstrate one of the most striking features of any scenario that explains the distortions in the CBR based on radiatively decaying neutrinos; namely, that recombination is delayed until very low temperatures. To establish this we will use Eq. (6.16) in the limit that $m_\nu/k_C>1$. To see that this is the appropriate limit we compute m_ν/k_C assuming the Hubble parameter $h=\frac{1}{2}$. We find

$$\frac{m_\nu}{k_C}=100\left[\frac{m_\nu}{1\ \text{MeV}}\right]\left[\frac{n_e}{10^{-9}n_\gamma}\right]\left[\frac{T(\tau)}{10^4T_0}\right], \tag{6.17}$$

which, as will be shown in the next section, is greater than one in the parameter range of interest. Thus the photon spectrum becomes

$$\lim_{t/\tau,m_\nu/k_C\gg 1} f_H(k,t)=\Theta(m_\nu/2-k)\frac{C}{32}\left[\frac{m_\nu^3}{k^2k_C(t/\tau)}\right] \times\exp-(k/2k_C)(t/\tau)^{1/2}\ . \tag{6.18}$$

With this expression for the photon spectrum, we can compute the effective number density of photons able to ionize atoms, $\tilde{n}_\gamma$, as defined by Eq. (4.14). At late times $(\epsilon_0/2k_C)(t/\tau)^{1/2}$ is very large, so we can evaluate $\tilde{n}_\gamma$ by integrating by parts and keeping only the surface term. This gives

$$\tilde{n}_\gamma\simeq\left[\frac{\tau}{t}\right]^{3/2} n_\nu(\tau)g(\epsilon_0)\exp\left[-\frac{\epsilon_0}{2k_C}(t/\tau)^{1/2}\right], \tag{6.19}$$

where $g(\epsilon_0)$, the gaunt factor evaluated at its end point, is 0.79. We must use this $\tilde{n}_\gamma$ to determine down to what temperature the equilibrium condition used in Eq. (4.16) obtains. We find that

$$\frac{\tilde{n}_\gamma\sigma_I}{\frac{\dot{R}}{R}}=6\times10^7\left[\frac{B}{10^{-5}}\right]\left[\frac{t_0}{t}\right]^{1/2}\exp\left[-\frac{\epsilon_0}{2k_C}(t/\tau)^{1/2}\right]. \tag{6.20}$$

Following the prescription of Sec. IV, we now set this ratio equal to 1 to determine the regime of applicability of the generalized Saha equation. We find the critical temperature, above which the Saha equation can be applied, to be

$$T_c=0.4T_0\left[\frac{n_e}{10^9n_\gamma}\left[\frac{T(\tau)}{10^4T_0}\right]^2\right]. \tag{6.21}$$

So for a large range of neutrino lifetimes τ, the generalized Saha equation derived in Sec. IV can be applied. If we define the recombination temperature to be that at which the number of free electrons is equal to the number of bound electrons ($r=\frac{1}{2}$), then Eq. (4.16) becomes

$$\frac{1}{2}=\frac{n_B(\tau)}{Bn_\nu(\tau)}\frac{\epsilon_0^2}{\sqrt{2\pi T_e(t_R)}m_e^{3/2}}\exp[(\epsilon_0/2k_C)(t_R/\tau)^{1/2}]\ , \tag{6.22}$$

where t_R is the recombination time. Solving we find

$$T(t_R)=1.1T_c\ , \tag{6.23}$$

where we have suppressed the very small dependence on B and $T(\tau)$. We will see in Sec. VII that adequate values of y can be obtained only if the decay temperature $T(\tau)$ is a bit higher than 10^4T_0. Equation (6.23) is changed very slightly at these higher temperatures, but, as seen from Eq. (6.21), the critical temperature T_c is very sensitive to $T(\tau)$. For realistic lifetimes, the recombination temperature is just above T_0, so the electrons and photons have formed atoms, but only very recently. For our purposes, it is correct to assume that the plasma is fully ionized in the regime of interest.

VII. THE y PARAMETER

All the elements are now in place for a calculation of y. We will calculate y in the two regimes: (i) freely streaming photons ($m_\nu/2<k_C$) and (ii) heavily scattered decay photons ($m_\nu/2>k_C$).

(i) $m_\nu/2<k_C$.

Here the distribution, Eq. (5.11), holds so

$$\rho_HT_H=\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}k^2 \times\left[\frac{2\pi^2Bn_\nu(\tau)}{(m_\nu/2)^2k}\frac{e^{-[k\phi/(m_\nu/2)]^2}}{\phi}\Theta(m_\nu/2-k)\right]$$

$$=\frac{Bn_\nu(\tau)m_\nu^2}{16\phi^5}[1-e^{-\phi^2}(1+\phi^2)]\ , \tag{7.1}$$

where we have used the timelike variable $\phi=(t/\tau)^{1/2}$. To calculate y, we rewrite Eq. (3.16) in terms of ϕ, using the efficiency as given in Eq. (3.17):

$$y\simeq\int_0^{\phi(\text{today})}\frac{d\phi}{\phi^2}\left[\frac{n_e(\tau)\sigma_T}{m_e(\dot{R}/R)(\tau)}\right]\frac{\rho_HT_H}{\rho_{\text{CBR}}}\ . \tag{7.2}$$

The quantity in large parentheses is by definition equal to $1/k_C$, where k_C is the critical energy introduced in Sec. VI. The CBR energy density ρ_{CBR} falls off as ϕ^{-4}, so inserting Eq. (7.1) into this expression for y leads to

$$y\simeq\left[\frac{Bn_\nu(\tau)m_\nu/2}{\rho_{\text{CBR}}(\tau)}\right]\left[\frac{m_\nu/2}{k_C}\right]\frac{1}{4} \times\int_0^{\phi(\text{today})}\frac{d\phi}{\phi^3}[1-e^{-\phi^2}(1+\phi^2)]$$

$$=\left[\frac{Bn_\nu(\tau)m_\nu/2}{\rho_{\text{CBR}}(\tau)}\right]\left[\frac{m_\nu/2}{k_C}\right]\frac{1}{8}\ . \tag{7.3}$$

