Secondary structure of a general multiloop amplitude in open-bosonic-string theory

Hitoshi Konno* and Takashi Suzuki
Institute of Physics, University of Tsukuba, Ibaraki 305, Japan
(Received 14 July 1989)

We investigate the structure of a general multiloop amplitude in open-bosonic-string theory. We show that any connected part of the S matrix can be constructed from three kinds of multiloop tadpole, one-loop nonplanar self-energy, and trees due to duality. A separate discussion for the unconnected part is also given. Some rules for calculating a general multiloop amplitude and explicit results for the nonorientable and nonplanar multiloop amplitudes are given in the framework of the Becchi-Rouet-Stora-Tyutin-invariant operator formalism.

It is a well-known fact that any Feynman-like diagram in string theory can be described as a combination of several primitive diagrams and trees due to duality.[1,2] Such a structure can be called a primitive structure. In the open-bosonic-string theory, the primitive diagrams are the following four diagrams: one-loop planar and nonorientable tadpoles, one-loop nonplanar self-energy, and two-loop nonplanar tadpole. The development of the Becchi-Rouet-Stora-Tyutin- (BRST-) invariant operator formalism[3-6] makes it possible to realize this rigorously in terms of operators. In previous papers[5-7] we completed the calculation of these operators and confirmed this fact in the planar case (Ref. 8 see also Refs. 9 and 10). Here we will go on in this program to general cases. For this purpose, we will make clear how these primitive operators appear in general amplitudes and show that there is a structure which may be called a secondary structure. Some rules for calculating general amplitudes will also be discussed.

We will show first that, due to duality, in general the connected part of the string S matrix is made from the following five secondary building blocks: three kinds of multiloop tadpoles (Fig. 1), one-loop nonplanar self-energy, and trees. Each of these three kinds of multiloop tadpoles is constructed from only one of the three primitive tadpoles: one-loop planar, one-loop nonorientable, and two-loop nonplanar tadpoles. Such a structure follows the next five lemmas.

Lemma 1. Three primitive tadpoles are mutually commutable.

Lemma 2. The product of two nonplanar self-energy diagrams is dual to the product of one nonplanar self-energy and one planar tadpole (Fig. 2).[2]

Lemma 3. The nonplanar self-energy diagram is commutable with a tadpole which is made by multiloop tadpoles with another nonplanar self-energy on its external leg (Fig. 3).

Lemma 4. The orientable tadpole can pass through the nonplanar self-energy, while the nonorientable part cannot (Fig. 4).

Corollary 4. The product of the orientable tadpole and the nonplanar self-energy is dual to the product of that tadpole and a planar tadpole (Fig. 5).

Lemma 5. The product of the one-loop nonorientable tadpole and the two-loop nonplanar tadpole is reduced to a product of three one-loop nonorientable tadpoles (Fig. 6).[2] (In Ref. 11, one can also find similar statements to our lemmas except for lemmas 3 and 4.)

These lemmas can be proved by using the duality of the four-point vertex in the orientable cases and also by using the relation illustrated in Fig. 7 in the nonorientable case (see, for example, Fig. 8, the proof of lemma 3). In the context of the BRST-invariant operator formalism, the four-point duality has already been shown by us,[6] and is satisfied off the mass shell. As for the last relation, we can however argue that it is only satisfied in the on-mass-shell amplitude. The reason for this is the following. The left-hand side of Fig. 7 is given by (the notations and conventions follow the ones in Ref. 6–8)

$$\langle V_{12E}|U^{(E)}\langle V_{F34}||R_{EF}\rangle T^{(4)}\ , \tag{1}$$

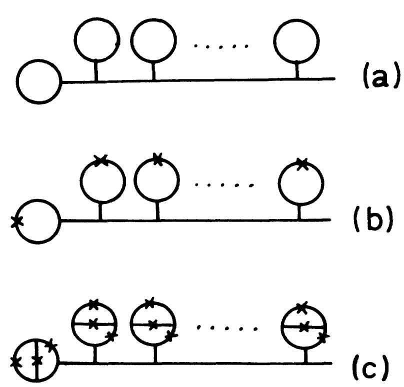

FIG. 1. (a) Planar multiloop tadpole, (b) nonorientable multiloop tadpole, (c) nonplanar multiloop tadpole.

