# Relativistic hydrodynamics of quark-gluon plasma and stability of scaling solutions

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The scaling solutions of the relativistic hydrodynamics are expected to play an important role in describing the expansion stage of a quark-gluon plasma which may be formed in nucleus-nucleus collisions at high energies. After summarizing some general properties of the scaling solutions, we study in detail their stability against small perturbations. In some typical cases of the two-dimensional scaling solution it is found that (i) the scaling solution is stable if the Reynolds number $R$ defined in terms of the viscosity coefficients is larger than a critical value $R_c$ $(=1)$, (ii) it is also stable for a long-wavelength perturbation if $R$ is small enough, and (iii) it becomes unstable when $R$ approaches $R_c$ from below. It is also shown that these results are related to the time dependence of the Reynolds number, the entropy density, and the temperature, and the point $R=R_c$ corresponds to a critical instant when the heating due to the dissipative processes balances with the cooling due to the expansion of the fluid. The stability of the scaling solution of the quark-gluon plasma is examined for typical ranges of the relevant parameters.

## I. INTRODUCTION

Extensive studies have been made recently on the evolution of the quark-gluon plasma (QGP) which is expected from lattice QCD calculations and will be realized in heavy-ion collisions. Many authors have used relativistic hydrodynamics[1,2] to describe the space-time development of the QGP. The equations of relativistic hydrodynamics have analytic solutions which are called "the scaling solutions.[3-5]" In particular, the $(1+1)$-dimensional scaling solutions have been often used to describe the expansion of the quark-gluon (or hadronic) fluid as the simplest model, either for the perfect fluid[4-6] or for the imperfect (viscous) fluid.[7-12] In more realistic models such as a cylindrically symmetric solution with both longitudinal and transverse expansions,[13-15] the scaling solutions have sometimes been assumed as the approximate solutions for the longitudinal expansion. It is, therefore, very important to answer the question whether or not the scaling solutions are stable. If the scaling solutions are unstable, small deviations from them will grow rapidly and they will no longer describe the whole process of expansion.

Baym *et al.*[13] showed that, for the perfect fluid, the $1+1$ scaling solutions are stable under small perturbations. In this paper we generalize their analysis and study the stability of the $1+1$ scaling solutions when the dissipative processes exist. It will be shown that the scaling solutions become unstable when the viscosity coefficients are large. The Reynolds number and the entropy density at the initial stage of fluid expansion play important roles. In Sec. II we review the relativistic hydrodynamics briefly according to the Landau-Lifshitz formalism.[2] In Sec. III we explain about the scaling solutions and their properties. The stability of the $(1+1)$-dimensional scaling solutions is examined in Sec. IV. The result is applied to the massless quark-gluon fluid in Sec. V. A summary and discussion are given in Sec. VI. Lyapunov's method[16] for stability analysis is explained in the Appendix.

## II. RELATIVISTIC HYDRODYNAMICS

### A. The four-velocity and the local three-frame projector

There are two typical formalisms to describe relativistic hydrodynamics. One was formulated by Eckart[1] and the other by Landau and Lifshitz.[2] The four-velocity is commonly defined by

$$u^{\mu}\equiv\left[\frac{1}{\sqrt{1-|\mathbf{v}|^2}},\frac{\mathbf{v}}{\sqrt{1-|\mathbf{v}|^2}}\right], \tag{2.1}$$

where $\mathbf{v}$ is the velocity of the fluid and we put the light velocity $c=1$. In Eckart's formalism, $\mathbf{v}$ is the velocity of the (conserved) number (such as the baryon-number) transport. On the other hand, in the formalism of Landau and Lifshitz, it is the velocity of the energy transport. We use the latter because it is more convenient than the former when the chemical potential $\mu$ is small.[10]

The four-velocity (2.1) satisfies the normalization

$$u^{\mu}u_{\mu}=1 . \tag{2.2}$$

In our convention, the metric tensor $g^{\mu\nu}$ is defined by

$$g^{00}=-g^{11}=-g^{22}=-g^{33}=1 \text{ and } g^{\mu\nu}=0 \text{ for } \mu\neq\nu . \tag{2.3}$$

The sign convention is opposite the one used in Ref. 2.

It is convenient to introduce the local three-frame projector[10] which is defined by

 

$$\Delta^{\mu\nu}\equiv g^{\mu\nu}-u^{\mu}u^{\nu} . \quad (2.4)$$

It has the property

$$u^{\nu}\Delta_{\nu\mu}=0 . \quad (2.5)$$

Any Lorentz four-vector $V^{\mu}$ is decomposed as

$$V^{\mu}=u^{\mu}V^{\nu}u_{\nu}+\Delta^{\mu\nu}V_{\nu} . \quad (2.6)$$

### B. The equation of motion

The energy-momentum tensor is

$$T^{\mu\nu}=\epsilon u^{\mu}u^{\nu}-p\Delta^{\mu\nu}+\Pi^{\mu\nu}=T_{p}^{\mu\nu}+\Pi^{\mu\nu} , \quad (2.7)$$

where $\epsilon$ and $p$ are the energy density and the pressure, respectively. The first term $T_{p}^{\mu\nu}$ is the ideal part (perfect-fluid part), and the dissipative part (viscous stress tensor) $\Pi^{\mu\nu}$ is given by

$$\Pi^{\mu\nu}=\eta(\nabla^{\mu}u^{\nu}+\nabla^{\nu}u^{\mu}-\tfrac{2}{3}\Delta^{\mu\nu}\nabla^{\rho}u_{\rho})+\zeta\Delta^{\mu\nu}\nabla^{\rho}u_{\rho} , \quad (2.8)$$

where $\eta$ and $\zeta$ are the shear viscosity and the bulk viscosity, respectively, and $\nabla^{\mu}$ is defined by[10]

$$\nabla^{\mu}\equiv\Delta^{\mu\nu}\partial_{\nu} . \quad (2.9)$$

Here we note the following fact. The factor $\frac{2}{3}$ on the right-hand side (RHS) of Eq. (2.8) is chosen so that this term is traceless. Notice that $\Delta^{\mu}_{\mu}=3=$the dimension of the space. This choice reflects the physical meaning of the shear viscosity.

The conserved number current is expressed as

$$J^{\mu}=nu^{\mu}+h^{\mu} \quad (2.10)$$

with

$$h^{\mu}=\kappa\left[\frac{nT}{\epsilon+p}\right]^2\nabla^{\mu}\left[\frac{\mu}{T}\right] , \quad (2.11)$$

where $T$, $n$, $\mu$, and $\kappa$ are the temperature, the number density, the chemical potential for $n$, and the heat conductivity, respectively. The current $h^{\mu}$ is induced by heat conduction.