The term in the first set of large parentheses is essentially the available fractional excess energy. When the photons produced by decays do not lose much energy by scattering off the background electrons, not all of this energy gets converted to the CBR. In fact, y is reduced here by a factor of $m_\nu/2k_C$. Using the expression for n_ν,y can also be written as

$$y=\frac{45\zeta(3)}{88\pi^4}B\frac{m_\nu}{T(\tau)}\frac{m_\nu/2}{k_C}\quad\text{for } (m_\nu/2k_C)<1\ . \tag{7.3a}$$

(ii) $m_\nu/2>k_C$.

Here we use the distribution, Eq. (6.14), to calculate

$$\rho_HT_H=\frac{Bn_\nu(\tau)m_\nu k_C}{4}e^{-\phi^2} \tag{7.4}$$

and then y follows:

$$y \simeq \left[\frac{B n_\nu(\tau) m_\nu/2}{\rho_{\rm CBR}(\tau)} \right] \frac{\sqrt{\pi}}{8} = \frac{45\zeta(3)}{88\pi^4} \sqrt{\pi} B \frac{m_\nu}{T(\tau)} \quad \text{for } (m_\nu/2k_C) > 1 \ . \tag{7.5}$$

Note the difference between Eqs. (7.3) and (7.5). In each case y is proportional to the fractional excess energy. When the photons produced by decays do not lose much energy by scattering off the background electrons ($m_\nu/2 < k_C$), the efficiency of converting this excess energy into the background is small, so y is reduced by a factor of $(m_\nu/2k_C)$. When the photons do scatter down ($m_\nu/2 > k_C$), though, the excess energy is efficiently transferred to the background, and there is no such reduction. A key feature of efficient energy transfer can be gleaned from Eq. (7.5). The y parameter is completely independent of the electron number density and the cross section in the limit of efficient transfer. As long as

$$n_e(\tau)\sigma_T \gtrsim \frac{m_e}{m_\nu/2} \frac{\dot{R}}{R}(\tau) \tag{7.6}$$

the excess energy in the CBR is independent of $n_e\sigma_T$.

We still have two loose ends to tie up. First, in deriving the equation for the decay photon spectrum (6.3), we assumed that f_H is much less than 1. It is straightforward to check that with the parameters needed to get a realistic value of y, this condition is indeed satisfied. Second, the validity of our expression for y depends upon an assumption we made in deriving the electron temperature. There, we neglected the interference term, Eq. (3.14), with respect to the other terms driving T_e. In Appendix C we show that, for realistic values of B, m_ν, and τ, the interference term is indeed small compared to the other terms driving T_e.

VIII. CONSTRAINTS ON RADIATIVELY DECAYING NEUTRINOS

It is not our purpose here to examine all the constraints on unstable neutrinos.[30] However, so as not to leave the false impression that a value of $y=0.02$ is easy to come by, we present here two constraints on radiatively decaying neutrinos. These significantly narrow the allowed values of m_ν, τ, and B; nevertheless, a small region remains from which $y=0.02$ can be achieved.

The first constraint is that the energy density due to the decay products of the heavy neutrino should not exceed the critical energy density. The condition with $\rho_B = 0.15\rho_C$ and $H_0 = 50$ km sec^{-1} Mpc^{-1} is[31]

$$\frac{m_\nu}{100 \text{ keV}} < \frac{T(\tau)}{1.1 \text{ eV}} \ . \tag{8.1}$$

The second constraint comes from ultraviolet astronomy. If a neutrino decayed radiatively as outlined in the previous few sections, the photons from decays should be detectable today. We must calculate the expected intensity and compare this with observations. We will focus on the constraint coming from a wavelength $\lambda_U = 2200$ Å ($k_U = 5.63$ eV). Experimentally the diffuse background at this wavelength satisfies[32]

$$\lambda_U I_\lambda < 5.3\times 10^4 \frac{\text{photons}}{\text{cm}^2 \text{sec sr}} \ . \tag{8.2}$$

The flux of photons is related to the distribution function we have been using by

$$\lambda_U I_\lambda = k_U \left[\frac{dn}{d\Omega\, dk} \right]_{k=k_U} = k_U \left[\frac{2}{(2\pi)^3} k_U^2 f(k_U) \right] \ . \tag{8.3}$$

Requiring that the photon flux from neutrino decay be less than the background then gives a constraint on the distribution function:

$$f(k_U) < 9.3\times 10^{-21} \ . \tag{8.4}$$

In Sec. VI we derived an expression (6.16) for $f(k)$ today, that is at $T_0 = 2.81$ K. We can use this to express the constraint, Eq. (8.4), as a constraint on m_ν, τ, and B. Specifically we require

$$B \left[\frac{T(\tau)}{m_\nu} \right]^3 \left[\frac{1+\sqrt{1+m_\nu^2 T_0/[2k_U k_C T(\tau)]}}{2} \right]^4 \left[(m_\nu/2k_C) + \left[\frac{k_U T(\tau)}{m_\nu T_0} \right] \{1+\sqrt{1+m_\nu^2 T_0/[2k_U k_C T(\tau)]}\} \right]^{-1} \times \exp\left[-\left[\frac{k_U T(\tau)}{m_\nu T_0} \right]^2 \{1+\sqrt{1+m_\nu^2 T_0/[2k_U k_C T(\tau)]}\}^2 \right] < 8.9\times 10^{-22} \ , \tag{8.5}$$

where we have used the fact that in a radiation-dominated universe $(t_0/\tau)^{1/2} = T(\tau)/T_0$.

