Effect of small-scale structure on the dynamics of cosmic strings

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The effective equation of state for a string with short-wavelength perturbations is shown to be $\mu T = \mu_0^2$, where μ and T are the mass density and tension in the rest frame of the string and μ_0 is the unperturbed string tension. The dynamics of "wiggly" strings in flat spacetime is discussed. The conclusions are in agreement with the results obtained by Carter using different methods.

I. INTRODUCTION

Recent numerical simulations of cosmic-string evolution[1–3] have led to a surprising discovery that the strings have a significant structure on scales much smaller than their correlation length. This structure is due to the kinks formed by intercommuting strings and contributes nearly a half to the total energy.

The purpose of this paper is to study the effect of small-scale wiggles on string dynamics. I shall disregard the cosmological expansion and string intercommuting and consider an idealized situation when the typical wavelength of the wiggles is much smaller than the characteristic scale of the string. To an observer who cannot resolve the wiggles, the string will appear as smooth, but the effective mass per unit length μ and tension T of the string will be different from those of an unperturbed string.

The dynamics of strings with small-scale wiggles was recently discussed by Carter.[4] He argued that the string equation of state should be "nondispersive" in the sense that the speeds of propagation of transverse and longitudinal waves along the string are the same. These speeds are given by

$$v_T = (T/\mu)^{1/2}\ , \quad v_L = (-dT/d\mu)^{1/2} \tag{1.1}$$

and the requirement $v_T = v_L$ leads to a unique equation of state:

$$\mu T = \mu_0^2 = \text{const}\ , \tag{1.2}$$

where μ_0 is the unperturbed string tension. μ and T have to be understood as mass per unit length and tension in the local rest frame of the string and are, in general, functions of position and time. Carter developed a formalism for studying the dynamics of elastic strings with an arbitrary equation of state.[4,5] In the nondispersive case (1.2) the equations of motion considerably simplify, and he was able to obtain their general solution.[4]

In this paper we shall study the dynamics of wiggly strings from a microscopic point of view. In the next section the string equation of state will be derived from first principles, starting with the Nambu equations of motion. The dynamics of wiggly strings is discussed in Sec. III. The conclusions are briefly stated in Sec. IV. They are in full agreement with those of Carter.

II. EQUATION OF STATE

We begin by reviewing the well-known formalism of string dynamics.[6,7] The spacetime trajectory of the string can be described by a vector function $\mathbf{x}(\sigma,t)$, where σ is a parameter along the string. The equations of motion for $\mathbf{x}(\sigma,t)$ can be written as

$$\ddot{\mathbf{x}} - \mathbf{x}'' = 0\ , \tag{2.1}$$

$$\dot{\mathbf{x}}\cdot\mathbf{x}' = 0, \quad \dot{\mathbf{x}}^2 + \mathbf{x}'^2 = 1\ , \tag{2.2}$$

where overdots and primes stand for derivatives with respect to t and σ, respectively. The last two equations are the gauge conditions that fix the parametrization of the string world sheet. In this gauge, the energy-momentum tensor of the string is given by

$$T^{\mu\nu}(\mathbf{x},t) = \mu_0 \int d\sigma (\dot{x}^\mu \dot{x}^\nu - x'^\mu x'^\nu)\delta^{(3)}(\mathbf{x} - \mathbf{x}(\sigma,t))\ , \tag{2.3}$$

where $x^0 = t$ and μ_0 is the string tension. The general solution of the string equations of motion (2.1) and (2.2) has the form

$$\mathbf{x}(\sigma,t) = \tfrac{1}{2}[\mathbf{a}(\sigma - t) + \mathbf{b}(\sigma + t)]\ , \tag{2.4}$$

where the functions $\mathbf{a}(\sigma)$ and $\mathbf{b}(\sigma)$ satisfy the constraints

$$\mathbf{a}'^2 = \mathbf{b}'^2 = 1 \tag{2.5}$$

and are otherwise arbitrary.