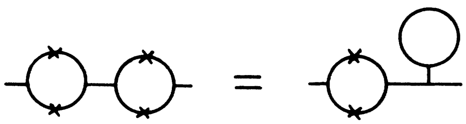

FIG. 2. The product of two nonplanar self-energies is dual to the product of one nonplanar self-energy and one planar tadpole.

where $U \equiv \tilde{\Omega}^{\dagger} T$, $\tilde{\Omega} \equiv \Omega e^{-L_{-1}}$. Using the relation $\langle V_{123}|\prod_{r=1,2,3}\Omega^{(r)\dagger} = \langle V_{132}|$, we can show that (1) turns out to be

$$\langle V_{12E}|T^{(E)}[\langle V_{F43}|\Omega^{(3)\dagger}\exp(L_1^{(F)})\exp(L_1^{(4)})|R_{EF}\rangle]U^{(4)} . \tag{2}$$

To make (2) match the right-hand side of Fig. 7, one must set the extra factors $\exp(L_1^{(r)}), r=F,4$ to one. This is however possible only in the on-shell amplitude. The factor $\Omega^{\dagger}$ on the third leg means that the third leg is twisted when it passes through the twisted line as one can easily see in Fig. 7, and does not affect any proofs because we always use that relation twice in any proof of the lemmas and $\Omega^{\dagger 2}=1$. All the lemmas thus can be proved in the on-shell amplitudes.

From lemma 1, a general multiloop tadpole, which is made by mixing three primitive tadpoles, can be decomposed to the three parts, each of which is made from only one of the primitive tadpoles. From lemma 2, we do not have to consider the product of more than two nonplanar self-energy insertions. From lemma 3 and corollary 4, in

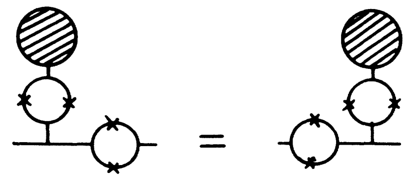

FIG. 3. Nonplanar self-energy is commutable with a tadpole made by multiloop tadpoles with another nonplanar self-energy on its external leg. The shadowed tadpole means a general multiloop tadpole made from three primitive tadpoles.

the orientable case the nonplanar self-energy does not appear except for the case in which at least one of the legs is contracted by the tree of the external particles. In the nonorientabe case, from lemmas 1, 3, 4, and 5, we do not have to consider the nonplanar two-loop tadpole.

As a result of these properties, we conclude that any connected part of the S matrix in the open-bosonic-string theory can be constructed from the three types of multiloop tadpoles, one-loop nonplanar self-energy and trees with the simplification mentioned above. One may call such a structure a secondary structure.

Let us next consider some typical cases and realize them in terms of operators. In these cases, we can make a rule which indicates what kinds of factor one should add to the on-shell amplitudes.

Let us first consider the product of two tadpoles. In general, a tadpole-type operator with l loops is given by the form

$$\langle \mathcal{T}^{(l)}| = \int \prod_{i=1}^{l} d^{26}k_i \int d\mu^{(l)} F^{(1)}[G^{(l)}]^{-26} F^{(2)}[G^{(l)}]^{2} \langle 0_a; q=3| \exp(E^{(l)})(b_{-1}-b_0+\Lambda^{(l)}b_0) , \tag{3}$$

$$F^{(s)}[G^{(l)}] \equiv \prod_{\gamma\in G^{(l)}}{}' \prod_{n=s}^{\infty} [1-w(\gamma)^n], \quad s=1,2 , \tag{4}$$

$$E^{(l)} = E^{(l)X} + E^{(l)\mathrm{gh}} ,$$

$$E^{(l)X} \equiv -\tfrac{1}{2}\sum_{i,j=1}^{l} k_i\cdot k_j (2\pi\,\mathrm{Im}\tau)_{ij} - \sum_{i=1}^{l}\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} A_n^i k_i - \sum_{n,m=1}^{\infty} \frac{a_n}{\sqrt{n}} Q_{nm} \frac{a_m}{\sqrt{m}} , \tag{5}$$

