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Topological fermionic string representation for Chern-Simons non-Abelian gauge theories

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We show that loop wave equations in non-Abelian Chern-Simons gauge theory are exactly solved by a conformally invariant topological fermionic string theory.

Recently, it was suggested in Ref. 1 that topological non-Abelian quantum field theories in three dimensions (3D) may be solved exactly by means of a noncritical fermionic string theory. The correctness of this string representation holds great potential for high-T_c superconductivity since it produces evidence in favor of a fermionic string picture for the fermionic magnons advocated in Ref. 2.

In this Rapid Communication we address the problem of solving exactly the Chern-Simons loop wave equation in the formalism proposed by the present author in Refs. 3 and 4.

Let us start our analysis by considering a set of multiplet scalar fields $\beta(x)$ interacting with an SU(N) non-Abelian Chern-Simons gauge theory (in the Euclidean sector) in 3D with a nongauged "flavor" group SO(M):

$$\mathcal{L}(\beta,\beta^\dagger,A_i^{(a)})=\tfrac{1}{4}\,|(\partial_i-gA_i)\beta|^2(x)+\epsilon^{ijk}\mathrm{Tr}[A_i(\partial_jA_k-\partial_kA_j+\tfrac{2}{3}[A_j,A_k]_-)](x),\ \ [i=1,2,3;(a)=1,\ldots,M]\,. \tag{1}$$

Physically the Lagrangian in Eq. (1) may be thought of as the effective Lagrangian obtained by integrating out the quark sector of the Weinberg-Salam electroweak theory at finite temperature and in the very-low-energy regime.[5] After integrating out the Gaussian action of the scalar field $\beta(x)$ and expressing the resulting functional determinant as a functional in the bosonic loop space (Refs. 4 and 6) we get the following expression for the theory's Euclidean vacuum energy (Ref. 7):

$$Z=\left\langle \exp\left(-\sum_{C_{xx}}\mathrm{Tr}^{(c)}\Phi^{\mathrm{CS}}[C_{xx}]\right)\right\rangle, \tag{2}$$

where $\Phi^{\mathrm{CS}}[C_{xx}]$ is the usual (normalized) Mandelstam loop defined by the loop C_{xx} and the Chern-Simons gauge field $A_i^{(a)}(x)$. The quantum average in Eq. (2) is defined by the pure Chern-Simons action of Eq. (1) and the sum over the loops $C_{xx}=\{X_i(\sigma),0<\sigma<T\}$ is given by the bosonic loop path integral

$$\sum_{C_{xx}}=-\int_0^\infty\frac{dT}{T}\int d^3x\int_{\substack{X(0)=x\\X(T)=x}}D^F[X(\sigma)]\exp\left(-\tfrac{1}{2}\int_0^T\dot{X}^2(\sigma)d\sigma\right). \tag{3}$$

In Ref. 1 the factorization (Ref. 6) of the averages of the products of Wilson loops on the basis of a diagrammatic analysis was presented. As a consequence of this result the nontrivial dynamical content of Eq. (2) is entirely given by the quantum Wilson loop which in turn is a matrix in the "flavor" space SO(M):

$$W_{(a)(b)}[C_{xx}]=\frac{1}{N}\left\langle \mathrm{Tr}^{(c)}P\left[\exp\left(i\oint_{C_{xx}}A_i(X(\sigma))dX_i(\sigma)\right)\right]\right\rangle. \tag{4}$$

In order to deduce a loop wave equation for $W_{(a)(b)}[C_{xx}]$, as in Ref. 4, Appendix A, we at first consider the covariant version of the loop C_{xx} by introducing an intrinsic metric $e(\sigma)$ on it,

$$C_{xx}=\{(X_i(\sigma),e(\sigma));\,0\le\sigma\le 2\pi;\,X_i(0)=X_i(2\pi)=x\}\,,$$

and replacing in Eq. (4) the tangent loop vector $dX_\mu(\sigma)$ by its covariant version $dX_i(\sigma)/e(\sigma)$. By shifting the $A_i(x)$ variable and introducing the Mandelstam scalar area derivative $\delta|\sigma|(X(\sigma'))$ at an arbitrary point $X(\sigma')\in C_{xx'}$ (Ref. 8) we get the following unrenormalized covariant loop equation ($\lambda=g^2N$):[1,4]