Conservation of energy-momentum and that of particle number yield, respectively,

$$\partial_{\mu}T^{\mu\nu}=0 \quad (2.12)$$

and

$$\partial_{\mu}J^{\mu}=0 . \quad (2.13)$$

It is convenient to decompose Eq. (2.12) using Eq. (2.6):

$$u_{\mu}\partial_{\nu}T^{\mu\nu}=0 , \quad (2.14)$$

$$\Delta_{\mu\nu}\partial_{\rho}T^{\nu\rho}=0 . \quad (2.15)$$

Equation (2.14) with Eqs. (2.7), (2.8), (2.10), (2.11), (2.13), the thermodynamical relation $\epsilon+p=Ts+\mu n$, and energy conservation yield the equation for the entropy flow:

$$\partial_{\nu}\left[su^{\nu}-\frac{\mu}{T}h^{\nu}\right]=-h_{\nu}\partial^{\nu}\left[\frac{\mu}{T}\right]+\frac{\Pi^{\rho\nu}}{T}\partial_{\rho}u_{\nu} , \quad (2.16)$$

where $s$ is the entropy density. The RHS of Eq. (2.16) represents the entropy production. The entropy is conserved for the perfect fluid ($\eta,\zeta,\kappa=0$). On the other hand, Eq. (2.15) yields the equation of motion

$$wu_{\nu}\partial^{\nu}u_{\mu}=\nabla_{\mu}p-\Delta_{\mu\nu}\partial_{\rho}\Pi^{\nu\rho} , \quad (2.17)$$

where $w$ is the enthalpy density defined by $w=\epsilon+p$. This equation corresponds to "the Navier-Stokes equation" in nonrelativistic hydrodynamics.

## III. ($M+1$)-DIMENSIONAL SCALING SOLUTION IN ($N+1$)-DIMENSIONAL SPACE-TIME

Equations (2.16) and (2.17) have analytic solutions independent of whether[3–6] or not[7–12] the dissipative processes are absent. They are called "the scaling solutions." They are sometimes used to describe the expansion of the hadronic or quark-gluon matter. In this section, we review the general properties of these solutions. For generality, we discuss the scaling solutions in $N+1$ space-time and we use

$$\Pi^{\mu\nu}=\eta\left[\nabla^{\mu}u^{\nu}+\nabla^{\nu}u^{\mu}-\frac{2}{N}\Delta^{\mu\nu}\nabla^{\rho}u_{\rho}\right]+\zeta\Delta^{\mu\nu}\nabla^{\rho}u_{\rho} \quad (3.1)$$

as the definition of the viscous tensor $\Pi^{\mu\nu}$ instead of Eq. (2.8). We remark that, in this definition, the part which contains the shear viscosity is traceless since $\Delta^{\mu}_{\mu}=N$ in $N+1$ space-time.

Consider the case in which the $N+1$ velocity $u^{\mu}$ has a nonvanishing time component and nonvanishing $M(\leq N)$ spatial ones, depending only on $x^{\mu}$ ($0\leq\mu\leq M$) in $N+1$ space-time. The ($M+1$)-dimensional scaling solutions in $N+1$ space-time is then given by

$$u^{\mu}=\begin{cases}x^{\mu}/\tau & \text{for } 0\leq\mu\leq M ,\\ 0 & \text{for } M<\mu\leq N ,\end{cases} \quad (3.2)$$

where

$$\tau\equiv\left[\sum_{j=0}^{M}x^{j}x_{j}\right]^{1/2} . \quad (3.3)$$

The temperature $T$ and the chemical potential $\mu$ are supposed to depend only on $\tau$:

$$T=T(\tau) \text{ and } \mu=\mu(\tau) . \quad (3.4)$$

Putting Eqs. (3.2) and (3.4) into Eq. (3.1) and the $N$-dimensional version of Eq. (2.11) and using the relations $(\partial\tau/\partial x^{\mu})=u_{\mu}$, $(\partial T/\partial x^{\mu})=(\partial T/\partial\tau)u_{\mu}$ and $(\partial\mu/\partial x^{\mu})=(\partial\mu/\partial\tau)u_{\mu}$, we get

$$\Pi^{\mu\nu}=\eta\left[\tilde{\Delta}^{\mu\nu}-\frac{M}{N}\Delta^{\mu\nu}\right]\frac{2}{\tau}+\zeta\Delta^{\mu\nu}\frac{M}{\tau} \quad (3.5)$$

and

$$h^{\mu}=0 , \quad (3.6)$$

where

$$\tilde{\Delta}^{\mu\nu}=\begin{cases}\Delta^{\mu\nu} & \text{for } 0\leq\mu,\nu\leq M ,\\ 0 & \text{otherwise} .\end{cases} \quad (3.7)$$

Equation (3.6) means that there is no heat conduction in the scaling solutions. Furthermore, from Eq. (3.5), it is easy to check that

$$wu_\nu\partial^\nu u_\mu=\Delta_{\mu\nu}\partial^\nu p=\Delta_{\mu\nu}\partial_\rho\Pi^{\nu\rho}=0\ . \tag{3.8}$$

The equation of motion which has the same form as Eq. (2.17) is automatically satisfied. The explicit $\tau$ dependence of $T(\tau)$ and $\mu(\tau)$ is determined by the entropy equation

$$\partial_\mu(su^\mu)=\frac{\partial(s\tau^M)}{\partial\tau}\tau^{-M}=\left[2M\left[1-\frac{M}{N}\right]\eta+M^2\zeta\right]\frac{1}{\tau^2T}$$
$$=\frac{Ms}{R\tau}\ , \tag{3.9}$$

where the Reynolds number $R$ is defined by[8]

$$R^{-1}\equiv\frac{2(1-M/N)\eta+M\zeta}{Ts\tau}\ . \tag{3.10}$$

Equation (3.9) with (3.10) yields the well-known results for $M=1$ and $N=3$ (Refs. 7–12). In the case of the perfect fluid, $R^{-1}$ and hence the RHS of Eq. (3.9) vanish. Thus the entropy density is proportional to $\tau^{-M}$ since $s\tau^M$ is a constant.

