Spacetimes admitting a three-parameter group of isometries and quasilocal gravitational mass

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Spacetimes admitting a three-parameter group of motions multiply transitive on two-dimensional spacelike orbits allow construction of a simple geometrical invariant out of the magnitudes of Killing fields, their gradients, and the metric. We show that this invariant covariantizes and geometrizes certain forms of "quasilocal mass" introduced and discussed in the literature, in particular, Misner's mass in spherical symmetry and Taub's "mass" in plane symmetry.

Let us suppose that $(M,g_{\mu\nu})$ represents an arbitrary spherically symmetric spacetime. That implies that there exists $\xi_{(i)}$, $i=1,2,3$ spacelike Killing vector fields obeying the algebra of SO(3), i.e.,

$$[\xi_{(i)},\xi_{(j)}]=\epsilon_{ijk}\xi_{(k)} \; . \tag{1}$$

Using the spacetime metric $g_{\mu\nu}$ as well as the above $\xi_{(i)}$, we form the following invariant:

$$m(J)=\frac{L}{2}(1-g^{\mu\nu}\nabla_\mu L\nabla_\nu L) \tag{2}$$

where

$$L^2=\tfrac{1}{2}g_{\mu\nu}(\xi^\mu_{(1)}\xi^\nu_{(1)}+\xi^\mu_{(2)}\xi^\nu_{(2)}+\xi^\mu_{(3)}\xi^\nu_{(3)}) \; . \tag{3}$$

By construction $m(J)$ is constant over any two-dimensional spacelike orbit J of the isometry group, i.e., it satisfies $\xi^\mu_{(i)}(\partial m/\partial x^\mu)\equiv 0$, $i=1,2,3$. Furthermore, it is identical to some expressions of quasilocal gravitational mass.

Let us, for example, consider a well-known spacetime: the Reissner-Nordström black hole. The metric can be written in the familiar form

$$ds^2=-f\,dt^2+\frac{dr^2}{f}+r^2(d\theta^2+\sin^2\theta\,d\varphi^2)\;, \qquad f=1-\frac{2m}{r}+\frac{e^2}{r^2} \tag{4}$$

and one can easily show that (2) reduces as follows:

$$m(r)=m-\frac{1}{2}\frac{e^2}{r} \; . \tag{5}$$

Thus it is immediately recognized that the geometrical quantity (2) is equal to the amount of mass-energy contained within a sphere of radius r. Furthermore, (5) indicates that at the limit of infinite spacelike distance from the origin $m(J)$ reduces to the familiar Arnowitt-Deser-Misner (ADM) mass. To see if the above property of $m(J)$ persists, we will evaluate (2) for a more general class of spherically symmetric spacetime. Without loss of generality we can always locally write the line element in the following form:

$$ds^2=-e^\nu dt^2+e^\lambda dr^2+R^2(d\theta^2+\sin^2\theta\,d\varphi^2) \qquad \nu=\nu(t,r),\;\; \lambda=\lambda(t,r),\;\; R=R(t,r) \; . \tag{6}$$

$t=$const foliates the spacetime by spacelike hypersurfaces while no physical meaning is attributed to r, which simply measures coordinate distances away from the center of symmetry.[1] An arbitrary orbit is parametrized by the coordinates (t,r) and using (6) it is easy to show that (2) reduces to the following expression:

$$m(t,r)=\frac{R}{2}\left[1+e^{-\nu}\left[\frac{\partial R}{\partial t}\right]^2-e^{-\lambda}\left[\frac{\partial R}{\partial r}\right]^2\right] \; . \tag{7}$$