In Fig. 3 we plot this constraint, Eq. (8.5), in $[m_\nu, T(\tau)]$ space. The dashed line here is what the constraint would look like in the absence of Compton scattering; that is, if we had used Eq. (6.13) to implement the constraint, Eq. (8.4). To see why the inclusion of Compton scattering changes the constraint, consider a free streaming photon with energy k_U today. If it was produced at a cosmic time t_P, then its energy at the time of production was just $k = k_U[T(t_P)/T_0]$. All the photons we are interested in were produced with an energy $k = m_\nu/2$, so if they really did travel freely after they were produced, we can determine t_P:

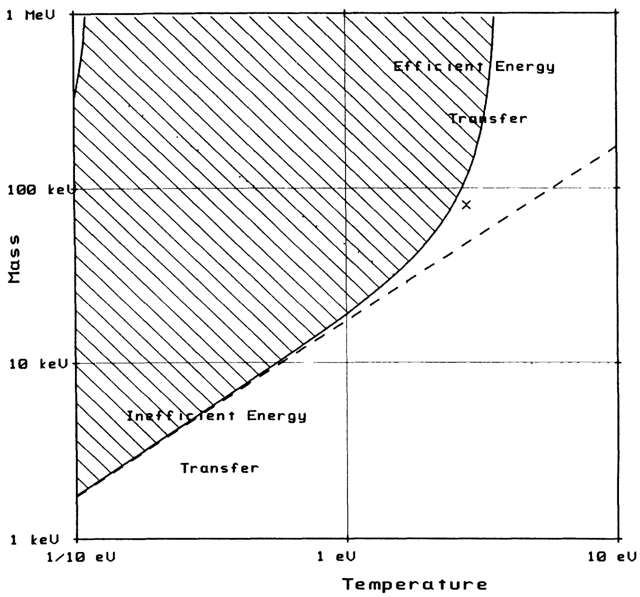

FIG. 3. Ultraviolet constraint (hatched region ruled out) coming from $k_U=5.63$ eV in the $[m_\nu, T_{\rm CBR}(\tau)]$ plane, where m_ν is the decaying neutrino mass and $T_{\rm CBR}(\tau)$ is the cosmic temperature at the time of decay. The radiative branching ratio is set to 3×10^{-5}. The dotted line separates the regions where Compton scattering is unimportant ("inefficient energy transfer") and where it is relevant ("efficient energy transfer"). The dashed line is what the constraint becomes in the absence of Compton scattering. X marks the point $[m_\nu=80$ keV, $T_{\rm CBR}(\tau)=2.8$ eV$]$ discussed in the text.

$$m_\nu/2=k_U[T(t_P)/T_0]$$

or, using radiation domination,

$$t_P=\tau\left[\left[\frac{T(\tau)}{1\ \text{eV}}\right]\left[\frac{50\ \text{keV}}{m_\nu}\right]\right]^2 . \qquad (8.6)$$

Let us take some examples. First, consider $T(\tau)=\frac{1}{10}$ eV and the neutrino mass $m_\nu=2.9$ keV. The time at which photons with energy k_U today were produced is $t_P=\tau[(\frac{1}{10})(\frac{50}{2.9})]^2$ or 3τ. By this time most of the neutrinos have already decayed; however the remaining fraction ($\sim e^{-3}\sim0.05$) is still large enough to oversaturate the UV constraint, as shown in Fig. 3. If, however, the neutrino mass is slightly smaller, say $m_\nu=1$ keV, then only those photons produced very late would have energy k_U today; all the others will have redshifted down. Therefore, $t_P=25\tau$ and essentially no neutrinos ($\sim e^{-25}\sim10^{-11}$) are left by this time. Indeed, Fig. 3 shows that the ultraviolet constraints do not rule out $T(\tau)=\frac{1}{10}$ eV and $m_\nu=1$ keV.

We now turn to the "efficient energy transfer" regime. Consider $T(\tau)=2.8$ eV and $m_\nu=40$ keV. Now Eq. (8.6) yields $t_P=12\tau$, so that there are too few ($e^{-12}\sim10^{-5}$) neutrinos around to produce an observable UV background. Therefore, these values are allowed. Finally, consider the same $T(\tau)$ but a slightly larger mass $m_\nu=80$ keV. If the photons traveled freely, then $t_P=3\tau$ and there should have been enough neutrinos around at t_P to swamp our UV detectors today. Indeed, the dashed line in Fig. 3 indicates that, in the absence of Compton scattering, this parameter set is ruled out. However, Compton scattering causes the photons produced at t_P to scatter down and lose energy: a photon that would otherwise have had $k=k_U$ today has lost energy and has $k\ll k_U$. So the only photons today with $k=k_U$ are those produced at $t\gg t_P$, by which time almost all of the neutrinos have decayed. Therefore, Fig. 3 shows that due to Compton scattering, this parameter set is allowed. The conclusion is that accounting for Compton scattering is crucial in deriving the correct constraints from present observations.

The hatched regions in Fig. 4 are those forbidden by energy-density considerations [Eq. (8.1)] and ultraviolet observations [Eq. (8.5)]. The solid curve corresponds to values of m_ν and $T(\tau)$ for which $y=0.02$. Here we have set the branching ratio $B=3\times10^{-5}$. Note that there are no parameter values in the "inefficient" region that give $y=0.02$ and satisfy the constraints. If the energy transfer is efficient, though, there is an allowed region.