For example, consider the case that $M=1$, $N=3$ with $\mu=0$, $s=4aT^3$ (massless particles) and $\frac{4}{3}\eta+\zeta=bT^3$ (weakly interacting massless particles), $a,b=$const. The solution of Eq. (3.9) is given by[7,9]

$$\frac{T}{T_i}=\left[\frac{\tau_i}{\tau}\right]^{1/3}\left\{1+\frac{b}{8a\tau_iT_i}\left[1-\left[\frac{\tau_i}{\tau}\right]^{2/3}\right]\right\}$$
$$=\left[\frac{\tau_i}{\tau}\right]^{1/3}\left\{1+\frac{R_i^{-1}}{2}\left[1-\left[\frac{\tau_i}{\tau}\right]^{2/3}\right]\right\}, \tag{3.11}$$

where $T_i$ and $R_i$ are the initial values of the temperature and the Reynolds number at the initial proper time $\tau=\tau_i$. For $M=N=3$ with $\mu=0$, $s=4aT^3$, and $\zeta=b'T^3$ ($b'$ is a constant), we obtain

$$\frac{T}{T_i}=\left[\frac{\tau_i}{\tau}\right]\left[1+\frac{3b'}{4a\tau_iT_i}\ln\left[\frac{\tau}{\tau_i}\right]\right]$$
$$=\left[\frac{\tau_i}{\tau}\right]\left[1+R_i^{-1}\ln\left[\frac{\tau}{\tau_i}\right]\right]\ . \tag{3.12}$$

A nonvanishing $R_i^{-1}$ makes the cooling rate smaller as expected. If $R_i^{-1}=0$, Eqs. (3.11) and (3.12) give[3,4]

$$\frac{T}{T_i}=\left[\frac{\tau_i}{\tau}\right]^{M/3}\ . \tag{3.13}$$

The role of $R$ can be examined by rewriting Eq. (3.9) as

$$\frac{\partial s}{\partial\tau}=(R^{-1}-1)\frac{Ms}{\tau}\ . \tag{3.14}$$

As is seen in this equation, the entropy density increases if $R<1$ and decreases if $R>1$. When $R=1$, the entropy production due to the dissipative processes balances with the dilution of the entropy caused by the expansion of the fluid. On the other hand, $R$ increases monotonically towards a limiting value $R_\infty$ during the expansion stage if $s,\eta,\zeta\propto T^3$ as is shown in Figs. 1 and 2. Therefore, there is a peak in $s$ (and hence in $T$) if $R_i<1$ and $R_\infty>1$, and no peak if $R_i>1$ or $R_\infty<1$.

In Fig. 1 the $\tau$ dependence of $T$, $s$, and $R^{-1}$ for $M=1$ and $N=3$ with two typical values of $R_i$ is compared with the perfect-fluid case. For $R_i=9.52$ which may correspond to QGP (see Sec. V.), $T$, $s$, and $R^{-1}$ monotonically

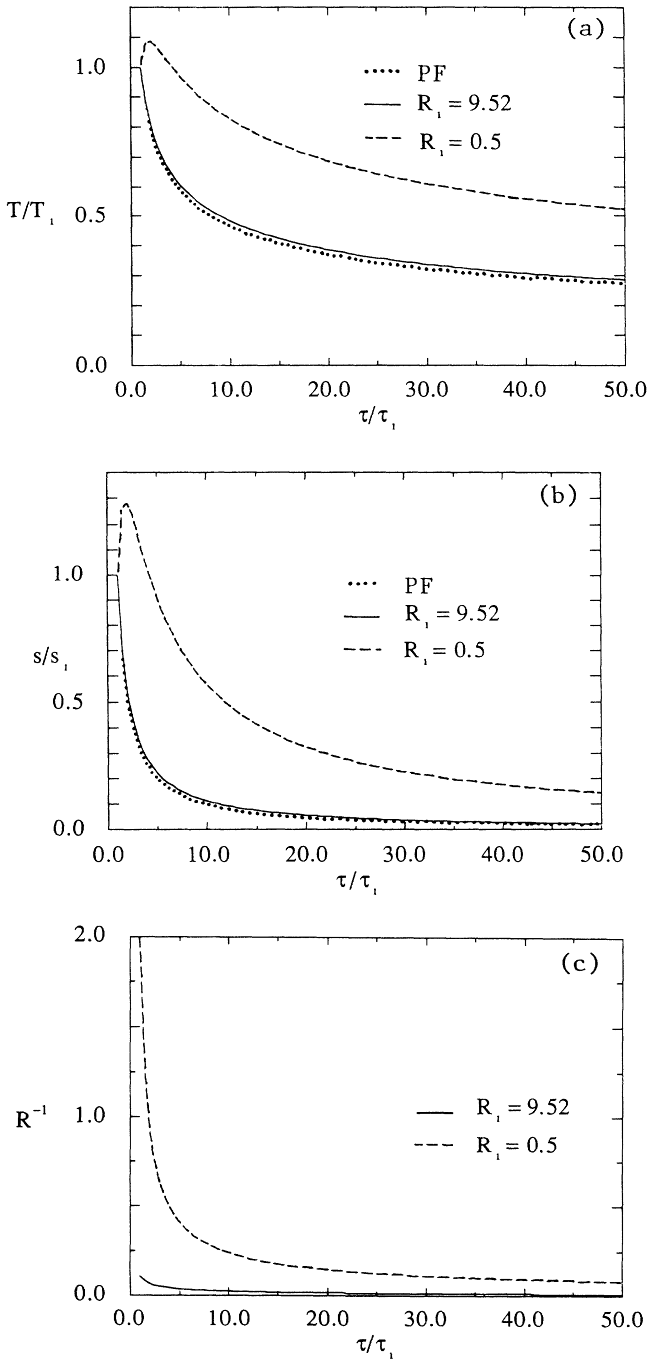


FIG. 1. The $\tau$ dependence of (a) temperature $T$, (b) entropy density $s$, and (c) the inverse of Reynolds number $R^{-1}$ in the scaling solution with $M=1$ and $N=3$ when $s,\eta,\zeta\propto T^3$. The initial conditions are $R_i=9.52$ (solid line) and 0.5 (dashed line). The dotted line corresponds to the case of the perfect fluid (PF).

decrease. For $R_i = 0.5$, $T$ and $s$ have peaks while $R^{-1}$ monotonically decreases. In both cases $R_\infty$ is infinite.

We also show the $\tau$ dependence of the $T$, $s$, and $R^{-1}$ in the case of $M = N = 3$ in Fig. 2. It is easily seen that the cooling is much faster than in the case of $M = 1$ and $N = 3$.