A static straight string is represented by $\mathbf{a}(\sigma) = \mathbf{b}(\sigma) = \mathbf{n}\sigma$, where $\mathbf{n}$ is a unit vector along the string. We shall consider a perturbed string

$$\mathbf{a}(\sigma) = k_1\sigma\mathbf{n} + \boldsymbol{\xi}_1(\sigma)\ , \quad \mathbf{b}(\sigma) = k_2\sigma\mathbf{n} + \boldsymbol{\xi}_2(\sigma) \tag{2.6}$$

with $k_1, k_2 = \text{const}$ and

$$\mathbf{n}\cdot\boldsymbol{\xi}_1(\sigma) = \mathbf{n}\cdot\boldsymbol{\xi}_2(\sigma) = 0\ . \tag{2.7}$$

The perturbations $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ represent the "wiggles" and can be pictured as a superposition of waves propagating along the string in opposite directions. The constraint equations (2.5) imply

$$\boldsymbol{\xi}'^2_{1,2} = 1 - k^2_{1,2} \tag{2.8}$$

and therefore

$$-1 \le k_1, k_2 \le 1 \; . \tag{2.9}$$

We shall assume that $k_1 + k_2 \neq 0$ and that the direction of $\mathbf{n}$ is chosen so that $k_1 + k_2 > 0$. (For $k_1 + k_2 = 0$ the string lies entirely in a plane perpendicular to $\mathbf{n}$.) The assumption that k_1 and k_2 are constant means that all physical properties are uniform along the string when averaged over the wiggles. A more general situation will be discussed at the end of this section.

To an observer who cannot resolve the wiggles, the string appears to be a straight line. It will be convenient to use Cartesian coordinates with x^1 axis along the string. Then $\mathbf{n} = (1,0,0)$, $\xi_1^1 = \xi_2^1 = 0$, and the effective energy-momentum tensor of the string has the form

$$T^{\mu\nu}_{\text{eff}}(\mathbf{x}, t) = \theta^{\mu\nu}\delta(y)\delta(z) \; . \tag{2.10}$$

The quantities $\theta^{\mu\nu}$ can be found by averaging the microscopic energy-momentum tensor (2.3) over a distance d and a time interval τ much greater than the typical wavelength and the oscillation period of the wiggles, respectively,

$$\theta^{\mu\nu} = (\tau d)^{-1} \int T^{\mu\nu}(\mathbf{x}, t) d^3x \, dt \; . \tag{2.11}$$

Here, the spatial integration is over a region between two parallel planes separated by a distance d (see Fig. 1) and the time integration is over a time interval $\Delta t = \tau$.

For the energy density component T^{00}, the spatial integration gives

$$\int T^{00}(\mathbf{x}, t) d^3x = \mu_0 \int d\sigma = \mu_0 \Delta\sigma \tag{2.12}$$

and the time averaging replaces $\Delta\sigma$ by its average value $\langle \Delta\sigma \rangle$. With the aid of Eqs. (2.6) for $\mathbf{a}(\sigma)$ and $\mathbf{b}(\sigma)$, the distance d can also be expressed in terms of $\langle \Delta\sigma \rangle$:

$$d = \left\langle \int x^{1\prime} d\sigma \right\rangle = \tfrac{1}{2}(k_1 + k_2)\langle \Delta\sigma \rangle \; . \tag{2.13}$$

Hence,

$$\theta^{00} = 2\mu_0 (k_1 + k_2)^{-1} \; . \tag{2.14}$$

In a similar way we obtain

$$\theta^{11} = -2\mu_0 k_1 k_2 (k_1 + k_2)^{-1} \; , \tag{2.15}$$

$$\theta^{01} = \mu_0 (k_2 - k_1)(k_1 + k_2)^{-1} \; . \tag{2.16}$$

Physically, it is clear that all other components of $\theta^{\mu\nu}$ should vanish. This is not difficult to verify. For example, for θ^{0i} with $i = 2,3$ we have

$$\theta^{0i} = \frac{\mu_0}{2\tau d} \int dt \int d\sigma [\xi_2^{i\prime}(\sigma + t) - \xi_1^{i\prime}(\sigma - t)] \; . \tag{2.17}$$

The rapidly oscillating functions ξ_1^i and ξ_2^i average out to zero, and thus $\theta^{02} = \theta^{03} = 0$. For θ^{ij} with $i,j = 2,3$ we obtain

$$\theta^{ij} = -\frac{\mu_0}{2\tau d} \int dt \int d\sigma (\xi_2^{i\prime}\xi_1^{j\prime} + \xi_2^{j\prime}\xi_1^{i\prime}) \; . \tag{2.18}$$

After a change of variables $u = \sigma - t$, $v = \sigma + t$, the double integral of $\xi_2^{i\prime}\xi_1^{j\prime}$ splits into a product of integrals $\int \xi_2^{i\prime} dv$ and $\int \xi_1^{j\prime} du$, each of which is equal to zero.

