Hot δ expansion

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The δ expansion, a novel perturbative scheme for quantum field theories, is investigated in the case of finite temperatures. For the exactly solvable example of the large-N limit of massless scalar four-dimensional ϕ^4 theory, the first two orders of the δ expansion are compared with ordinary perturbation theory which at finite temperature is infrared singular beyond one-loop order. The δ expansion gives results which are nonperturbative with respect to the coupling constant and which compare favorably with the exact solution. Most importantly, the δ series naturally solves the general infrared problem of perturbative high-temperature field theories.

Ordinary weak-coupling perturbation theory is deficient in the case of massless quantum field theories at finite temperature, and generally so in the high-temperature limit where the bare masses are negligible compared with the temperature. Typically, an obstruction appears in the perturbative calculation at some order in the coupling constant, depending on the quantity under consideration, beyond which infrared divergences block further calculations.[1,2] As a prominent example, the magnetic screening mass in perturbative thermal quantum chromodynamics is completely incalculable due to this obstruction.

A novel perturbative scheme called the δ expansion has been proposed and elaborated recently[3] which uses an artificial parameter δ to expand a theory in a power series, thereby liberating the physical coupling constants from appearing in lowest powers only. It has been successfully applied to scalar[3] and fermionic[4] field theories, and, most recently, progress has been made in the case of Abelian gauge theories.[5]

In this Brief Report we shall consider a massless scalar four-dimensional ϕ^4 $[(\phi^4)_4]$ theory with global symmetry O(N), defined by the Lagrangian

$$L=\tfrac{1}{2}(\partial_\mu\phi)^2+\frac{\lambda}{N}\phi^4\,,\qquad \phi^2=\phi_a\phi_a,\quad a=1,\dots,N\,. \tag{1}$$

In the limit $N\to\infty$ this theory can be solved exactly so that we can compare the conventional weak-coupling expansion and the δ expansion with the exact solution. Finite temperature induces a mass $m(\lambda)\propto T$ for the scalar fields ϕ and conventional perturbation theory is only able to determine $m(\lambda)$ to order λ^1; infrared divergences set in at two loops.

With $N\to\infty$ the Schwinger-Dyson equation for the propagator can be solved to give the implicit gap equation[6]

$$m^2=4\lambda I(m^2)\,, \tag{2}$$

where

$$I(m^2)=\int\frac{d^4p}{(2\pi)^4}\,\frac{2\pi\delta(p^2+m^2)}{e^{|p_0|/T}-1}=\frac{T^2}{2\pi^2}\int_{m/T}^{\infty}dx\,\frac{[x^2-(m/T)^2]^{1/2}}{e^x-1}\,. \tag{3}$$

Notice that there are no infrared divergences in the exact solution (2).

For I the following series representation[7] will be useful for $m/T<2\pi$:

$$\frac{2\pi^2}{T^2}I(m^2)=\sum_{k=1}^{\infty}\frac{m}{kT}K_1(km/T)$$
$$=\frac{\pi^2}{6}-\frac{\pi m}{2T}-\frac{1}{4}(m/T)^2\left[\ln\frac{m}{4\pi T}+\gamma-\frac{1}{2}\right]-\frac{1}{4}(m/T)^2\sum_{n=1}^{\infty}(-1)^n\frac{(2n)!}{(n+1)!n!}\zeta(2n+1)\left[\frac{m}{4\pi T}\right]^{2n}, \tag{4}$$

where $K_1(x)$ is the modified Bessel function of the second kind and order 1, γ is Euler's constant, and $\zeta(x)$ is the Riemann ζ function.

Solving Eqs. (2) and (4) for $m^2(\lambda)$ and small λ yields

$$m^2(\lambda)=\frac{T^2}{3}\left[\lambda-\frac{\sqrt{3}}{\pi}\lambda^{3/2}+\frac{\lambda^2}{4\pi^2}\left[7-2\gamma-\ln\frac{\lambda}{48\pi^2}\right]\right] +O(\lambda^{5/2})\ . \tag{5}$$

Clearly, a perturbative expansion in powers of λ is bound to fail—only the first derivative of (5) with respect to λ exists at $\lambda=0$. A conventional perturbation series is a formal Taylor series in integer powers of λ, so the result in (5) is inaccessible to conventional perturbative methods. The failure of conventional perturbation theory to reproduce (5) manifests itself by infrared divergences appearing in higher-order diagrams.