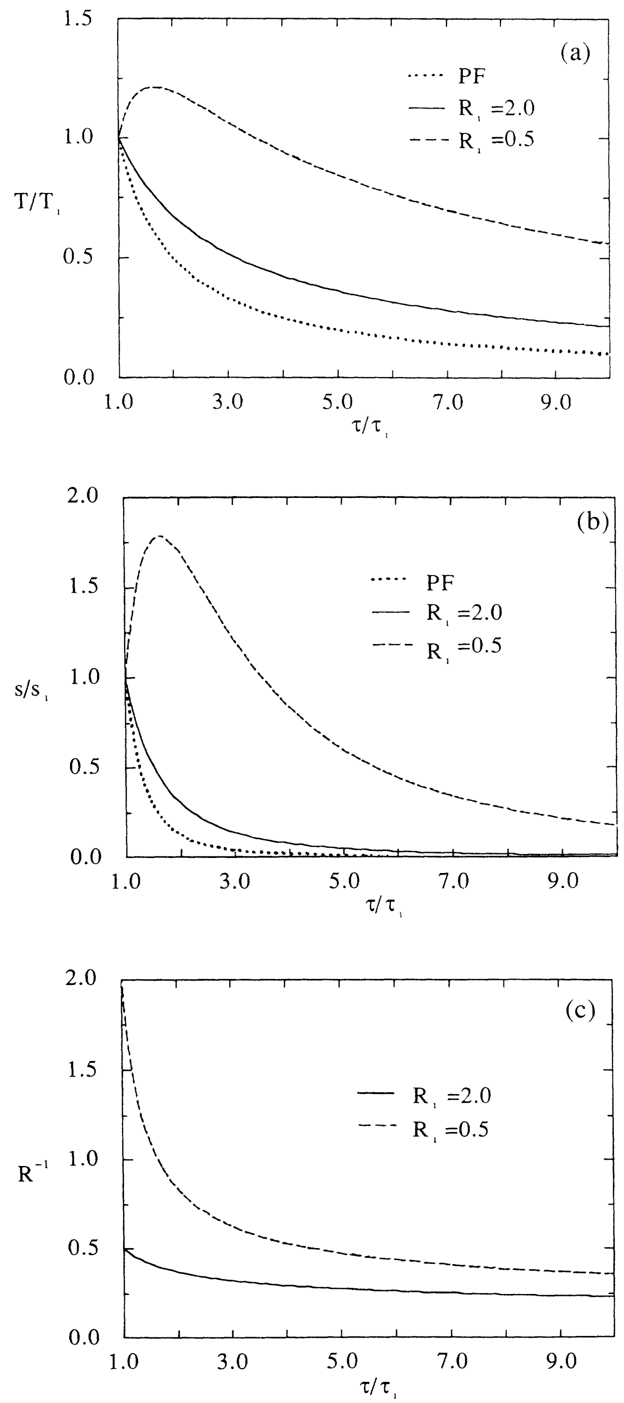


FIG. 2. The $\tau$ dependence of (a) temperature $T$, (b) entropy density $s$, and (c) the inverse of Reynolds number $R^{-1}$ in the scaling solution with $M = N = 3$ when $s, \zeta \propto T^3$. The initial conditions are $R_i = 2.0$ (solid line) and 0.5 (dashed line). The dotted line corresponds to the case of the perfect fluid (PF).

## IV. STABILITY OF THE SCALING SOLUTION

In this section we discuss the stability of the $(1+1)$-dimensional scaling solution in $(1+1)$- and $(3+1)$-dimensional space-time when the dissipative processes exist. The stability means that small deviations from the scaling solutions damp through the whole development and ensures that the scaling solutions can be realized in realistic physical problems such as heavy-ion collisions. Since we do not consider the transverse expansion, we treat $(1+1)$-dimensional flow in $(3+1)$-dimensional space-time as if it was the flow in $(1+1)$-dimensional space-time. In our treatment, the only difference between the two cases lies in the definition of the viscosity coefficients as will be shown later. Our approach corresponds to the work done by Baym *et al.* for the perfect fluid in $1+1$ dimensions (Ref. 13).

### A. Relativistic hydrodynamics in 1+1 dimensions

First, we transform the coordinate variable $x^\mu = (x^0, x^1) = (t, z)$ into the proper time $\tau$ and a light-cone variable $y$ which are defined by

$$\tau \equiv \sqrt{t^2 - z^2} \quad \text{and} \quad y \equiv \tfrac{1}{2} \ln \left[ \frac{t+z}{t-z} \right] . \tag{4.1}$$

It is easy to check that

$$\begin{bmatrix} \partial_0 \\ \partial_1 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial t} \\ \dfrac{\partial}{\partial z} \end{bmatrix} = \begin{bmatrix} \cosh y & -\sinh y \\ -\sinh y & \cosh y \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial \tau} \\ \dfrac{1}{\tau} \dfrac{\partial}{\partial y} \end{bmatrix} . \tag{4.2}$$

Further, we use the rapidity $\theta$ instead of $u^\mu$:

$$u^\mu = (\cosh\theta, \sinh\theta) . \tag{4.3}$$

It is convenient to introduce the differential operators $D$ and $\nabla$ defined by

$$\begin{bmatrix} D \\ \nabla \end{bmatrix} \equiv \begin{bmatrix} \cosh(\theta - y) & \sinh(\theta - y) \\ \sinh(\theta - y) & \cosh(\theta - y) \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial \tau} \\ \dfrac{1}{\tau} \dfrac{\partial}{\partial y} \end{bmatrix} . \tag{4.4}$$

Notice that $D = u^\mu \partial_\mu$. Because $\nabla^\mu u^\nu = \Delta^{\mu\nu} \partial^\rho u_\rho = \Delta^{\mu\nu} \nabla\theta$, we get

$$\Pi^{\mu\nu} = \xi \Delta^{\mu\nu} \nabla\theta , \tag{4.5}$$

where

$$\xi = \begin{cases} \zeta & \text{for } M = 1, N = 1 , \\ \tfrac{4}{3}\eta + \zeta & \text{for } M = 1, N = 3 . \end{cases} \tag{4.6}$$

This result indicates that $\frac{4}{3}\eta + \zeta$ in the $3+1$ case works just like a bulk viscosity in the $1+1$ case. In other words, there is only one kind of viscosity in $(1+1)$-dimensional space-time because of the relation $\nabla^\mu u^\nu = \Delta^{\mu\nu} \partial^\rho u_\rho$. The entropy equation (2.16) and the equation of motion (2.17) are now expressed as

$$D\ln s+\nabla\theta=\frac{\xi}{Ts}(\nabla\theta)^2+\frac{\mu}{Ts}\partial^\mu h_\mu \tag{4.7}$$

and

$$D\theta+\frac{s\nabla T+n\nabla\mu}{Ts+\mu n}=\frac{\xi}{Ts+\mu n}[(\nabla\ln\xi)(\nabla\theta)+(D\theta)(\nabla\theta)+(\nabla\nabla\theta)]\ . \tag{4.8}$$