However, the right-hand side of the above is identical to what Misner and collaborators[2,3] Cahill and McVitie[4] define as the mass-energy contained within a sphere of coordinate radius r at time t. These authors also relate $m(t,r)$ to the curvature tensor of (6) in the following way:

$$m(t,r)=\tfrac{1}{2}R^3R_{\theta\varphi}{}^{\theta\varphi} \; . \tag{8}$$

The noncovariant character of the right-hand side of (7) and (8) is rather displeasing. In this regard it is appropriate to point out that the authors of Ref. 2 suggest a covariant definition for the right-hand side of (8). Specifically, they indicate that one can write the equivalent of (8) as follows:

$$m(t,r)=\frac{R}{8}\alpha^{\mu\nu}\alpha^{\kappa\lambda}R_{\mu\nu\kappa\lambda} \; , \tag{9}$$

where the $\alpha_{\mu\nu}$ is a two-dimensional properly normalized antisymmetric tensor associated with the intrinsic tangent space of the (t,r) orbit. Furthermore, $R\equiv(g_{\theta\theta})^{1/2}$ acquires an invariant definition by relating it to the proper area of the (t,r) orbit. We believe, though, that the new expression (2) provides an additional and easier to handle covariant representation of Misner's quasilocal gravitational mass.

At this point and for later use let us indicate a connection between the proper area A of the orbits and the magnitudes of the Killing fields $\xi_{(i)}$, $i=1,2,3$. It is captured in the following formula:

$$A=\int_{s^2} da = 2\pi[g(\xi_{(1)},\xi_{(1)})+g(\xi_{(2)},\xi_{(2)})+g(\xi_{(3)},\xi_{(3)})] \; . \tag{9a}$$

da stands for the infinitesimal surface elements, while the right-hand side represents the magnitudes of $\xi_{(i)}$. The derivation is rather trivial. In the coordinate system employed in (6) we have

$$A=\int_{s^2} da = 4\pi g^{1/2}\left(\frac{\partial}{\partial\varphi},\frac{\partial}{\partial\varphi}\right)\frac{g^{1/2}(\partial/\partial\theta,\partial/\partial\theta)}{\sin\theta}$$

$$=4\pi g^{1/2}(\xi_{(3)},\xi_{(3)})\frac{g^{1/2}(\sin\phi\xi_{(1)}+\cos\phi\xi_{(2)},\sin\phi\xi_{(1)}+\cos\phi\xi_{(2)})}{\sin\theta} \; ,$$

where we have eliminated the basis vectors $\partial/\partial\theta,\partial/\partial\varphi$ in favor of $\xi_{(i)}$ and the right-hand side of the above is to be evaluated in an arbitrary point of the orbit under consideration. Further, if one expands and eliminates the angle ϕ via

$$\sin^2\phi=\frac{g(\xi_{(1)},\xi_{(1)})-g(\xi_{(2)},\xi_{(2)})+g(\xi_{(3)},\xi_{(3)})}{2g(\xi_{(3)},\xi_{(3)})}$$

the result is equality (9a).

In order to obtain further insight and particularly to provide some physical reasons as to why $m(J)$ should be viewed as a "form of mass," let us assume that (6) describes the geometry of a collapsing configuration.[5] We consider the progress of outgoing congruence of light emitted orthogonally from a given orbit.

Local convergence to the future of J requires that along the rays we should have

$$AL^2=AR^2<0 \; , \tag{10}$$

where

$$A=e^{-\nu/2}\frac{\partial}{\partial t}+e^{-\lambda/2}\frac{\partial}{\partial r}$$

is tangent to the outgoing null rays. Equation (10) is guaranteed when the corresponding orbit J obeys the following condition:

$$g^{\mu\nu}\nabla_\mu L\nabla_\mu L<0 \; . \tag{11}$$

One can easily show now that (11) implies

$$\frac{2m(J)}{L}>1 \tag{12}$$

and vice versa. Inequality (12) justifies us attributing $m(J)$ properties of mass. Whenever it is satisfied, light emitted from such orbits finds itself in a strong field enforcing an initially outgoing congruence to start to reconverge. Of course (12) is a local condition and reveals no information about the future of the congruence. Its future depends whether or not certain inequalities upon the Ricci tensor hold true along the entire future of the congruence.