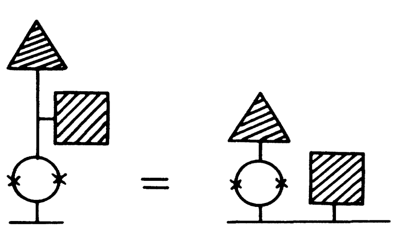

FIG. 4. Only the orientable part of tadpole can pass through nonplanar self-energy. The shadowed square and triangle tadpoles mean orientable and nonorientable multiloop tadpoles, respectively.

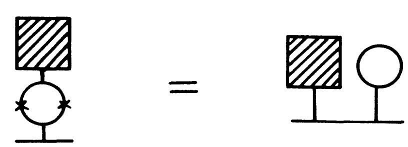

FIG. 5. The product of orientable tadpole and nonplanar self-energy is dual to the one of that tadpole and planar tadpole.

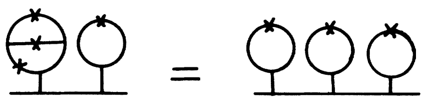

FIG. 6. Two-loop nonplanar tadpole coupled with nonorientable tadpole is dual to three nonorientable tadpoles.

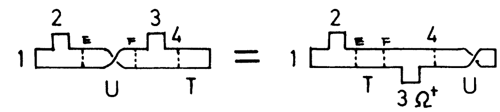

FIG. 7. The third leg can move to the left of the twist accompanying the extra twist.

$$E^{(l)\,\mathrm{gh}} \equiv -\sum_{n,m\geq 2} c_n B_{nm} b_m \,, \tag{6}$$

where k_i are loop momenta. $F^{(s)}[G^{(l)}]$ are the partition functions associated with the Schottky group $G^{(l)}$ generated by l projective transformations $P_i^{(l)}$. $F^{(1)}$ and $F^{(2)}$ correspond to the orbital and ghost contributions, respectively.[4] It is worthwhile to note the similarity of $F^{(s)}$ with the Selberg zeta function $Z(s)$ (Ref. 12). The projective transformations $P_i^{(l)}$ are defined recursively from the ones appearing in the primitive tadpoles[6–8] [see, for example, the rules (9)]. The measure of moduli $d\mu^{(l)}$ and the coefficient $\Lambda^{(l)}$ are also defined recursively from the primitive ones [see, for example, (8) and (10)–(12)]. The coefficients A_n^i, Q_{nm}, B_{nm} are as usual the differential coefficients of the first Abelian integral $\phi^{(i)}(y,x)$, the logarithm of the prime form $E(x,y)$ divided by $x-y$, and the automorphic form $\mathcal{G}_{x,y}^{(l)}$ of weight 2 in x and -1 in y, all of which are defined in terms of the Schottky generators.[13,4,7,8]

Note that expression (3) is adaptable to orientable planar, nonorientable, orientable nonplanar and their mixed multiloop tadpoles with suitable $G^{(l)}$, $d\mu^{(l)}$, and $\Lambda^{(l)}$. The reason why their expressions are superficially the same is the fact that their Riemann doubles[14] are the same, if they have the same number of loops. They are distinguished by the differences in the locations of the isometric circles and the signs of the multipliers associated with the Schottky generators.