From now on, we assume $\mu=\kappa=0$ and introduce the quantity $\Phi$ defined by

$$d\Phi=c_s d\ln s=\frac{d\ln T}{c_s}\quad(\text{at }\mu=0)\ , \tag{4.9}$$

where $c_s$ is the sound velocity. By using this quantity instead of $T$ and $s$, Eqs. (4.7) and (4.8) are reduced to a symmetric form as

$$D\Phi+c_s\nabla\theta=c_sR^{-1}\tau(\nabla\theta)^2 \tag{4.10}$$

and

$$D\theta+c_s\nabla\Phi=R^{-1}\tau[(\nabla\ln\xi)(\nabla\theta)+(D\theta)(\nabla\theta)+(\nabla\nabla\theta)]\ , \tag{4.11}$$

where

$$R^{-1}\equiv\frac{\xi}{Ts\tau}\ . \tag{4.12}$$

This definition of the Reynolds number is in accordance with Eq. (3.10) (Ref. 8). We remark that Eqs. (4.10) and (4.11) are valid not only for the scaling solutions but also for the general (1+1)-dimensional flows. Equations (4.10) and (4.11) are cast into the following convenient form:

$$\left[\tau\frac{\partial}{\partial\tau}+\tanh(\theta-y\pm\theta_s)\frac{\partial}{\partial y}\right](\Phi\pm\theta)=R^{-1}\tau^2[\cosh(\theta-y)\pm c_s\sinh(\theta-y)]^{-1}\times\{c_s(\nabla\theta)^2\pm[(\nabla\ln\xi)(\nabla\theta)+(D\theta)(\nabla\theta)+(\nabla\nabla\theta)]\}\ , \tag{4.13}$$

where $\theta_s=\text{arctanh}c_s$.

Putting into Eq. (4.13) the scaling solutions Eqs. (3.2) and (3.4) which are equivalent to

$$\theta(\tau,y)=y \tag{4.14}$$

and

$$\Phi(\tau,y)=\Phi_0(\tau)\ , \tag{4.15}$$

we get

$$\tau\frac{\partial\Phi_0}{\partial\tau}=(R_0^{-1}-1)c_s^0\ . \tag{4.16}$$

From now on, the subscript or superscript zero means that the quantity is calculated using the scaling solution. Equation (4.16) corresponds to Eq. (3.14). It implies that $\Phi_0(\tau)$ increases as $\tau$ increases if $R_0<1$ and decreases if $R_0>1$. At the point $R_0=1$, $\Phi_0$ reaches a maximum which corresponds to the peak in Figs. 1(a) and 1(b).

### B. The stability of the scaling solutions

Next we consider small deviations of $\theta$ and $\Phi$ from the scaling solutions. We write the solutions as

$$\theta(\tau,y)=y+\delta\theta(\tau,y) \tag{4.17}$$

and

$$\Phi(\tau,y)=\Phi_0(\tau)+\delta\Phi(\tau,y)\ . \tag{4.18}$$

Putting them into Eq. (4.13), retaining only the lowest-order terms of $\delta\Phi(\tau,y)$ and $\delta\theta(\tau,y)$, we get

$$\left[\tau\frac{\partial}{\partial\tau}\pm c_s^0\frac{\partial}{\partial y}\right](\delta\Phi\pm\delta\theta)\pm(1-c_s^{02})\delta\theta+\left[\frac{\partial c_s}{\partial\Phi}\right]_0\delta\Phi$$
$$=R_0^{-1}c_s^0\left[\left[\frac{\partial\ln(R^{-1}\tau)}{\partial\Phi}\right]_0\delta\Phi+\frac{1}{c_s^0}\left[\frac{\partial c_s}{\partial\Phi}\right]_0\delta\Phi+2\frac{\partial(\delta\theta)}{\delta y}\right]\mp R_0^{-1}c_s^{02}\delta\theta$$
$$\pm R_0^{-1}\left[\left[\frac{\partial\ln\xi}{\partial\Phi}\right]_0\left[\tau\frac{\partial\Phi_0}{\partial\tau}\delta\theta+\frac{\partial(\delta\Phi)}{\partial y}\right]+\tau\frac{\partial(\delta\theta)}{\partial\tau}+\frac{\partial^2(\delta\theta)}{\partial y^2}\right]\ . \tag{4.19}$$

From Eq. (4.19) with the Fourier transformation

$$\delta s(\tau,y)=s_0\delta\Phi(\tau,y)/c_s^0 = \int dk \exp(iky)\delta s(\tau,k) \quad (4.20)$$

and

$$\delta\theta(\tau,y)=\int dk \exp(iky)\delta\theta(\tau,k) , \quad (4.21)$$

we obtain

$$\tau\frac{\partial}{\partial\tau}\begin{bmatrix} \frac{\delta s(\tau,k)}{s_0} \\ i\delta\theta(\tau,k) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} \frac{\delta s(\tau,k)}{s_0} \\ i\delta\theta(\tau,k) \end{bmatrix} = \mathbf{A}\begin{bmatrix} \frac{\delta s(\tau,k)}{s_0} \\ i\delta\theta(\tau,k) \end{bmatrix} \quad (4.22)$$

for $R_0\neq 1$ where

$$A_{11}=c_s^0 R_0^{-1}\left[\frac{\partial \ln(R^{-1}\tau)}{\partial\Phi}\right]_0 , \quad (4.23)$$

$$A_{12}=-k(1-2R_0^{-1}) , \quad (4.24)$$

$$A_{21}=kc_s^0\left[c_s^0-R_0^{-1}\left[\frac{\partial \ln\xi}{\partial\Phi}\right]_0\right]\Big/(1-R_0^{-1}) \quad (4.25)$$

and

$$A_{22}=-\left[(1-c_s^{02})+c_s^{02}R_0^{-1} + c_s^0 R_0^{-1}(1-R_0^{-1})\left[\frac{\partial\ln\xi}{\partial\Phi}\right]_0 + k^2 R_0^{-1}\right]\Big/(1-R_0^{-1}) . \quad (4.26)$$

For $R_0=1$, we have

$$\tau\frac{\partial}{\partial\tau}\left[\frac{\delta s(\tau,k)}{s_0}\right] = c_s^0\left\{\left[\frac{\partial\ln(R^{-1}\tau)}{\partial\Phi}\right]_0 + k^2\left[c_s^0-\left[\frac{\partial\ln\xi}{\partial\Phi}\right]_0\right]\Big/(1+k^2)\right\}\frac{\delta s(\tau,k)}{s_0} \quad (4.27)$$

and

$$\delta\theta(\tau,k)=\left\{-ikc_s^0\left[c_s^0-\left[\frac{\partial\ln\xi}{\partial\Phi}\right]_0\right]\Big/(1+k^2)\right\} \times\frac{\delta s(\tau,k)}{s_0} . \quad (4.28)$$

If we put $R_0^{-1}=0$, Eqs. (4.22)–(4.26) reduce to the result for the perfect fluid obtained by Baym *et al.*[13]