IX. CONCLUSION

Observations are beginning to detect small spectral distortions in the cosmic-background radiation (CBR). Since there seems to be a possibility that these distortions

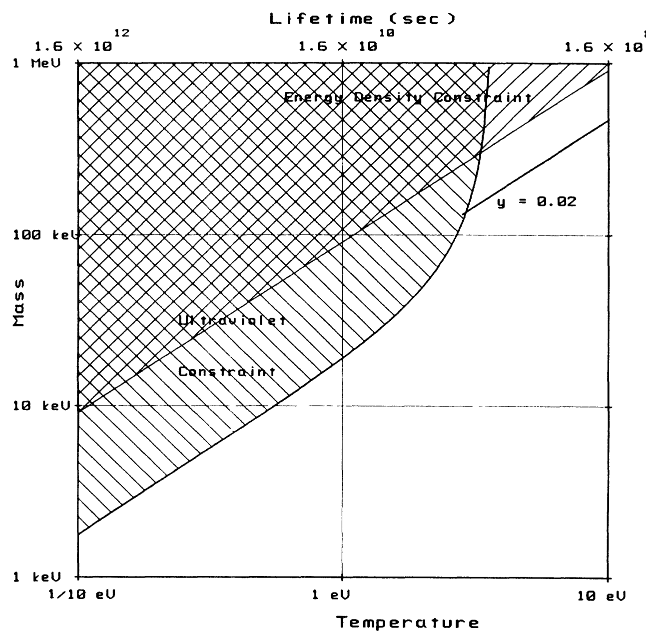

FIG. 4. The parameter y is 0.02 along the solid curve. The hatched regions are ruled out by ultraviolet observations and the energy density constraint. The branching ratio has again been set to 3×10^{-5}. Note that an acceptable y comes only from the "efficient energy transfer" region of parameter space. Neutrinos with masses above 1 MeV must be analyzed relativistically; temperatures above 10 eV require the treatment to include inelastic processes. Our work is limited, therefore, to temperatures and masses beneath these values. The time-temperature relationship is derived using radiation domination and setting the Hubble rate today to $H_0=50$ km sec^{-1} Mpc^{-1}.

were caused by hot electrons inverse Compton scattering off the background photons, we feel it is worthwhile to reexamine this mechanism. The treatment given here has yielded the following new results: (i) a novel solution for the distorted spectrum that depends on the Zel'dovich-Sunyaev y parameter; (ii) an expression for the electron temperature when the electrons are simultaneously cooled by the background photons and heated by another source of photons; (iii) a recipe for computing the ionization fraction, the number of free electrons; (iv) a demonstration that any excess energy, in the form of photons, emitted into the Universe at a cosmic time τ is efficiently converted to the CBR if the energy of the quanta satisfy

$$E > k_C \equiv \frac{m_e(\dot{R}/R)(\tau)}{n_e(\tau)\sigma_T} . \tag{9.1}$$

Throughout the last several sections, we have illustrated the last point by working within the framework of the radiatively decaying neutrino scenario, which, although feasible, is far from compelling. Thus, the most interesting question provoked by the distortions remains unanswered: what new particle physics is responsible for the excess energy observed in the cosmic background radiation?

ACKNOWLEDGMENTS

Much of the work on this paper was done at the Aspen Center for Physics. We are very grateful to the Center for its hospitality. We are especially grateful to L. Brown for many useful suggestions and a critical reading of the manuscript. In addition we would like to thank G. Feinberg, A. Lange, and A. Loeb for several valuable conversations. S.D. was supported in part by the U.S. Department of Energy under Grant No. DE-FG02-84ER40158 with Harvard University.

APPENDIX A

Because the method used in solving Eqs. (5.8) and (6.6) may be unfamiliar, we would like to illustrate it in detail in a simple example. Consider

$$\left[\frac{\partial}{\partial t} - \frac{\dot{R}}{R}k\frac{\partial}{\partial k}\right] f(k,t) = g(k,t) , \tag{A1}$$

where $g(k,t)$ is a given function of the indicated arguments. The equation would be readily solvable if only we could introduce a parameter $\lambda(k,t)$ such that

$$\frac{d}{d\lambda} f(\lambda(k,t)) = \left[\frac{\partial}{\partial t} - \frac{\dot{R}}{R}k\frac{\partial}{\partial k}\right] f(k,t) . \tag{A2}$$

Hence we must have

$$\frac{d\lambda}{dt} = 1 \tag{A3}$$

and

$$\frac{d\lambda}{dk} = \frac{-1}{k(\dot{R}/R)} = \frac{dt}{dk} , \tag{A4}$$

where the last equality follows by virtue of Eq. (A3). Then, it follows that

$$\frac{dR}{dk} = \frac{dR}{dt}\frac{dt}{dk} = -\frac{R}{k} . \tag{A5}$$

Thus Eq. (A2) holds along any trajectory in the (R,k) plane with

$$k = \frac{\text{const}}{R(t)} \tag{A6}$$

and

$$\lambda = t - t_0 . \tag{A7}$$

Thus we can write Eq. (A2) as

$$\frac{df}{d\lambda} = g\left[\frac{\text{const}}{R(t(\lambda))}, \lambda + t_0\right] \tag{A8}$$

so that

$$f(\lambda) = f(\lambda=0) + \int_0^\lambda d\lambda' g\left[\frac{\text{const}}{R(t(\lambda'))}, \lambda' + t_0\right] . \tag{A9}$$

Hence

$$f(k,t) = f(kR(t)/R(t_0), t_0) + \int_{t_0}^t dt' g\left[\frac{R(t)}{R(t')}k, t'\right] . \tag{A10}$$

(We would like to thank G. Feinberg for discussions on this equation.)