In the δ expansion, instead of using λ as an expansion parameter, one introduces an artificial perturbation parameter δ which characterizes the degree of nonlinearity of the interactions rather than their strength:

$$\begin{aligned} L^{(\delta)}&=\tfrac{1}{2}(\partial_\mu\phi)^2+\lambda M^2\phi^2\left[\frac{\phi^2}{NM^2}\right]^\delta \\ &=\tfrac{1}{2}(\partial_\mu\phi)^2+\tfrac{1}{2}(2\lambda M^2)\phi^2 \\ &\quad+\lambda M^2\phi^2\sum_{n=1}^{\infty}[\ln(\phi^2/NM^2)]^n\frac{\delta^n}{n!}\ , \end{aligned} \tag{6}$$

where M is an arbitrary mass parameter introduced to keep λ dimensionless.

The δ expansion apparently leads to a highly nonpolynomial Lagrangian. Although it is usually quite difficult to calculate with nonpolynomial Lagrangians, Ref. 3 formulates simple diagrammatic rules that determine the coefficients of a power series [8] in δ. These rules will not be repeated here. We only remark that these new diagrammatic rules are manifestly free of infrared difficulties because now the propagators are massive with squared mass $2\lambda M^2$. Since M is arbitrary, we can use this parameter to optimize the series expansion in δ. We will consider the first two orders of the δ series and demonstrate their numerical accuracy by comparing them with the exact result.

In the first order of the δ expansion one obtains

$$\begin{aligned} m_{(1)}^2&=2\lambda M^2+\delta[2\lambda M^2(1+L)],\quad \delta=1\ , \\ L&\equiv \ln I(2\lambda M^2)/M^2\ . \end{aligned} \tag{7}$$

This result depends on the arbitrary parameter M, which we propose to determine by an optimization procedure called the principle of minimal sensitivity[9] (PMS); to wit, we will require that

$$\frac{\partial}{\partial M^2}m_{(K)}^2=0 \tag{8}$$

for the result of order K in the δ expansion.

At $K=1$, this determines M through the implicit equation

$$M^2=I(2\lambda M^2)\exp[1+2\lambda M^2 I'(2\lambda M^2)/I(2\lambda M^2)]\ , \tag{9}$$