If $s,\xi\propto T^3$ (and hence $c_s$ is constant), these equations are written in a more compact form. For $R_0\neq 1$ Eqs. (4.23)–(4.26) reduce to

$$A_{11}=-c_s^{02}R_0^{-1} , \quad (4.29)$$

$$A_{12}=-k(1-2R_0^{-1}) , \quad (4.30)$$

$$A_{21}=kc_s^{02}(1-3R_0^{-1})/(1-R_0^{-1}) , \quad (4.31)$$

and

$$A_{22}=-[(1-c_s^{02})+c_s^{02}R_0^{-1}+3c_s^{02}R_0^{-1}(1-R_0^{-1}) + k^2R_0^{-1}]/(1-R_0^{-1}) . \quad (4.32)$$

Equations (4.27) and (4.28) become

$$\tau\frac{\partial}{\partial\tau}\left[\frac{\delta s(\tau,k)}{s}\right] = -c_s^{02}\left[1+\frac{2k^2}{1+k^2}\right]\frac{\delta s(\tau,k)}{s_0} \quad (4.33)$$

and

$$\delta\theta(\tau,k)=\frac{2ikc_s^{02}}{1+k^2}\frac{\delta s(\tau,k)}{s_0} . \quad (4.34)$$

Now we discuss about the stability of the scaling solution. We define that the solution is "stable" if any small deviations damp through the time development of the solution. If the solution is not stable we call it "unstable." However, before considering the stability through the whole process of the development, we study whether or not the deviations grow for given $R_0$ and $k$. For simplicity we assume $s,\xi\propto T^3$ and $c_s^2=\frac{1}{3}$. This condition is approximately satisfied in the case of the weakly interacting fluid composed of massless particles. First, we consider the case $R_0=1$. In this case the deviations damp since $-c_s^{02}[1+2k^2/(1+k^2)]<0$ in Eqs. (4.33) and (4.34). Next we consider the case $k=0$. The two components of Eq. (4.22) decouple, and $\delta s/s_0$ damps since $A_{11}$ is negative, while $\delta\theta$ damps if $A_{22}<0$. The condition $A_{22}<0$ yields $R_0>R_c=1$ or $R_0<R_c'\sim 0.58$. When $R_0\neq 1$ and $k\neq 0$ the situation is more complicated since the two components of Eq. (4.22) do not decouple and the matrix $\mathbf{A}$ depends on $\tau$. We study the stability of the scaling solution using Lyapunov's method[16] (see the Appendix) on the $R_0^{-1}$-$k$ plane. The result are summarized in Fig. 3.

In the region where $R_0\geq 1$ (region I) the magnitude of the deviations damps for any $k$. It also damps for fixed $k$ when $R_0$ is small enough (region IV) and grows for fixed $R_0$ when $k$ is large enough (region II). The stability has not been confirmed in region III. As is seen later, we can study the stability of the scaling solution through the whole process without the knowledge of region III. One has $R_0\geq R_{0i}$ ($R_{0i}$ is the initial Reynolds number), since $R_0$ monotonically increases through the expansion of the fluid as seen in Fig. 1(c). For fixed $k$, the scaling solutions move from the right to the left on the $R_0^{-1}$-$k$ plane shown in Fig. 3. The scaling solution is stable for $R_{0i}\geq 1$. This is consistent with the result for the perfect fluid.[13] On the other hand, if $R_{0i}<1$, the scaling solution becomes very unstable as $R_0$ approaches to 1. The growth of the magnitude of the deviations diverges like $\tau^{1/\epsilon}$ as $R_0^{-1}\to 1+\epsilon$ for small $\epsilon>0$. Therefore, the line $R_0^{-1}=1$ on the $R_0^{-1}$-$k$ plane is a critical line for the stability.

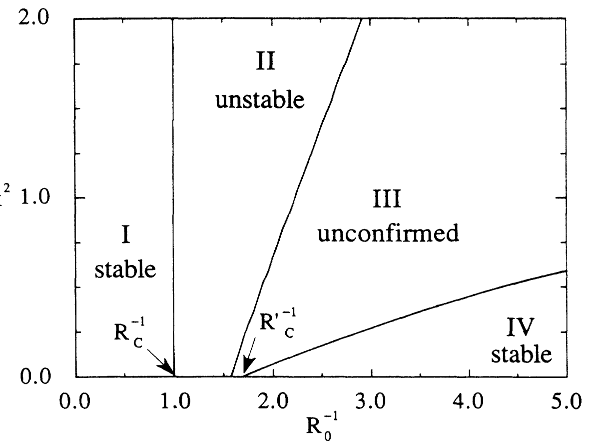


FIG. 3. The stability of the $(1+1)$-dimensional scaling solution examined on the $R_0^{-1}$-$k^2$ plane. In regions I and IV ($R_c=1$, $R_c'\sim 0.58$), the scaling solutions are stable, while they are unstable in region II. The stability in region III is not confirmed yet.

The singularity at $R_0=R_c=1$ is explained as follows. The equation of motion for $M=1$ is rewritten as

$$(\epsilon+p-\xi\partial^\rho u_\rho)u_\nu\partial^\nu u_\mu=\nabla_\mu(p-\xi\partial^\rho u_\rho) \ . \tag{4.35}$$

The factor $p-\xi\partial^\rho u_\rho$ works as an effective pressure, and hence the factor $\epsilon+p-\xi\partial^\rho u_\rho$ works as an effective enthalpy density. In the $(1+1)$-dimensional scaling solution with $\mu=0$, this effective enthalpy density reduces to $w(1-R_0^{-1})$. It is positive if $R_0>1$ while negative if $R_0<1$. The effective enthalpy density vanishes when $R_0=1$. We also remark that the critical Reynolds number $R_c=1$ corresponds to the peak of the temperature and the entropy density as seen in Figs. 1(a) and 1(b). The entropy production due to the viscosity balances with the dilution of the entropy due to the expansion of fluid when $R_0=R_c$.

## V. APPLICATION TO THE QGP FLUID

### A. QGP fluid

In this section we examine the stability condition of the $(1+1)$-dimensional scaling solution found in Sec. IV for the QGP fluid. First, we set the values of the parameters used in Sec. IV. We neglect heat conductivity for simplicity, and put $\mu=0$ which may be well realized in the central region $(y\sim 0)$ of high-energy heavy-ion collisions. We assume the following relations which are approximately satisfied in the case of weakly interacting massless particles:

$$s=4aT^3,\quad c_s^2=\tfrac{1}{3},\quad \tfrac{4}{3}\eta+\zeta=bT^3 \ . \tag{5.1}$$

For the values of $a$ and $b$ we employ the results by Hosoya and Kajantie.[9] Using the relativistic kinetic theory for the QGP matter, they have obtained

$$a\approx 1.75+1.15N_f \ ,$$
$$1/b\approx\begin{bmatrix}12\\2.7\end{bmatrix}\alpha_s^2\ln(1/\alpha_s)\quad\text{for } N_f=\begin{bmatrix}0\\2\end{bmatrix} \ , \tag{5.2}$$

where $N_f$ is the number of the quark flavor and $\alpha_s$ is the QCD coupling. If we put $N_f=2$ and assume $\alpha_s=0.8$ in Eq. (5.2), we get $a\approx 4.05$ and $b\approx 2.59$. As is mentioned in Ref. 9, $\alpha_s$ has a logarithmic dependence on $T$. However, we treat $b$ as a constant.