APPENDIX B

In Sec. V we derived an expression for the photon phase-space density arising from neutrino decays. Much work has been done on this subject and we would like to use this appendix to make some further points and connect our work with that of previous authors. To that end, we return to Eq. (5.8), which determines the phase-space density. Only now, we do not assume a radiation-dominated universe; rather we let

$$R(t) = R(\tau)(t/\tau)^n . \tag{B1}$$

A radiation-dominated universe corresponds to $n = \frac{1}{2}$ while a matter-dominated universe has $n = \frac{2}{3}$. Equation (5.8) becomes

$$\left[\frac{\partial}{\partial t} - \frac{\dot{R}}{R}k\frac{\partial}{\partial k}\right] f_H(k,t) = \frac{Bn_\nu(\tau)\pi^2}{\tau(m_\nu/2)^2}\left[\frac{\tau}{t}\right]^{3n} e^{-t/\tau} \times \delta(k - m_\nu/2) . \tag{B2}$$

We could go through the same discussion as in Sec. V; instead let us use the work of Appendix A. Specifically, using Eq. (A10), we find

$$f_H(k,t) = \int_0^t dt' \frac{B\pi^2 n_\nu(\tau)}{\tau(m_\nu/2)^2}\left[\frac{\tau}{t'}\right]^{3n} e^{-t'/\tau} \times \delta(k(t/t')^n - (m_\nu/2)) . \tag{B3}$$

The delta function can be rewritten as

$$\delta(k(t/t')^n-(m_\nu/2))=\frac{t'^{n+1}}{knt^n}\delta(t'-t(2k/m_\nu)^{1/n}) \ . \quad \text{(B4)}$$

With the delta function the integral is trivial, and we are left with

$$f_H(k,t)=\frac{B\pi^2 n_\nu(\tau)}{n(m_\nu/2)^{1/n}k^{3-1/n}}\left[\frac{\tau}{t}\right]^{3n-1} \times\exp-(t/\tau)(2k/m_\nu)^{1/n}\Theta(m_\nu/2-k) \ . \quad \text{(B5)}$$

Equation (B5) agrees with Eq. (5.11) when $n=\frac{1}{2}$ as it must. For a matter-dominated universe, $n=\frac{2}{3}$ and we have

$$f_H(k,t)|_{\text{MD}}=\frac{3B\pi^2 n_\nu(\tau)}{2(m_\nu/2)^{3/2}k^{3/2}}\left[\frac{\tau}{t}\right] \times\exp[-(t/\tau)(2k/m_\nu)^{3/2}]\Theta(m_\nu/2-k) \ . \quad \text{(B6)}$$

Often in the literature, an expression is given for the photon flux; that is, the number of photons with energy between k and $k+dk$ passing through a given area in a given time. This is just the number density of such photons times their velocity c:

$$c\,dn_H=c\frac{2}{(2\pi)^3}d^3k f_H(k,t) \quad \text{(B7)}$$

so that the flux per energy interval per steradian today (t_0) is

$$c\frac{dn_H}{d\Omega\,dk}\bigg|_{\text{MD}}=\frac{c}{4\pi}\frac{Bn_\nu(t_0)k^{1/2}}{(m_\nu/2)^{3/2}}\left[\frac{3t_0}{2\tau}\right] \times\exp[-(t_0/\tau)(2k/m_\nu)^{3/2}]\Theta(m_\nu/2-k) \ . \quad \text{(B8)}$$

Here $n_\nu(t_0)$, the neutrino number density in the absence of decays, is 109 cm^{-3}. Stecker,[33] for example, has discussed the detection of neutrinos with lifetimes longer than the age of the Universe, so that the exponential factor in Eq. (B8) can be set to 1. Equation (B8) then reduces to Stecker's equation (4a).

Finally, we note that it is illuminating to rewrite the general solution for $f_H(k,t)$ as

$$f_H(k,t)=-B\pi^2 n_\nu(\tau)(\tau/t)^{3n}\Theta(m_\nu/2-k)\frac{1}{k^2}\frac{\partial}{\partial k} \times\exp[-(t/\tau)(2k/m_\nu)^{1/n}] \ . \quad \text{(B9)}$$

Equation (B9) is particularly useful to use to see how the number density of decay photons builds up with time. For

$$n_H(t)=2\int\frac{d^3k}{(2\pi)^3}f_H(k,t)=Bn_\nu(t)(1-e^{-t/\tau}) \ . \quad \text{(B10)}$$

This expression clearly illustrates that each heavy neutrino has a probability (B) of producing one photon when it decays, so that when the all heavy particles have decayed, the hot photon number density is simply Bn_ν.

APPENDIX C

In this appendix, we justify an approximation that led to our final expression for y. Specifically, we demonstrate that the interference term in the equation which determines the electron temperature (3.14) is much smaller than the other two terms driving T_e, and therefore we were justified in neglecting it. We need to compare the interference term in Eq. (3.14) to one of the other terms, say $\rho_H T_H$. The ratio is

$$\frac{I}{\rho_H T_H}=\frac{\int\frac{d^3k}{(2\pi)^3}k^2(e^{k\phi/T(\tau)}-1)^{-1}\{Cm_\nu/(2k\phi)\exp[-(2k\phi/m_\nu)^2]\Theta(m_\nu/2-k)\}}{\int\frac{d^3k}{(2\pi)^3}k^2\{Cm_\nu/(2k\phi)\exp[-(2k\phi/m_\nu)^2]\Theta(m_\nu/2-k)\}}$$

$$=\frac{\int_0^{m_\nu/2}dk\,k^3(e^{k\phi/T(\tau)}-1)^{-1}\exp[-(2k\phi/m_\nu)^2]}{\int_0^{m_\nu/2}dk\,k^3\exp[-(2k\phi/m_\nu)^2]} \ . \quad \text{(C1)}$$