From Eqs. (2.14) and (2.15) we see that the string energy density θ^{00} can take any value in the range $\mu_0 < \theta^{00} < \infty$ and that $-\theta^{00} < \theta^{11} < \theta^{00}$. The stress component θ^{11} represents tension when k_1 and k_2 are both positive and pressure when they have opposite signs. θ^{00} and $-\theta^{11}$ can be identified with the proper mass density μ and proper tension T only when $k_1 = k_2 (\equiv k)$, so that $\theta^{01} = 0$ and the string is at rest. In this case,

$$\mu = k^{-1}\mu_0 \; , \quad T = k\mu_0 \; , \tag{2.19}$$

in agreement with the equation of state (1.2). In the general case, in order to find μ and T, one has to perform a Lorentz transformation in the x direction. Alternatively, μ and $-T$ can be invariantly defined as eigenvalues of $\theta^{\mu\nu}$. Their product is given by the Lorentz-invariant determinant

$$-\mu T = \theta^{00}\theta^{11} - (\theta^{01})^2 \; . \tag{2.20}$$

Substituting $\theta^{\mu\nu}$ from (2.14)–(2.16) into (2.20), we recover Carter's equation of state (1.2):

$$\mu T = \mu_0^2 \; . \tag{2.21}$$

It is easily verified that $\theta^{\mu\nu} l_\mu l_\nu \ge 0$ with a null vector $l^\mu = (l, l, 0, 0)$. This implies

$$\mu \ge T \; . \tag{2.22}$$

Until now we assumed that k_1 and k_2 in Eq. (2.6) are constant. In a more general situation they are slowly varying functions of σ. Slow variations means that the function changes very little on the characteristic length and time scale of the wiggles. In fact, k_1 and k_2 are the magnitudes of $\mathbf{a}'$ and $\mathbf{b}'$ averaged over the wiggles. It is easily seen that the analysis of this section is directly applicable to the case of slowly varying k_1 and k_2. In this case μ and T are functions of σ and t, and the equation of state (2.21) applies locally.

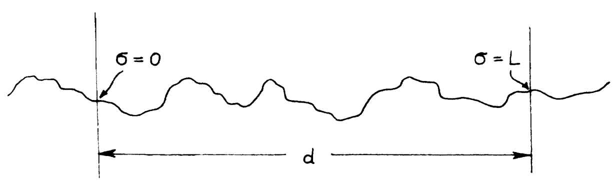

FIG. 1. The string energy-momentum tensor is averaged over a scale d much greater than the typical wavelength of the wiggles.

III. LOOP DYNAMICS

At the microscopic level, all possible loop trajectories are described by Eqs. (2.4) and (2.5). These equations allow a simple geometric interpretation: the functions $\mathbf{a}(\sigma)$ and $\mathbf{b}(\sigma)$ in Eq. (2.4) describe two arbitrary curves in three dimensions with σ being the length parameter along the curves. For a closed loop in its center-of-mass frame, the functions $\mathbf{a}(\sigma)$ and $\mathbf{b}(\sigma)$ are periodic,

$$\mathbf{a}(\sigma+L)=\mathbf{a}(\sigma), \quad \mathbf{b}(\sigma+L)=\mathbf{b}(\sigma) , \tag{3.1}$$

and thus the two curves have the same length L. It is called the invariant length of the loop and is equal to M/μ, where M is the loop's mass.

We consider the situation when the curves $\mathbf{a}(\sigma)$ and $\mathbf{b}(\sigma)$ have small-scale wiggles of wavelength much smaller than L. We are interested in the motion of a smoothed loop described by the $\mathbf{a}$ and $\mathbf{b}$ curves averaged over the wiggles. The averaged functions

$$\mathbf{A}(\sigma)=\langle \mathbf{a}(\sigma)\rangle , \quad \mathbf{B}(\sigma)=\langle \mathbf{b}(\sigma)\rangle \tag{3.2}$$

do not satisfy the constraint equations (2.5). Instead, we have

$$|\mathbf{A}'(\sigma)|\leq 1 , \quad |\mathbf{B}'(\sigma)|\leq 1 . \tag{3.3}$$

The averaging here is over a scale large compared to the typical wavelength of the wiggles, but small compared to the characteristic scale of the smoothed loop. The quantities $|\mathbf{A}'|$ and $|\mathbf{B}'|$ are similar to $|k_1|$ and $|k_2|$ of Sec. II. Their values are determined by the density and the size of the wiggles. In the most general case, $\mathbf{A}(\sigma)$ and $\mathbf{B}(\sigma)$ can be arbitrary periodic functions satisfying Eq. (3.3), and the formula

$$\langle \mathbf{x}(\sigma,t)\rangle=\tfrac{1}{2}[\mathbf{A}(\sigma-t)+\mathbf{B}(\sigma+t)] \tag{3.4}$$

represents the motion of *some* wiggly loop. The variety of possible motions for wiggly strings is much richer than that for Nambu strings, since $|\mathbf{A}'|$ and $|\mathbf{B}'|$ are no longer constrained to lie on a unit sphere. In a generic case, the curves $\mathbf{A}'(\sigma)$ and $-\mathbf{B}'(\sigma)$ do not intersect, and the loops do not develop cusps. (Of course, the cusps do occur at the microscopic level, but their scale is determined by the size of the wiggles.) As I already mentioned in the Introduction, these results coincide with the conclusions reached by Carter using quite different methods.[4]