Finally, one can easily show that for every asymptotically flat at spacelike or future null infinity spacetime $M(J)$ reduces to the ADM or Bondi mass, respectively.

Because of the above described properties of Killing fields for spherically symmetric spacetimes, one wonders whether similar properties hold for the other classes of spacetimes admitting groups of isometries. It is well known that besides the spherically symmetric class of spacetimes there exists another two classes admitting a three-parameter group of spacelike motions multiply transitive on two-dimensional spacelike orbits.[6] They are the plane symmetric and "hyperbolically" symmetric classes. The Killing fields respectively satisfy the following algebra:

$$[\xi_{(1)},\xi_{(2)}]=0,\quad [\xi_{(2)},\xi_{(3)}]=-\xi_{(1)},\quad [\xi_{(3)},\xi_{(1)}]=\xi_{(2)} \; ,$$

$$[\xi_{(1)},\xi_{(2)}]=\xi_{(3)},\quad [\xi_{(2)},\xi_{(3)}]=-\xi_{(1)},\quad [\xi_{(3)},\xi_{(1)}]=\xi_{(2)} \; .$$

However, one crucial difference between the spherically symmetric class and the two classes described above is that the corresponding orbits for the latter are open topologically two planes. Let us, for example, take a closer look at plane-symmetric spacetimes. We can always write the line element in the following form:

$$ds^2=-e^{2\phi}dt^2+e^{2\psi}dz^2+R^2(dx^2+dy^2) \; ,$$

$$\phi=\phi(t,z),\quad \psi=\psi(t,z),\quad R=R(t,z) \tag{13}$$

implying that the associated infinitesimal generators of Euclidean motions on the $t=$const, $z=$const space have the form

$$\xi_{(1)}=\delta^\nu_x\frac{\partial}{\partial x^\nu} \; ,$$

$$\xi_{(2)}=\delta^\nu_y\frac{\partial}{\partial x^\nu} \; , \tag{14}$$

$$\xi_{(3)}=x\delta^\nu_y\frac{\partial}{\delta x^\nu}-y\delta^\nu_x\frac{\partial}{\delta x^\nu} \; .$$

Using this explicit representation of $\xi_{(i)}$, we form the following quantity:[7]

$$M(t,z)=-\frac{L}{2}g^{\mu\nu}\nabla_\mu L\nabla_\nu L \ , \tag{15}$$

where

$$L^2=\tfrac{1}{2}[g_{\mu\nu}(\xi^\mu_{(1)}\xi^\nu_{(1)}+\xi^\mu_{(2)}\xi^\nu_{(2)})] \ . \tag{16}$$

It is easy to show that a combination of (13), (15), and (16) yields

$$M(t,z)\equiv\frac{R}{2}\left[e^{-2\phi}\left[\frac{\partial R}{\partial t}\right]^2-e^{-2\psi}\left[\frac{\partial R}{\partial z}\right]^2\right] . \tag{17}$$

This quantity formally plays a role similar to the corresponding Misner's mass in the spherically symmetric case. The similarity is justified by the following easily verifiable relation between M and the curvature tensor of (13):