We perform the top integration by breaking it up into two regimes: $k<(k_I/\phi)$ and $k>(k_I/\phi)$, where k_I is some intermediate value of momentum with the property

$$T(\tau)\ll k_I\ll m_\nu/2 \ . \quad \text{(C2)}$$

In the first regime, $k<(k_I/\phi)$, the argument of the exponential $(2k\phi/m_\nu)^2<(2k_I/m_\nu)^2\ll 1$, so the integral becomes

$$\int_0^{k_I/\phi}dk\,k^3(e^{k\phi/T(\tau)}-1)^{-1}\exp[-(2k\phi/m_\nu)^2] \simeq\left[\frac{T(\tau)}{\phi}\right]^4\int_0^{k_I/T(\tau)}d\mu\frac{\mu^3}{e^\mu-1} \simeq 3!\zeta(4)\left[\frac{T(\tau)}{\phi}\right]^4 \quad \text{(C3)}$$

in the limit $k_I \gg T(\tau)$. In the second part of the integral,

$$\frac{k\phi}{T(\tau)} > \frac{k_I}{T(\tau)} \gg 1 , \tag{C4}$$

so that the background photon distribution becomes simply $\exp[-k\phi/T(\tau)]$, leaving

$$\int_{k_I/\phi}^{\infty} dk\, k^3 \exp\{-(2k\phi/m_\nu)^2-[k\phi/T(\tau)]\} = \frac{T(\tau)k_I^3}{\phi^4} e^{-k_I/T(\tau)}\left[1+O\left[\frac{T(\tau)}{k_I}\right]\right] . \tag{C5}$$

The exponential suppression shows this term to be negligible with respect to the first, so

$$\frac{I}{\rho_H T_H} \simeq \frac{3!\zeta(4)[T(\tau)/\phi]^4}{(m_\nu/2\phi)^4[1/2-\frac{1}{2}e^{-\phi^2}(1+\phi^2)]} = 12\zeta(4)\left[\frac{T(\tau)}{m_\nu/2}\right]^4 \frac{1}{1-e^{-\phi^2}(1+\phi^2)} \ll 1 \tag{C6}$$

since $T \ll m_\nu/2$. Therefore, we are justified in taking the electron temperature to be as given in Eq. (3.15). We should mention that Eq. (C6) also gives an idea of when the interference term cannot be neglected. When the extra photons injected into the Universe have energies comparable to the energies of the CBR photons [the equivalent in this scenario would be $m_\nu/2 \sim T(\tau)$ instead of $m_\nu/2 \gg T(\tau)$], then it no longer makes sense to break up the photon distribution into two parts as we did in Eq. (3.12).

APPENDIX D

The purpose of this appendix is to discuss the evolution of the electron temperature T_e as a function of time in the radiatively decaying neutrino scenario. To this end we employ the approximate expression [Eq. (3.15)]

$$T_e - T \simeq \frac{\rho_H}{\rho_{\rm CBR}} T_H , \tag{D1}$$

where T is the CBR temperature, $\rho_{\rm CBR}$ is the CBR energy density, ρ_H is the energy density of the hot photons, and T_H is their temperature.

It is instructive to evaluate this expression first in the approximation that the electron cooling can be ignored. This will help us appreciate the role of the cooling. In this approximation we can use Eq. (7.1) for $\rho_H T_H$. Using the definition of ϕ, this becomes

$$\rho_H T_H = \frac{Bn_\nu(\tau)m_\nu^2}{16}\left[\frac{\tau}{t}\right]^{5/2}\{1-e^{-t/\tau}[1+(t/\tau)]\} , \tag{D2}$$

where B is the branching ratio into photons and $n_\nu(\tau)$ is the neutrino number density at $t=\tau$ ignoring decays. To find T_e we must divide this by the CBR energy density

$$\rho_{\rm CBR} = \frac{\pi^2}{15}T(\tau)^4\left[\frac{\tau}{t}\right]^2 . \tag{D3}$$

Thus, the ratio of the electron temperature to the CBR temperature is

$$\frac{T_e}{T} = 1+\frac{Bn_\nu(\tau)m_\nu^2}{16(\pi^2/15)T(\tau)^5}\{1-e^{-t/\tau}[1+(t/\tau)]\} . \tag{D4}$$

But the neutrino number density can be expressed in terms of T:

$$n_\nu(\tau) = \frac{3}{11}n_\gamma = \frac{3}{11}\frac{2\zeta(3)}{\pi^2}T(\tau)^3 \tag{D5}$$

so we find that

$$\frac{T_e}{T} = 1+630\{1-e^{-t/\tau}[1+(t/\tau)]\}\left[\frac{B}{10^{-5}}\right] \times \left[\frac{m_\nu}{100\ \rm keV}\right]^2\left[\frac{1\ \rm eV}{T(\tau)}\right]^2 . \tag{D6}$$

Hence if we ignore the Compton scattering of the hot photons, T_e/T rises monotonically to its asymptotic value as time evolves. The free electrons are simultaneously heated and cooled by the hot photon distribution and the CBR photons. Since the average energy of a typical photon in each of these distributions falls off as $1/R(t)$ (due to the redshift), it is reasonable that the electron temperature also drops as $1/R(t)$ once all the hot photons have been produced. Equivalently, $T_e/T \rightarrow$ constant as t/τ gets big.