Next, we discuss the relevant range of the wave number $k$ which is introduced in the Fourier transform of Eqs. (4.20) and (4.21). There is a cutoff at large $k$ in the case of the realistic fluid. Fixing $\tau$ $(d\tau=0)$, we get

$$dz=\left.\frac{\partial z}{\partial y}\right|_{\tau:\text{fixed}}dy=t\,dy \ . \tag{5.3}$$

From the uncertainty principle, $k_{\max}$ is roughly estimated as

$$k_{\max}\sim\frac{1}{(\Delta y)_{\min}}=\frac{t}{(\Delta z)_{\min}} \ , \tag{5.4}$$

where $(\Delta y)_{\min}$ and $(\Delta z)_{\min}$ are the minimum scales of $y$ and $z$. It is obvious that $(\Delta z)_{\min}$ cannot be smaller than the mean distance $n_p^{-1/3}$ between the nearest particles, where $n_p$ is the particle number density estimated as

$$n_p\sim\frac{\epsilon}{m_t}\sim\frac{\epsilon}{T} \ , \tag{5.5}$$

$m_t$ being the transverse mass of the particle. Then, we get

$$k_{\max}\sim\frac{t}{(\Delta z)_{\min}}\lesssim t(\epsilon/T)^{1/3} \ . \tag{5.6}$$

For $\epsilon\sim 1$ GeV/fm$^3$, $T\sim 300$ MeV, and $t\sim 1-10$ fm, we get

$$k_{\max}\lesssim 1.5-15 \ . \tag{5.7}$$

### B. Region I

As seen in Sec. IV, the scaling solution is stable when it is in region I of Fig. 3. The stability condition is given by

$$1\le R_{0i}\quad\text{or}\quad \tau_iT_i\ge\frac{b}{4a}\quad(\text{for any } k) \ . \tag{5.8}$$

If we choose the initial condition as $\tau_i=1$ fm and $T_i=300$ MeV, we get $R_{0i}\approx 9.52$ and Eq. (5.8) is satisfied. For this value of $R_{0i}$, the development of the temperature, the entropy density, and the Reynolds number has already been given in Fig. 1. If we use a rather small value of $\alpha_s$, for example, $\alpha_s=0.3$, we get $R_{0i}\approx 7.22$ which also satisfies Eq. (5.8). In Fig. 4 we show the region where Eq. (5.8) is satisfied on the $\alpha_s$-$\tau_iT_i$ plane. The scaling solution may be unstable if $\alpha_s$ is very small. The region of possible instability enlarges as $N_f$ increases.

### C. Region II and III

The scaling solutions are unstable in region II. The higher-order effects are important to see the behavior of the actual solution in this region. It is not at all clear at present whether this region is relevant to QGP. A reli-

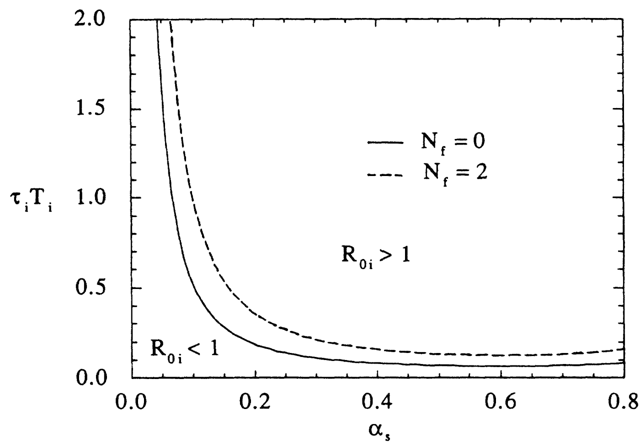


FIG. 4. The region which satisfies $R_{0i} \geq 1$ is shown on the $\alpha_s$-$\tau_i T_i$ plane. The solid line is the condition $R_{0i} = 1$ when $N_f = 0$. The dashed line is the condition $R_{0i} = 1$ when $N_f = 2$.

able estimate of viscosity coefficients near the transition point is necessary. The stability of region III may be important if $k_{\max}$ is not very small.

### D. Region IV

The deviations from the scaling solution damp in region IV of Fig. 3. The stability condition is approximately expressed as

$$R_0 \lesssim (5.4k^2 + 1.7)^{-1} . \quad (5.9)$$

As seen in Eq. (5.7), $k$ has the upper bound. Equation (5.9) is satisfied if $R_0 \lesssim (0.07-0.001)$. It may be possible that, if the scaling QGP fluid is realized in region IV, hadronization takes place before the solution enters the unstable region II.

## VI. SUMMARY AND DISCUSSION

The results obtained in this paper are summarized as follows.

(1) We have formulated the scaling solutions of the relativistic hydrodynamics in any spatial dimension.

(2) In the scaling solution, when $R_0 = 1$, the cooling of the fluid due to the expansion of the flow balances with the entropy production which is caused by the dissipative processes.

(3) Assuming $\mu = \kappa = 0$, we have transformed the $1+1$ hydrodynamic equations of the viscous fluid into the form which are convenient for studying the stability of the scaling solution.

(4) Using these equations, we have studied the stability of the scaling solutions against small perturbations in detail when the dissipative processes exist.

Further results are obtained by assuming that $s = 4aT^3$, $c_s^2 = \frac{1}{3}$, and $\frac{4}{3}\eta + \zeta = bT^3$ (weakly interacting massless particles).

(5) For small perturbations along the two (time and longitudinal) directions, the $(1+1)$-dimensional scaling solutions are stable in region I of Fig. 3 (when $R_0 \geq R_c = 1$) and in region IV, while they are unstable in region II.

(6) The change of the sign of the effective enthalpy density $w - \xi\partial^\mu u_\mu$ at $R = R_c = 1$ plays a crucial role to separate regions I and II.

(7) The condition for the $(1+1)$-dimensional scaling solution being in region I through the whole development is given by

$$\tau_i T_i \geq \frac{b}{4a} . \quad (6.1)$$

Condition (6.1) is likely realized in the massless weakly interacting quark-gluon fluid, unless the QCD coupling $\alpha_s$ is very small. It is thus very probable that the scaling solutions have a realistic meaning for the problems of QGP. On the other hand, it is not yet clear whether or not the stability region IV has a physical relevance for the case of the QGP fluid.