Now the product of two tadpole-type operators are evaluated as

$$\langle \mathcal{T}^{(l+l')}|_I = \langle \mathcal{T}^{(l)}|_E (b_0^E - b_1^E) \int_0^1 \frac{dx}{x(1-x)} \hbar^{(E)}(x) \langle \mathcal{T}^{(l')}|_K (b_0^K - b_1^K) \int_0^1 \frac{dy}{y(1-y)} \hbar^{(K)}(y) \langle V_{LFI} \| R_{KL} \rangle | R_{EF} \rangle \,. \tag{7}$$

The resultant expression is given by applying the following rule to (3): adding factors $d\mu^{(l')}, dx/x^2, dy/y^2$, and changes

$$\begin{aligned} &\prod_{i=1}^{l} d^{26}k_i \ \text{ to } \ \prod_{i=1}^{l+l'} d^{26}k_i \,, \\ &\Lambda^{(l)} \ \text{ to } \ \Lambda^{(l)}\Lambda^{(l')} \left[\frac{x}{1-x}\right]^2 \left[\frac{y}{1-y}\right]^2 , \\ &G^{(l)} \ \text{ to } \ G^{(l+l')} \,. \end{aligned} \tag{8}$$

Here the $l+l'$ generators of the new Schottky group $G^{(l+l')}$ are given in terms of the old ones $P_i^{(l)}$ and $P_i^{(l')}$, which belong to $G^{(l)}$ and $G^{(l')}$, as follows:

$$\begin{aligned} &P_i^{(l+l')} = \Omega^\dagger \hbar(x)^{-1} P_i^{(l)} [\Omega^\dagger \hbar(x)^{-1}]^{-1} \,, \\ &\alpha_i^{(l+l')} = [\Omega^\dagger \hbar(x)^{-1}](\alpha_i^{(l)}), \text{ etc.}, \quad i = 1,2,\ldots,l \,, \\ &P_i^{(l+l')} = \Omega \hbar(y)^{-1} P_i^{(l')} [\Omega \hbar(y)^{-1}]^{-1} \,, \\ &\alpha_i^{(l+l')} = [\Omega \hbar(y)^{-1}](\alpha_i^{(l')}), \text{ etc.} \,, \\ &\qquad\qquad i = l+1, l+2, \ldots, l+l' \,. \end{aligned} \tag{9}$$

Here $P_1^{(1)}$, $\alpha_1^{(1)}$, and $\beta_1^{(1)}$ are given by

$$p_1^{(1)} = \begin{bmatrix} \pm\omega & 1 \\ 0 & 1 \end{bmatrix}, \quad \alpha_1^{(1)} = \frac{1}{1-(\pm\omega)}, \quad \beta_1^{(1)} = \infty \,,$$

where plus and minus correspond to the planar and the nonorientable cases, respectively. As examples, the measure factors of the three kinds of multiloop tadpole in Fig. 1 are evaluated as

$$\begin{aligned} &d\mu_{\mathrm{P}}^{(l)} = \prod_{i=1}^{l} \frac{dw_i(1-w_i)}{w_i^2} \prod_{j=1}^{l-1} \frac{dx_j dy_j}{x_j^2 y_j^2} \,, \\ &\Lambda_{\mathrm{P}}^{(l)} = \prod_{i=1}^{l} w_i \prod_{i=1}^{l-1} \left[\frac{x_i}{1-x_i} \frac{y_i}{1-y_i}\right]^2 , \end{aligned} \tag{10}$$

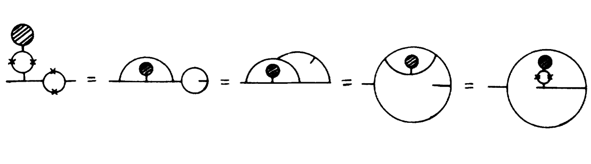

FIG. 8. Lemma 3 is proved by making repated use of duality of the four-point vertex.