$$M(t,z)=\frac{R^3}{2}R_{xy}{}^{xy} \ .$$

In addition, if one uses the left-hand side of (17), then the Einstein nonvacuum field equations have a simple and elegant form.[8] However, because of plane symmetry we have now lost a physical interpretation of $M(J)$ as a quantity measurable from spacelike infinity, characterizing the gravitational field of a bounded source. If one imagines some plane-symmetric distribution of matter, then, again due to plane symmetry, asymptotic flatness is lost. Therefore, we cannot associate in the familiar way an ADM type of mass to the gravitational field. However, we might attempt to interpret (15) as a "local mass." To make our point, let us again consider a locally collapsing plane symmetric spacetime. For such spacetimes, an observer "attached" to a specific orbit J feels that the proper area R of a small coordinate rectangle is a decreasing function of his proper time.[9] Similar analysis as the one in the spherically symmetric case leads us to the conclusion that both congruences of light emitted orthogonally from a given orbit J locally converge to the future of J provided that L [see (16)] obeys (11), i.e.; the same inequality as in the case of spherical symmetry. In view of (15) and the positiveness of L, this implies $M>0$. Conversely, if in this collapsing configuration there exist orbits characterized by $M>0$, then light is locally trapped. Thus M appears to affect the expansion of null congruences. We may note that for Taub's static vacuum plane-symmetric solution of Einstein's equation,[8] $M(J)$ is a negative constant, i.e., behaves as (2) does for a negative mass Schwarzschild solution. For the other known vaccum plane-symmetric solution of Einstein's equations, i.e., the special case of Kasner metric, M is positive. However, a physical interpretation is unclear.

Although not enough results are known, we expect the above discussion to be extended without any difficulties to the hyperbolic class of spacetimes. Finally, we should mention that Thorne's C energy of cylindrically symmetric spacetimes[10] also finds a covariant representation by means of the two-spacelike Killing vectors, which generate translations and rotations along and around the symmetry axis.

We can conclude from the above discussion that infinitesimal generators of isometries in combination with the metric offer the possibility of constructing nonlocal invariants that may be of physical importance.

Note added: After this paper was submitted for publication I became aware of the recent work of Poisson and Israel.[11] In their fascinating treatment of the mass inflation of spherical black holes they also advanced a covariant definition of what we have called Misner's mass. Their definition is essentially identical to ours except that they use the proper area of SO(3) orbits instead of the norm of the Killing vector fields. Of course, in view of (9a) the two definitions are equivalent to each other. I have also become aware of the work of Berezin *et al.*[12] Although in their work one can see seeds for a covariant definition of m, no explicit covariant formula has actually been written.

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[1]It is assumed that there exists a unique center of spherical symmetry, i.e., that $t\equiv$const hypersurfaces are nested by a sequence of spheres and furthermore, $R(0,t)=0$; however, similar considerations hold if a Kantowski-Sachs or a Friedmann-Robertson-Walker model of spherical symmetry is considered. [The latter model, although homogeneous and isotropic, allows orbits invariant under SO(3).]

[2]C. W. Misner and D. H. Sharp, Phys. Rev. **136**, B571 (1966).

[3]C. W. Hernandez and C. W. Misner, Astrophys. J. **143**, 452 (1966).

[4]M. E. Cahill and G. S. McVittie, J. Math. Phys. **11**, 1362 (1970).

[5]By that term we mean that $e^{-\nu/2}(\partial R/\partial t)<0$ and, furthermore, as in Ref. 3 we take $e^{-\lambda/2}(\partial R/\partial r)>0$.

[6]A. Z. Petrov, *Einstein Spaces* (Pergamon, New York, 1969); M. A. H. MacCallum, in *General Relativity: An Einstein Centenary Survey,* edited by S. W. Hawking and W. Israel (Clarendon, Oxford, 1979).

[7]We prefer to examine (15) rather than (2) because the former is normalized to zero in flat spacetime.

[8]A. H. Taub, in *General Relativity,* edited by L. O'Raifeartaigh (Clarendon, Oxford, 1972). In the above ariticle an overview of plane symmetry is presented and references to earlier work are cited.

[9]We will assume, as in spherical symmetry, that $e^{-\psi}(\partial R/\partial z)>0$. In this way, some plane crossing singularities are also avoided.

[10]K. S. Thorne, Phys. Rev. **138**, B251 (1965).

[11]E. Poisson and W. Israel, Phys. Rev. D **41**, 1796 (1990).

[12]N. A. Berezin, V. A. Kuzmin, and I. I. Tkachev, Phys. Rev. D **36**, 2919 (1987).