and the resulting $m_{(1)}(\lambda)$ is plotted in Fig. 1. The first-

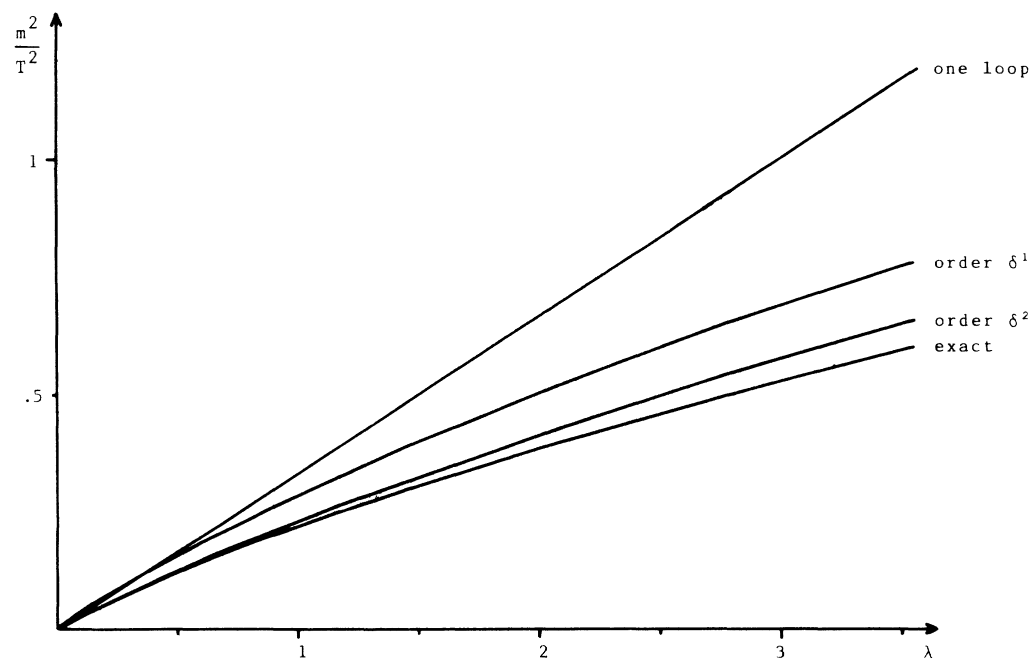

FIG. 1. A comparison of the exact solution in Eq. (2) with three approximations to the exact solution, the conventional one-loop approximation, the first-order δ series, and the second-order δ series, all plotted as functions of the coupling constant λ. Observe that the one-loop approximation veers away from the exact solution as λ increases but that the approximations obtained from the δ expansion are almost uniformly accurate (cf. Fig. 2).

order result is a highly nontrivial function of the coupling constant λ and is thus capable of containing nonperturbative information (with respect to λ).

Let us compare the result in (9) with that obtained by using conventional perturbation theory. To first order in λ, ordinary perturbation theory gives simply $m^2(\lambda)=\lambda/3$; beyond first order in λ, the conventional loop expansion runs into infrared divergences. Nevertheless, this first-order result expresses the correct behavior in the limit $\lambda \to 0$, whereas, to first order in the δ expansion,

$$M^2 \to \frac{e}{12}T^2 \Longrightarrow m^2(\lambda) \to \frac{e}{6}\lambda T^2 , \qquad (9')$$

and the relative error for small λ is approximately 36%. However, as Fig. 1 shows, the error of the one-loop approximation explodes for large values of the coupling constant, and the first-order δ expansion is superior beyond $\lambda \approx 0.3776$. Moreover, the relative error between the first-order δ result $m^2_{(1)}(\lambda)$ and the exact result appears to be almost uniform in λ, thus providing a good qualitative picture of the true nonperturbative behavior. This is most conspicuously demonstrated in Fig. 2 where the percentage relative error between the respective approximation and the exact soluton is plotted over λ.

The figures also display the result of the second-order calculation in δ, which is determined by

$$m^2_{(2)} = \lambda M^2[4+4L+L^2+4\lambda M^2(1+L)I'/I] , \qquad (10)$$

where $I=I(2\lambda M^2)$. By invoking the PMS, M is fixed implicitly by

$$0=\frac{\partial}{\partial M^2}m^2_{(2)} = 2L+L^2+12\lambda M^2(1+L)I'/I - 8\lambda^2 M^4 L I'^2/I^2 + 8\lambda^2 M^4(1+L)I''/I . \qquad (11)$$

In fact, there are now two distinct solutions to (11): one of which corresponds to $L \to -2$ when $\lambda \to 0$, implying a completely wrong asymptotic behavior $m^2_{(2)} \to 0 \times \lambda$, and another one with $L \to 0$ and $m^2_{(2)} \to \lambda T^2/3$, which surprisingly, is exact to order λ. The result based on the second solution is plotted in Figs. 1 and 2. Note that it gives a significant improvement over the first-order δ result, and it is now better than the conventional perturbative result for all values of the coupling constant.