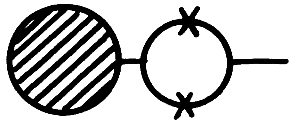

FIG. 9. Another typical combination.

$$d\mu_{\rm NO}^{(l)}=\prod_{i=1}^{l}\frac{dw_i(1+w_i)}{w_i^2}\prod_{j=1}^{l-1}\frac{dx_jdy_j}{x_j^2y_j^2}\ ,$$
$$\Lambda_{\rm NO}^{(l)}=\prod_{i=1}^{l}(-w_i)\prod_{i=1}^{l-1}\left[\frac{x_i}{1-x_i}\frac{y_i}{1-y_i}\right]^2\ , \tag{11}$$

$$d\mu_{\rm NP}^{(2l)}=\prod_{i=1}^{l}\frac{dw_i}{w_i^2}\frac{d\xi_i}{\xi_i}\frac{d\mu_id\nu_i(1-\mu_i\nu_i)}{\mu_i^2(1-\mu_i)\nu_i^2(1-\nu_i)}\prod_{j=1}^{l-1}\frac{dx_jdy_j}{x_j^2y_j^2}\ ,$$
$$\Lambda_{\rm NP}^{(2l)}=\prod_{i=1}^{l}\Delta_i^2\prod_{i=1}^{l-1}\left[\frac{x_i}{1-x_i}\frac{y_i}{1-y_i}\right]^2,\quad \Delta_i\equiv\frac{\mu_i\nu_i}{1-\mu_i\nu_i}w_i\ , \tag{12}$$

where we used a nonorientable tadpole in Ref. 6 and a nonplanar two-loop tadpole in Ref. 7 with a gauge transformed measure $d\mu_{\rm NP}^{(2)}$ and $\Delta_{\rm NP}^{(2)}$ by $S(f)$. (We have also used the new notation μ_i,ν_i instead of x_iy_i in Ref. 7.)

Next consider another typical combination appearing in the nonorientable case (see lemmas 3 and 4), the combination of a nonplanar self-energy operator with a tadpole-type operator (Fig. 9):

$$\langle \mathcal{T}^{(l+1)}|_I=\langle \mathcal{T}^{(l)}|_E(b_0^E-b_1^E)\times\int_0^1\frac{dx}{x(1-x)}\hbar^{(E)}(x)\langle \Sigma_{\rm NP}^{(1)}|_{FI}|R_{EF}\rangle\ . \tag{13}$$

In this case the resultant tadpole is given by applying the following rule to (3): add the factors

$$\frac{dx}{x^2}\frac{dw}{w_{\rm NP}^2}\frac{d\xi_{\rm NP}}{\xi_{\rm NP}}\left[1-w_{\rm NP}\Lambda^{(l)}\left[\frac{x}{1-x}\right]^2\right] \tag{14}$$

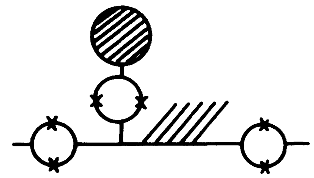

FIG. 10. General diagram with insertion of some particle states.

and change

$$(b_{-1}-b_0+\Lambda^{(l)}b_0)\ \text{to}\ (b_{-1}^I-b_0^I+w_{\rm NP}b_0^I)\ ,\qquad G^{(l)}\ \text{to}\ G^{(l+1)}\ . \tag{15}$$

Here the new Schottky group $G^{(l+l')}$ is generated by the following $l+1$ projective transformations:

$$P_i^{(l+1)}=[\hbar(x)\Gamma_{\xi_{\rm NP}}]^{-1}P_i^{(l)}[\hbar(x)\Gamma_{\xi_{\rm NP}}]\ ,\quad i=1,2,\ldots,l\ , \tag{16}$$
$$P_{l+1}^{(l+1)}=P_{\rm NP}$$

with fixed points

$$\alpha_i^{(l+1)}=[\hbar(x)\Gamma_{\xi\rm NP}]^{-1}(\alpha_i^{(l)})\ ,$$
$$\beta_i^{(l+1)}=[\hbar(x)\Gamma_{\xi_{\rm NP}}]^{-1}(\beta_i^{(l)}),\quad i=1,2,\ldots,l\ , \tag{17}$$
$$\alpha_{l+1}^{(l+1)}=\alpha_{\rm NP},\quad \beta_{l+1}^{(l+1)}=\beta_{\rm NP}\ ,$$

respectively. Here the quantities with subscripts NP mean that they come from the nonplanar self-energy operator.[15,8] The more complicated case, for example, is illustrated in Fig. 10, which is also easily constructed in an analogous way, and is left to the reader to verify.