Let us now consider some interesting special cases. If the wiggles are statistically the same on $\mathbf{a}$ and $\mathbf{b}$ curves and are uniformly distributed along each curve, then

$$|\mathbf{A}'(\sigma)|=|\mathbf{B}'(\sigma)|\equiv k=\text{const.} \tag{3.5}$$

These equations are quite similar to the constraint equations for the Nambu strings, Eq. (2.5). Given a solution of Eq. (2.5), we can obtain a solution of (3.5) as

$$\mathbf{A}(\sigma)=\mathbf{a}(k\sigma), \quad \mathbf{B}(\sigma)=\mathbf{b}(k\sigma) . \tag{3.6}$$

Hence, in this case the motion of wiggly strings is the same as that of Nambu strings, except it is slowed down by a factor of $k<1$.

Another interesting case is when one of the curves, say, $\mathbf{B}(\sigma)$, shrinks to a point. We can choose the coordinates so that this point is at $\mathbf{x}=0$. Then $\langle \mathbf{x}(\sigma,t)\rangle=\frac{1}{2}\mathbf{A}(\sigma-t)$, and it is easily seen that the shape of the loop does not change in time. When $\mathbf{B}(\sigma)$ shrinks to a point, $\mathbf{B}'(\sigma)$ also vanishes. This corresponds to $k_2=0$ in the case of a "wiggly straight string" discussed in Sec. II. From Eq. (2.15) we see that for $k_2=0$ the tension vanishes. This explains why the loop with $\mathbf{B}(\sigma)=0$ remains static.

Finally, we consider a more specific example when $\mathbf{A}(\sigma)$ and $\mathbf{B}(\sigma)$ are two concentric circles:

$$\begin{aligned} \mathbf{A}(\sigma)&=R_A\left[\hat{\mathbf{e}}_1\cos\left[\frac{2\pi\sigma}{L}\right]+\hat{\mathbf{e}}_2\sin\left[\frac{2\pi\sigma}{L}\right]\right] , \\ \mathbf{B}(\sigma)&=R_B\left[\hat{\mathbf{e}}_1\cos\left[\frac{2\pi\sigma}{L}\right]+\hat{\mathbf{e}}_2\sin\left[\frac{2\pi\sigma}{L}\right]\right] , \end{aligned} \tag{3.7}$$

with R_A, $R_B\leq L/2\pi$. It is easily shown that the radius of the loop described by these functions changes in time according to

$$r=\tfrac{1}{2}\left[R_A^2+R_B^2+2R_AR_B\cos\frac{4\pi t}{L}\right]^{1/2} . \tag{3.8}$$

It oscillates in the range

$$\tfrac{1}{2}|R_A-R_B|\leq r\leq\tfrac{1}{2}(R_A+R_B) \tag{3.9}$$

with a period $\Delta t=L/2$. For $R_A=R_B=R$, Eq. (3.8) reduces to

$$r=R\cos(2\pi t/L) .$$

In the case of a Nambu string, the motion of a circular loop is given by $r=R\cos(t/R)$. It has the same form as (3.10), but its period is $\Delta t=\pi R<L/2$. In the case when $R_B\to 0$, Eq. (3.9) gives $r=R_A/2=\text{const}$, and the loop is static.

IV. CONCLUSIONS

We have shown that the effective equation of state for strings with small-scale structure is

$$\mu T=\mu_0^2 , \tag{4.1}$$

where μ is the mass per unit length and T is the tension in the local rest frame of the string, and μ_0 is the unperturbed string tension.

The motion of the string averaged over small-scale wiggles is described by

$$\langle \mathbf{x}(\sigma,t)\rangle=\tfrac{1}{2}[\mathbf{A}(\sigma-t)+\mathbf{B}(\sigma+t)] , \tag{4.2}$$

where $\mathbf{A}(\sigma)$ and $\mathbf{B}(\sigma)$ are arbitrary functions satisfying

$$|\mathbf{A}'(\sigma)|\leq 1 , \quad |\mathbf{B}'(\sigma)|\leq 1 . \tag{4.3}$$

These conclusions are in full agreement with the results obtained by Carter[4] using different methods.

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