The ambiguity we have encountered in implementing

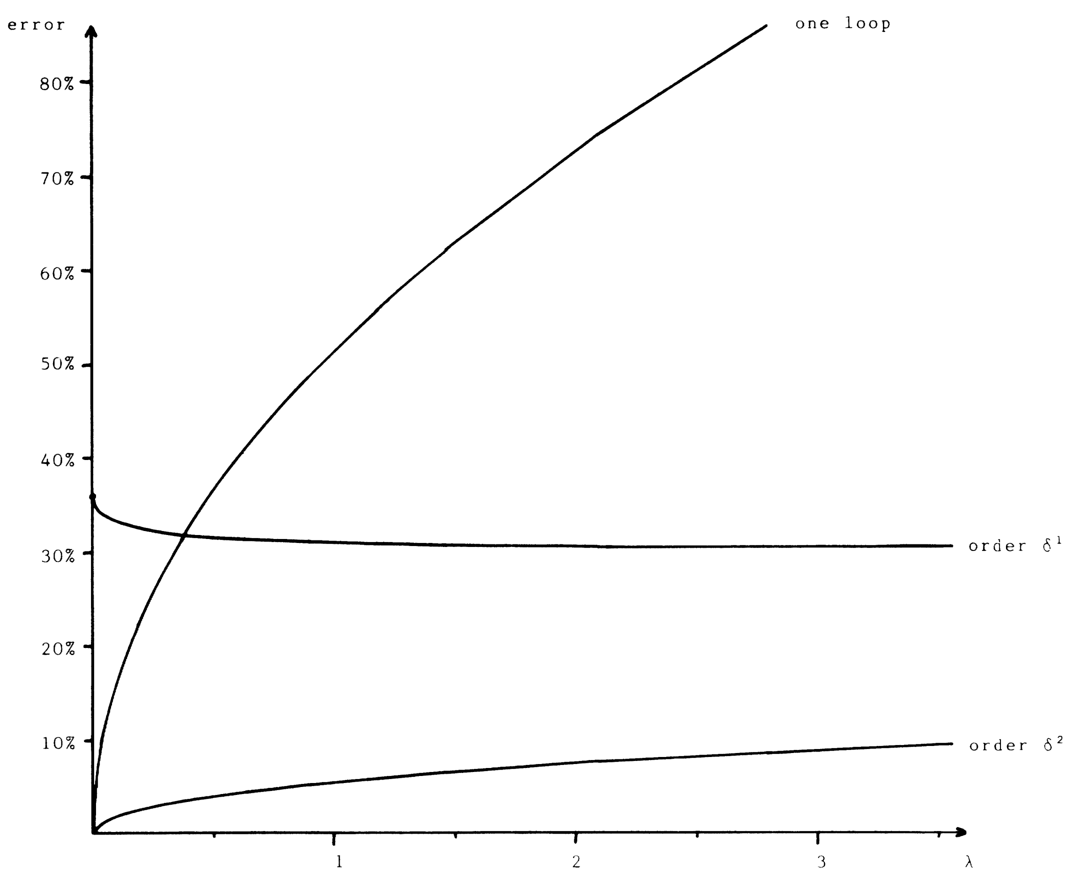

FIG. 2. The relative errors between the exact solution in Eq. (2) and the three approximations to the exact solution shown in Fig. 1. This figure shows clearly that the relative error of the first-order δ-series approximation depends only weakly on λ. The one-loop approximation, on the other hand, rapidly becomes poor with increasing λ. The second-order δ-series approximation is extremely accurate for all values of λ.

the PMS at second order in δ can in fact be resolved by considering the analogous issue at higher orders. In the limit $\lambda \to 0$ the Kth-order δ result is given by the comparatively simple expression

$$m^2_{(K)} \to 2\lambda M^2 \left\{ 1 + \sum_{n=1}^{K} \left[\frac{L^{n-1}}{(n-1)!} + \frac{L^n}{n!} \right] \right\} ,$$
$$L = \ln \frac{I(0)}{M^2} , \tag{12}$$

and the PMS condition in this limit reads

$$L^K + K L^{K-1} = 0 , \tag{13}$$

which for all $K \geq 2$ has two solutions: $L = -K$ and $L = 0$. These solutions correspond to extrema of the order 1 and $K-1$, respectively. In the limit $K \to \infty$ the result should become completely independent of M, so only the second solution is acceptable.

In conclusion, the lowest-order results in the δ expansion optimized by the PMS provide a good approximation to the exact solution. Even more significantly, the δ expansion automatically solves the infrared problems that generally occur in ordinary perturbation theory for massless field theories at finite temperature and also for massive field theories in the high-temperature limit.

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