Now we present the general expression of an amplitude with on-shell external ground-state tachyons, which is evaluated by attaching the desired particle states to the external legs of the operators:

$$\langle \mathcal{T}^{(l)}|_E(b_0^E-b_I^E)\int\frac{dz}{z(1-z)}\hbar^{(E)}(z)\langle V_{1,2,\ldots,F,\ldots,N+1}\|R_{EF}\rangle\prod_{i=1(\neq F)}^{N+1}|p_i;-\rangle$$
$$=\int d\tilde\mu\int\frac{\prod_{i=1(\neq F)}^{N+1}dz_i}{dV_{abc}}F^{(1)}[G^{(l)}]^{-26}F^{(2)}[G^{(l)}]^2(\det 2\pi\,{\rm Im}\tau)^{-13}\prod_{r<s}\left[E(z_s,z_r)\exp\left[-\pi\int_{z_r}^{z_s}d\phi_i({\rm Im}\tau)_{ij}^{-1}\int_{z_r}^{z_s}d\phi_j\right]\right]^{p_r\cdot p_s}, \tag{18}$$

where

$$\langle V_{1,2,\ldots,F,\ldots,N+1}| \prod_{i=1(\neq F)}^{N+1} |p_i-\rangle$$

$$= \int \frac{\prod_{i=1}^{N+1} dz_i}{dV_{abc}} \frac{z_{F+1}-z_{F-1}}{(z_{F+1}-z_{\bar{r}})(z_F-z_{F-1})} {}_F\langle \tilde{0},0_a| \exp\left[\sum_{r=1(\neq F)}^{N+1} (a^F|zr)\right](b_{-1}^F-b_1^F)\text{: } \exp(a^{F\dagger}|U_F-1|a^F)\text{: } . \tag{19}$$

Here $\langle \mathcal{T}^{(l)}|$ is a general tadpole operator constructed by using the rules discussed above. The measure $d\tilde{\mu}$ is also constructed recursively by using (10)–(12) and (14). In particular, in the cases where $\langle \mathcal{T}^{(l)}|$ are constructed from the three tadpoles in Fig. 1, the variables x_i, y_i, $i=1,2,\ldots,l-1$, z and z_F in the measures $d\tilde{\mu}$ can be expressed by the fixed points $\bar{\alpha}_i^{(l)}, \bar{\beta}_i^{(l)}$ of

$$\bar{P}_i^{(l)} = [\kappa(z)U_F]^{-1} P_i^{(l)} \kappa(z) U_F, \quad i=1,2,\ldots,l$$

in the same way as the planar case[15,8] (in the nonplanar case, one should choose l projective transformations, each of which corresponds to the nonplanar self-energy part in it[7])

$$\frac{(z_{F+1}-z_{F-1})dz_F}{(z_{F+1}-z_F)(z_F-z_{F-1})} \frac{dz}{z^2} \prod_{i=1}^{l-1} \frac{dx_i dy_i}{x_i^2 y_i^2} = \prod_{i=1}^{l} \frac{d\bar{\alpha}_i^{(l)} d\bar{\beta}_i^{(l)} (1\pm w_i)}{(\bar{\alpha}_i^{(l)}-\bar{\beta}_i^{(l)})^2} , \tag{20}$$

where one should choose plus or minus according to whether ω_i corresponds to the orientable or nonorientable loops. Thus the measures $d\tilde{\mu}$ in these cases are given by

$$d\tilde{\mu}_{\mathrm{P}} = \prod_{i=1}^{l} \frac{d\omega_i(1-\omega_i)^2}{\omega_i^2} \frac{d\bar{\alpha}_i^{(l)} d\bar{\beta}_i^{(l)}}{(\bar{\beta}_i^{(l)}-\bar{\alpha}_i^{(l)})^2} , \tag{21}$$