$$\frac{\delta}{\delta|\sigma|(X(\sigma'))}W_{(a)(b)}[C_{X(0),X(2\pi)}]=\lambda\int_0^{2\pi}\frac{dX_i(\sigma)}{e(\sigma)}\frac{dX_j(\sigma')}{e(\sigma')}X_k(\sigma')\epsilon_{ijk}\delta^{(3)}(X(\sigma)-X(\sigma'))$$
$$\times W_{(a)(c)}[C_{X(0),X(\sigma)}]W_{(c)(b)}[C_{X(\sigma),X(2\pi)}] \tag{5}$$

where the line integral $\int_0^{2\pi}$ means that only the nontrivial self-intersection loop points $X_l(\sigma)=X_l(\sigma')$ with $\sigma\neq\sigma'$ contribute to the integrand in Eq. (5) since the condensate term $\langle F^2(x)\rangle$ vanishes identically in Chern-Simons gauge theories (see Appendix A of Ref. 4).

In order to solve Eq. (5) by means of a string theory let us consider an arbitrary (but fixed) 3D surface

$$\Sigma=\{\phi_i(\sigma,\zeta);\, 0\le\sigma\le 2\pi;\, 0\le\zeta\le T;\, i=1,2,3\}$$

possessing as a boundary the loop C_{xx} [this will always be possible if Σ is a homology three-sphere (Ref. 1)].

Let us introduce in Σ an $O(M)$ (neutral) spinor structure $\psi_{(a)}(\sigma,\zeta)$ together with a metric structure $\{g_{\mu\nu}(\sigma,\zeta);\mu,\nu=1,2\}$. We, thus, consider the following $O(M)$ fixed-area string propagator (the reader should compare this with the QCD[SU(∞)] string propagator of Ref. 4):

$$G_{(a)(b)}(C_{xx};A)=\int D^c[g_{\mu\nu}]D^c[\psi_{(a)}]\psi_{(a)}(0,0)\bar\psi_{(b)}(2\pi,0)\times\delta\left[\int_0^{2\pi}d\sigma\int_0^T d\zeta\sqrt{g(\sigma,\zeta)}-A\right]\exp\left[-\int_0^{2\pi}d\sigma\int_0^T d\zeta(\bar\psi D\!\!\!/_g\psi)(\sigma,\zeta)\right]$$
$$\times\exp\left[-\lambda \fint_0^{2\pi}d\sigma\frac{\dot X_l(\sigma)}{e(\sigma)}\fint_0^{2\pi}d\sigma'\int_0^T d\zeta'[\sqrt{g(\sigma',\zeta')}(\bar\psi\psi)(\delta',\zeta')\times\delta^{(3)}(\phi_l(\sigma',\zeta')-X_l(\sigma))\epsilon^{ljk}T_{jk}(\phi_l(\sigma',\zeta'))]\right] \quad (6)$$

where $D\!\!\!/_g$ denotes the covariant Dirac operator associated with the intrinsic metric $g_{\mu\nu}$,

$$T_{jk}(\phi(\sigma',\zeta'))=[(1/\sqrt{h})\epsilon^{\mu\nu}\partial_\mu\phi^j\partial_\nu\phi^k](\sigma',\zeta')$$

is the (normalized) orientation tensor of the surface Σ at the point $\phi_l(\sigma',\zeta')$ and $\fint$ means that only the nontrivial self-intersection points of the surface Σ with its boundary C_{xx} contribute. The intrinsic metric $g_{\mu\nu}$ satisfies the boundary condition $\lim_{\zeta\to 0^+}\sqrt{g(\sigma,\zeta)}=e(\sigma)$ and the intrinsic fermions $\psi(\sigma,\zeta)$ satisfy the Neumann condition $\lim_{\zeta\to 0^+}\partial_\sigma\psi(\sigma,\zeta)\equiv 0$.

Let us remark that the λ-interaction term in Eq. (6) for nondynamical fermions, $(\bar\psi_{(a)}\psi_{(a)})(\sigma,\zeta)=\mu=$const, is topologically invariant, being an entanglement index of the loop C_{xx} with respect to the surface Σ. As a result our string propagator depends functionally only on the topological class of the Σ surface. This is one of the reasons that we do not consider surface fluctuations in the above-written string propagator.