The qualitative features of the result of Fig. 3 including the value of $R_c$ do not change even if $c_s^2 \neq \frac{1}{3}$ as long as $c_s$ is a nonzero constant.

A kinetic theory argument of Ref. 10 shows that the following inequality should hold in order to apply the Navier-Stokes theory:

$$\eta < \tfrac{1}{4}\epsilon\tau \quad \text{(up to a factor of two uncertainty)} . \quad (6.2)$$

This inequality is rewritten as

$$\eta < \tfrac{3}{4}aT^4\tau \quad (6.3)$$

for a gas of weakly interacting massless particles. On the other hand, our stability condition $R_0 \geq 1$ reads

$$\eta \leq 3aT^4\tau - \tfrac{3}{4}\zeta \quad (6.4)$$

for the same situation with $M = 1$, $N = 3$. This inequality is satisfied and hence the scaling solution is stable if the inequality (6.3) is satisfied and $\zeta$ is smaller than $3aT^4\tau$.

We remark that the critical Reynolds number $R_c (=1)$ is different from the one known in the nonrelativistic hydrodynamics in the following points. (i) The viscosity $\xi$ which is used in Sec. IV works as a bulk viscosity in $1+1$ dimension. On the other hand, the Reynolds number used in the nonrelativistic hydrodynamics is defined in terms of a shear viscosity and the transverse motion plays an important role for the stability of the flow. (ii) In the nonrelativistic hydrodynamics the flow is stable (unstable) when the Reynolds number is smaller (larger) than the critical value. This behavior is opposite to the one which we found around $R_c$. In this point, another critical value $R_c'$ ($\approx 0.58$) which we have also found in Sec. IV is more analogous to the critical Reynolds number in the nonrelativistic cases rather than $R_c$.

There are still many open questions about the stability of the scaling solution.

(1) The results (2) and (6) mentioned above have apparently different origins. There may be a deeper connection between them.

(2) In this paper, we have examined the stability up to the lowest order of small perturbations. The higher-order effects may play an important role in the instability region II. The numerical calculations may be indispensable.

(3) The effect of transverse motion should be studied. If it is taken into account, a critical Reynolds number which exactly corresponds to the one known in the nonrelativistic case may be found.

(4) The effect of boundary or finiteness of the fluid should be studied further.[14,17]

(5) For the quark-gluon fluid the effect of hadronization is important. In particular, it is expected that the behavior of the transport coefficients around the transition point has a great importance for the whole development of the system.[18]

(6) The stability should be checked under more general conditions; e.g., $\mu\neq 0$, $\kappa\neq 0$, more general equation of state, etc. It is also very interesting to study the stability of the scaling solution for general $M$ and $N$, in particular, the case for $M=N=3$.

(7) The stability of region III is under investigation, as it may be important in some cases.

Kajantie, Raitio, and Ruuskanen solved the hydrodynamic equations numerically for the $(1+1)$-dimensional perfect fluid and compared the results with the scaling solution.[19,20] Recently, Akase *et al.* have studied the stability of the scaling solution numerically for some case with viscous terms.[12] However, the equation of state and viscosities used in Ref. 12 are not the same as ours.

## ACKNOWLEDGMENTS

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## APPENDIX

We consider the differential equation

$$\frac{d\mathbf{X}(t)}{dt}=\mathbf{A}(t)\mathbf{X}(t)\ , \tag{A1}$$

where $\mathbf{X}(t)=(X_1,X_2)$ is a two-component real vector and $\mathbf{A}(t)$ is a real $2\times 2$ matrix. Equation (A1) has the solution that $\mathbf{X}(t)=\mathbf{0}$. We study whether or not the deviations from the solution $\mathbf{X}(t)=\mathbf{0}$ at some $t$ develop as $t$ increases. We transform Eq. (A1) as

$$\frac{d\mathbf{X}'(t)}{dt}=\mathbf{A}'(t)\mathbf{X}'(t)\ , \tag{A2}$$

where

$$\mathbf{X}(t)'=\mathbf{B}\mathbf{X}(t)\equiv\begin{bmatrix}1 & \alpha\\ 0 & \beta\end{bmatrix}\begin{bmatrix}X_1\\ X_2\end{bmatrix}$$

and (A3)

$$\mathbf{A}'(t)=\mathbf{B}\,\mathbf{A}(t)\mathbf{B}^{-1}\ .$$

We remark that $\alpha$ and $\beta$ $(\neq 0)$ are constants. By defining $V(t)\equiv X_1'^2+X_2'^2=(X_1+\alpha X_2)^2+\beta^2X_2^2$, one has

$$\frac{dV(t)}{dt}=2\mathbf{X}'(t)^T\frac{d\mathbf{X}'(t)}{dt}=2\mathbf{X}'(t)^T\mathbf{A}'(t)\mathbf{X}'(t)$$
$$=\mathbf{X}'(t)^T[\mathbf{A}'(t)^T+\mathbf{A}'(t)]\mathbf{X}'(t)\ . \tag{A4}$$

We remark that $V(t)\geq 0$ and the equality is satisfied only when $X_1=0$ and $X_2=0$ (i.e., $\mathbf{X}=\mathbf{0}$). Increasing $V(t)$ with increasing $t$ will signify the instability. We can also show that

$$\lambda_{\min}(t)V(t)\leq\mathbf{X}'(t)^T[\mathbf{A}'(t)^T+\mathbf{A}'(t)]\mathbf{X}'(t)$$
$$\leq\lambda_{\max}(t)V(t)\ , \tag{A5}$$

where $\lambda_{\min}(t)$ and $\lambda_{\max}(t)$ are the smallest and the biggest eigenvalue of the symmetric matrix $\mathbf{A}'(t)^T+\mathbf{A}'(t)$, respectively. From Eqs. (A4) and (A5) we get

$$\lambda_{\min}(t)V(t)\leq\frac{dV(t)}{dt}\leq\lambda_{\max}(t)V(t)\ . \tag{A6}$$

From Eq. (A6), it can be shown that the solution $\mathbf{X}'(t)=\mathbf{0}$ is stable [hence $\mathbf{X}(t)=\mathbf{0}$ is stable] if $\lambda_{\max}(t)\leq 0$ and the solution $\mathbf{X}'(t)=\mathbf{0}$ is unstable [hence $\mathbf{X}(t)=\mathbf{0}$ is unstable] if $\lambda_{\min}(t)>0$. These inequalities on $\lambda_{\max}(t)$ and $\lambda_{\min}(t)$ may be realized if one chooses $\alpha$ and $\beta$ appropriately. This procedure is a simple example of Lyapunov's method and $V(t)$ is called Lyapunov's function.[16]


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