$$d\tilde{\mu}_{\mathrm{NO}} = \prod_{i=1}^{l} \frac{d\omega_i(1+\omega_i)^2}{\omega_i^2} \frac{d\bar{\alpha}_i^{(l)} d\bar{\beta}_i^{(l)}}{(\bar{\beta}_i^{(l)}-\bar{\alpha}_i^{(l)})^2} , \tag{22}$$

$$d\tilde{\mu}_{\mathrm{NP}} = \prod_{i=1}^{l} \frac{d\omega_i(1-\omega_i)}{\omega_i^2} \frac{d\xi_i}{\xi_i} \frac{d\mu_i d\nu_i (1-\mu_i\nu_i)}{\mu_i^2 \nu_i^2 (1-\mu_i)(1-\nu_i)} \times \frac{d\bar{\alpha}_i^{(l)} d\bar{\beta}_i^{(l)}}{(\bar{\beta}_i^{(l)}-\bar{\alpha}_i^{(l)})^2} . \tag{23}$$

Next let us consider the vacuum amplitudes. Because of the lack of a BRST-invariant physical vacuum state in the formalism, the lowest-order vacuum amplitudes illustrated in Fig. 11 should be considered separately from the higher-order ones

Lemma 6. Any vacuum amplitudes except for the lowest-order ones in the planar, nonorientable and nonplanar cases can be obtained by gluing two tadpoles, which are made from primitive tadpoles and nonplanar self-energy.

The proof follows the fact that from any vacuum diagram one can obtain a self-energy-type diagram by cutting one of the internal lines and using lemmas 1–5.

The general expression of the $l+l'$-loop vacuum amplitude is thus obtained as

$$\mathcal{Z}^{(l+l')} = \langle \mathcal{T}^{(l)}|_E (b_0^E - b_1^E) \int_0^1 \frac{dx}{x(1-x)} \kappa^{(E)}(x) \langle \mathcal{T}^{(l')}|_F R_{EF}\rangle$$

$$= \int d\mu^{(l)} \frac{dx}{x^2} d\mu^{(l')} (\det 2\pi\, \mathrm{Im}\tau)^{-13} F^{(1)}[G^{(l+l')}]^{-26} F^{(2)}[G^{(l+l')}]^2 \left[1-\Lambda^{(l)}\Lambda^{(l')}\left[\frac{x}{1-x}\right]^2\right] , \tag{24}$$

where the new Schottky group $G^{(l+l')}$ is generated by the new $l+l'$ projective transformations such as (9).

Because of lemma 6, it is meaningful to calculate a nonplanar two-loop vacuum amplitude [Fig. 11(c)]. The calculation is performed as

$$\mathcal{Z}_{\mathrm{NP}}^{(2)} = \mathrm{Tr}^{16}\mathrm{Tr}^{34} \left[\langle V_{123}|\langle V_{4'56'}| \prod_{i=1}^{3} (b_0^i - b_1^i) \int_0^1 \frac{dx_i}{x_i(1-x_i)} \kappa^{(i)}(x_i)|R_{4'4}\rangle|R_{25}\rangle|R_{6'6}\rangle\right]$$

$$= \frac{1}{2}\int_0^1 \prod_{i=1}^{3} \frac{dw_i(1-w_i)}{w_i^{3/2}} (\det 2\pi\, \mathrm{Im}\tau)^{-13} F^{(l)}[G^{(2)}]^{-26} F^{(2)}[G^{(2)}]^2 , \tag{25}$$

where the two Schottky generators are given by

$$P_1 = \begin{bmatrix} 1-X_2(1-X_1) & -1 \\ X_1X_2 & 0 \end{bmatrix} , \qquad P_2 = \begin{bmatrix} X_3(1-X_2) & -X_3 \\ 1 & -1 \end{bmatrix} \tag{26}$$

with $X_i \equiv x_i - 1/x_i \le 0$. The new variables ω_i, $i=1,2,3$ are the multipliers of P_1, P_2, and $P_3 \equiv P_1^{-1}P_2$ defined by