Now we consider Compton scattering, an effect which drastically changes the evolution of T_e/T. For $\rho_H T_H$, we use Eq. (7.4):

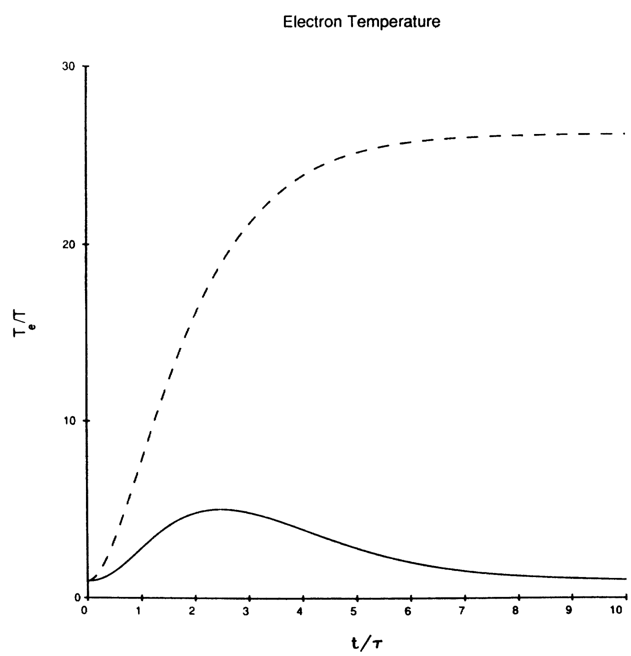

FIG. 5. The electron temperature as a function of time. Here we have set the branching ratio B to 10^{-5}; the neutrino mass $m_\nu = 100$ keV; and the decay temperature $T(\tau)=5$ eV. The solid line is the correct expression including the scattering of decay photons off the electrons; the dotted line shows how the temperature would behave if scattering were neglected.

$$\rho_H T_H = \frac{B n_\nu(\tau) m_\nu m_e (\dot{R}/R)(\tau)}{4 n_e(\tau) \sigma_T} e^{-t/\tau} , \qquad \text{(D7)}$$

where we have used the definition of k_C [Eq. (6.4)]; $n_e(\tau)$ is the free electron number density at $t=\tau$ and σ_T is the Thompson cross section. To evaluate T_e/T, we use Eq. (D3) and express the Hubble rate at τ in terms of the known rate today

$$\dot{R}/R(\tau) = 1.06\times 10^{-33}\ \text{eV} \left[\frac{T(\tau)}{2.3\times 10^{-4}\ \text{eV}} \right]^2 , \qquad \text{(D8)}$$

where we have set the Hubble constant today to 50 km sec^{-1} Mpc^{-1}. Then,

$$\frac{T_e}{T} = 1 + \frac{B n_\nu(\tau) m_\nu m_e (\dot{R}/R)(\tau)}{4 n_e(\tau) \sigma_T (\pi^2/15) T^5} e^{-t/\tau}$$
$$= 1 + 620 \left[\frac{t}{\tau} \right]^{5/2} e^{-t/\tau} \left[\frac{B}{10^{-5}} \right] \left[\frac{m_\nu}{100\ \text{keV}} \right]$$
$$\times \left[\frac{1\ \text{eV}}{T(\tau)} \right]^3 \left[\frac{10^{-9} n_\gamma}{n_e} \right] . \qquad \text{(D9)}$$

This expression has the satisfactory property that T_e/T begins as unity, rises to a maximum when $t/\tau = 5/2$ and then decreases, reflecting the fact that the hot photons which were raising the electron temperature have scattered down to lower energies. Figure 5 illustrates the difference that Compton scattering of the hot photons makes in the evolution of the electron temperature.

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*Permanent address.

[1]The basic early Universe references are Ya. B. Zel'dovich and R. A. Sunyaev, Astron. Zh. **46**, 775 (1969); R. A. Sunyaev and Ya. B. Zel'dovich, Astrophys. Space Sci. **7**, 20 (1970); R. A. Sunyaev and Ya. B. Zel'dovich, Comments Astrophys. Space Phys. **2**, 660 (1970). A very useful review can be found in Ya. B. Zel'dovich and I. D. Novikov, *The Structure and Evolution of the Universe* (Vol. II of *Relativistic Astrophysics*) (The University of Chicago Press, Chicago and London, 1983), see especially Chap. 8 written in collaboration with R. A. Sunyaev.

[2]R. A. Sunyaev and Ya. B. Zel'dovich, Comments Astrophys. Space Phys. **4**, 173 (1972).

[3]An excellent review is to be found in J. M. Uson and D. T. Wilkinson, in *Galactic and Extra Galactic Radio Astronomy*, 2nd ed., edited by G. Verschaur and K. Kellerman (Springer, New York, 1988), Chap. 14.

[4]Uson and Wilkinson (Ref. 3) note that these observations yield

$$30\ \text{km sec}^{-1}\text{Mpc}^{-1} \le H_0 \le 200\ \text{km sec}^{-1}\text{Mpc}^{-1} ,$$

which is consistent with conventional measurements.

[5]T. Matsumoto *et al.*, Astrophys J. **324**, 567 (1988). One of us (J.B.) is grateful to Andrew Lange for several discussions of this experiment.

[6]The measured isotropy for these new points is verified to within a few percent—an upper bound. No $\delta T/T$ distortion of any point on the spectral curve has been observed to the 10^{-4} level. The cosmological character of the new points could be certified if the Doppler shifts can be observed for them.

[7]See F. C. Adams *et al.*, Astrophys. J. **344**, 24 (1989), for a clear discussion of this and other mechanisms of producing Zel'dovich-Sunayev distortions.

[8]A. S. Kompaneets, Zh. Eksp. Teor. Fiz. **31**, 876 (1957) [Sov. Phys. JETP **4**, 730 (1957)].