It is important to point out that it is inconsistent to consider string solutions for Eq. (5) which have surface fluctuations since these fluctuations will lead one to consider second-order loop wave equations for $W_{(a)(b)}[C_{xx}]$ as in QCD [SU(ω)] which is not the case in Chern-Simons gauge theory since it has a nondynamical content $\langle\nabla_i F_{ik}(x)W[C_{xx}]\rangle\equiv 0$. However, the area A induced by the intrinsic fluctuating metric $g_{\mu\nu}$ still is a variable quantity since the metric structure on Σ is fluctuating in Eq. (6). So, our string representation differs from that suggested in Ref. 1, Eq. (22). Another important remark to be pointed out is related to the conformal invariance of the $O(M)$ string propagator in Eq. (6). This propagator has its conformal anomaly canceled if $M=26$, producing, thus, a noncritical string.

Let us show that $G_{(a)(b)}(C_{xx},A)$ satisfies the same loop equation, Eq. (5). In order to write the area equation for $G_{(a)(b)}(C_{xx},A)$ we evaluate its area partial derivative as in Eqs. (4)–(7) of Ref. 3:

$$\frac{\partial}{\partial A}G_{(a)(b)}(C_{xx},A)=-\lim_{\zeta\to 0^+}\left[\int D^c[g_{\mu\nu}]\delta\left[\int_0^{2\pi}d\sigma\int_0^T d\zeta\sqrt{g(\sigma,J)}-A\right]\left[-\frac{1}{2\sqrt{g}g^{00}}\frac{\overleftrightarrow{\delta}}{\delta g_{00}(\bar\sigma,\zeta)}\right]I_{(a)(b)}[\psi_{(a)},-g_{\mu\nu}]\right], \quad (7)$$

where the pure fermionic string propagator is

$$I_{(a)(b)}[\psi,g_{\mu\nu}]=\int D^c[\psi]\psi_{(a)}(0,0)\bar\psi_{(b)}(2\pi,0)\exp\left[-\int_0^{2\pi}d\sigma\int_0^T d\zeta(\bar\psi D\!\!\!/_g\psi)(\sigma,\zeta)\right]$$
$$\times\exp\left[-\lambda\fint_0^{2\pi}d\sigma\frac{\dot X_l(\sigma)}{e(\sigma)}\fint_0^{2\pi}d\sigma'\int_0^T d\zeta'\sqrt{g(\sigma',\zeta')}(\bar\psi\psi)(\sigma',\zeta')\delta^{(3)}\times(\phi_l(\sigma',\zeta')-X_l(\sigma))e^{ljk}T_{jk}(\phi_l(\sigma',\zeta'))\right]. \quad (8)$$

By canceling the conformal anomaly by choosing $M=26$ and evaluating the boundary limit of Eq. (8) as Eqs. (27)–(29b) of Ref. 4 we get the following result for the right-hand side of Eq. (7):

$$\frac{\partial}{\partial A}G_{(a)(b)}(C_{X(0);X(2\pi)};A)=\lambda\fint_0^{2\pi}dX_i(\sigma)dX_j(\sigma')X_k(\sigma')\epsilon_{ijk}\delta^{(3)}(X_l(\sigma)-X_l(\sigma'))\times G_{(a)(c)}(C_{X(0)X(\sigma)};A)G_{(c)(b)}(C_{X(\sigma)X(2\pi)};A). \quad (9)$$

The above-written equation coincides with the Chern-Simons loop wave equation in the loop proper-time gauge.

As in our earlier proposed formal string-wave-equation calculus we have worked with singular quantities. A useful regularization scheme for these self-suppressing terms is analyzed in Ref. 9 (for a perturbative regularization scheme see Ref. 10). Work on this regularization problem will be reported elsewhere.

[1]M. A. Awada, Phys. Lett. B **221**, 21 (1989).

[2]I. E. Ozaloshinski, A. M. Polyakov, and P. B. Wiegmann, Phys. Lett. A **127**, 112 (1988); X. G. Wen and A. Zee, Phys. Rev. Lett. **62**, 1937 (1989).

[3]Luiz C. L. Botelho, Phys. Rev. D **40**, 660 (1989).

[4]Luiz C. L. Botelho, J. Math. Phys. **30**, 2160 (1989).

[5]A. Niemi and G. Semenoff, Phys. Rev. Lett. **51**, 277 (1983).

[6]Yu. Makeenko and A. A. Migdal, Nucl. Phys. **B188**, 269 (1981).

[7]Edward Witten, Commun. Math. Phys. **121**, 351 (1989).

[8]P. Olesen and J. L. Petersen, Nucl. Phys. **B181**, 157 (1981).

[9]X. Fustero, R. Gambini, and A. Trias, Phys. Rev. Lett. **62**, 1964 (1989).

[10]M. Kardar and D. R. Nelson, Phys. Rev. Lett. **58**, 1289 (1987).