(a) (b) (c)

FIG. 11. Three lowest-order vacuum amplitudes in the planar, nonorientable, and nonplanar cases.

the equations

$$\frac{w_i}{(1+w_i)^2}=\frac{X_iX_{i+1}}{[1-X_{i+1}(1-X_i)]^2}, \quad i=1,2,3 \ . \tag{27}$$

Expression (25) is superficially the same as in the planar case.[9] However, in contrast with that case, it can be shown that the three multipliers in it cannot simultaneously reach one. This property is consistent with the view that three tubes corresponding to a closed-string propagation cannot be simultaneously pulled out in Fig. 11(c). It is also interesting that the integration region of $\mathcal{Z}_{\rm NP}^{(2)}$ is exactly determined as $0\le\omega_i\le 1$, $i=1,2,3$ as in the planar case. The relation of this to the fundamental region of the modular group is however not clear at this stage.

According to the properties of duality, we have given a classification of the open-bosonic-string amplitudes based on the primitive diagrams. As sample calculations, we have given expressions for nonorientable and nonplanar multiloop amplitudes with on-shell tachyons and the nonplanar two-loop vacuum amplitude.

Recently, some discussions which support the finiteness of the open-bosonic-string amplitude with $SO(2^{13})$ internal gauge symmetry have appeared.[16–19] We believe that our results will be useful for confirming such a property in the higher-order perturbation. The analysis of singularity structure in any amplitudes is now under investigation.

We would like to thank Dr. Kobayashi for useful discussions and Professor L. Clavelli for reading the manuscript. One of us (H.K.) wishes to thank Professor B. Harms for calling his attention to Refs. 18 and 19.

*Present address: Department of Physics and Astronomy, University of Alabama, Box 1921, Tuscaloosa, AL 35487-1921.

[1]K. Kikkawa, B. Sakita, and M. A. Virasoro, Phys. Rev. **184**, 1701 (1969).
[2]D. Gross, A. Neveu, J. Scherk, and J. Schwarz, Phys. Lett. **31B**, 592 (1970).
[3]P. DiVecchia, M. Frau, A. Lerda, and S. Sciuto, Nucl. Phys. **B298**, 527 (1988); Phys. Lett. B **199**, 49 (1987).
[4]H. Konno, Ph.D. thesis, University of Tsukuba, 1989.
[5]T. Kobayashi, H. Konno, and T. Suzuki, Phys. Rev. D **38**, 1150 (1988).
[6]T. Kobayashi, H. Konno, and T. Suzuki, Phys. Lett. B **211**, 86 (1988).
[7]T. Kobayashi, H. Konno, and T. Suzuki, Report No. UTHEP-192 (unpublished).
[8]H. Konno, Phys. Lett. B **212**, 165 (1988).
[9]G. Cristofano, F. Nicodemi, and R. Pettorino, Phys. Lett. B **200**, 292 (1988); Int. J. Mod. Phys. A (to be published).
[10]G. Cristofano, R. Musto, F. Nicodemi, and R. Pettorino, Phys. Lett. B **211**, 417 (1988).
[11]M. Kaku and L.-P. Yu, Phys. Rev. D **3**, 3020 (1971).
[12]E. D'Hoker and D. H. Phong, Rev. Mod. Phys. **60**, 917 (1988).
[13]V. Alessandrini and D. Amati, Nuovo Cimento **4A**, 793 (1971).
[14]S. Mandelstam, in *Unified String Theories,* edited by M. Green and D. Gross (World Scientific, Singapore, 1986).
[15]E. Cremmer, Nucl. Phys. **B31**, 477 (1971).
[16]S. Weinberg, Phys. Lett. B **187**, 278 (1987).
[17]N. Marcus and A. Sagnotti, Phys. Lett. B **188**, 58 (1987).
[18]L. Clavelli, P. H. Cox, and B. Harms, Phys. Rev. D **36**, 3747 (1987).
[19]J. Govaerts, Phys. Lett. B **220**, 77 (1989).