[9]A. P. Lightman and G. B. Rybicki, *Radiative Processes in Astrophysics* (Wiley-Interscience, New York, 1979), p. 213, present a derivation in which they assume that the electrons have a Maxwell-Boltzmann distribution. J. Bernstein, in *Kinetic Theory in the Expanding Universe* (Cambridge University Press, New York, 1988), Chap. 8, gives a derivation in which this assumption is relaxed. The electron temperature is *defined* by the relation

$$\int \frac{d^3p}{(2\pi)^3} \frac{p^2}{2m_e} g_e(p) \equiv \tfrac{3}{2} T_e n_e ,$$

where n_e is the electron number density per spin degree of freedom and $g_e(p)$, the electron distribution function need not be Maxwellian. The most general derivation is to be found in L. S. Brown, University of Washington report (unpublished). Brown begins by examining the photon correlation functions in the plasma including quantum fluctuations. He shows that the Kompaneets equation holds even in the quantum-mechanical regime provided that the electrons are in thermal equilibrium. In this limit, which is relevant to our regimes, there are no effects of scattering from plasma oscillations.

[10]See Bernstein (Ref. 9), Chap. 3.

[11]Our analysis differs from stronger claims on the validity of this expansion. B. J. T. Jones and G. Steigman, Mon. Not. R. Astron. Soc. **183**, 585 (1978), for example, claim that it is valid for $y<1$, while Ref. 7 implies it is valid for $T_e > T$. On the other hand, Ya. B. Zel'dovich *et al.*, Zh. Eksp. Teor. Fiz. **62**, 1217 (1972) [Sov. Phys. JETP **35**, 645 (1972)] also claim the expansion is valid only when $x^2 y < 1$.

[12]This expression was first derived in Zel'dovich and Sunyaev (Ref. 1). See also K. Tomita *et al.*, Prog. Theor. Phys. **43**, 1511 (1970).

[13]See, for example, Zel'dovich *et al.* (Ref. 11); K. L. Chan and B. J. T. Jones, Astrophys. J. **198**, 245 (1975). The expression given on page 216 of the review by Zel'dovich and Novikov (Ref. 1) is not correct. We shall derive the correct version below.

[14]J. Bernstein and L. S. Brown, Aspen Center for Physics, 1988 (unpublished).

[15]To derive this identity, expand $f(x)$ in powers of x and use the canonical commutation relations.

[16]Because of the evenness of the integrand, we can also write (2.36) as

$$f(x,y) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} ds\, e^{-s^2} f(e^{2s\sqrt{y}+3y}x, 0) .$$

[17]See, for example, Zel'dovich and Sunyaev (Ref. 1).

[18]G. Smoot *et al.*, Astrophys. J. **331**, 653 (1988); A. Kogut *et al.*, Bull. Am. Phys. Soc. **34**, 1175 (1989); M. Bensadoun

et al., *ibid.* **34**, 1175 (1989); P. Crane *et al.*, Astrophys. J. **346**, 136 (1989); D. M. Meyer *et al.*, *ibid.* **343**, L1 (1989).

[19]Bernstein (Ref. 9), Chap. 7.

[20]It is tempting here to use this analysis to rule out radiating cosmic-string scenarios, since the excess radiation has typical energies much lower than $T_{\rm CBR}$. However, it seems unlikely that the radiation emitted by strings is able to propagate and therefore the Boltzmann treatment given here may not apply. However, the fundamental questions raised by our analysis are how is the energy coming from strings able to *raise* the electron temperature and what is the time scale on which this energy transfer take place?

[21]The concept of the efficiency of the energy transfer was explored by A. Dar, A. Loeb, and S. Nussinov, Astrophys. J. **338**, L41 (1989); G. B. Field and T. Walker, Phys. Rev. Lett. **63**, 117 (1989). The new part of our analysis is the time dependence of ϵ.

[22]See C. G. Lacey and G. B. Field, Astrophys. J. **330**, L1 (1988). They point out the difficulty of injected large amounts of energy into the CBR at late times. The difficulty stems in large part from the drop in the efficiency.

[23]Ordinarily, neglecting helium leads to an error of order 6%. However, with the spectrum of high-energy photons needed to raise T_e, it seems likely that all helium atoms will be ionized as well.

[24]Lightman and Rybicki (Ref. 9), Chap. 10.

[25]The original Fukugita scenario was published in M. Fukugita, Phys. Rev. Lett. **61**, 1046 (1988). For modifications to this scenario, see J. Bernstein and S. Dodelson, *ibid.* **62**, 1804 (1989) and M. Fukugita, *ibid.* **62**, 1805 (1989).

[26]For a different treatment, see J. Silk and A. S. Stebbins, Astrophys. J. **269**, 1 (1983).

[27]Usually a heavy particle has an average momentum of order $\sqrt{MT}$ due to the Boltzmann factor: $e^{-p^2/2MT}$. A heavy neutrino which has a mass smaller than 1 MeV (the decoupling temperature) has a phase-space density $F(p) \sim e^{-p/T_\nu}$ [see J. Bernstein and G. Feinberg, Phys. Lett. **101B**, 39 (1981)]. Therefore, its average momentum is of order T_ν.

[28]See S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), Chap. 15.

[29]In Fukugita's model the Universe starts out matter dominated due to the heavy neutrinos. At a time $t \simeq \tau$ the relativistic decay products dominate the Universe, so it becomes radiation dominated after this time. A similar calculation to that in the text can be carried out assuming matter domination; the distortion parameter is smaller by a factor of 1.25.

[30]Fukugita and collaborators have explored many of the constraints on decaying neutrinos. After this work was completed, we received M. Fukugita and M. Kawasaki, TU Report No. 339, 1989 (unpublished); in addition to discussing the constraints in some detail, they also treat numerically many of the issues we have discussed and come to very similar conclusions.

[31]Bernstein (Ref. 9); Fukugita (Ref. 30).

[32]M. Fukugita, M. Kawasaki, and T. Yanogida, Astrophys. J. **342**, L1 (1989).

[33]F. W. Stecker, Phys. Rev. Lett. **63**, 117 (1980).