Field theory and the nonrelativistic quark model: A parametrization of the meson masses

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We apply the method developed in previous papers [Phys. Rev. D **40**, 2997 (1989); **40**, 3111 (1989)] to write a general parametrization of the masses of the lowest meson nonets. We compare the mass formulas obtained in this way with those used currently in a nonrelativistic quark model (NRQM). As in the previous work, the aim is to separate the features specific to the NRQM from those that follow from a relativistic field theory satisfying a few rather general conditions (satisfied by QCD). For the meson masses it turns out that the formulas obtained from field theory have essentially the same structure and number of parameters as those in the NRQM. This explains the good fit that the NRQM provides in this case; the internal v/c of the $q\bar{q}$ pair is irrelevant. As a by-product, the mixing angle for the $I=0$ mesons (e.g., η and η') predicted from the masses turns out to be model independent to first order in flavor breaking. We discuss the question of the linear versus quadratic or, say, square-root mixing angles.

I. INTRODUCTION

This paper is a continuation of two papers[1,2] addressed to the following problem: to separate in the predictions of the nonrelativistic quark model (NRQM) the features specific to the model from those that follow from a relativistic field theory under rather general conditions (which QCD satisfies). It is assumed that such a field theory is constructed in terms of quark and neutral gluon fields and has, as QCD, a strong Hamiltonian free of λ-flavor matrices,[3] except for λ_8. To answer the above question we display the parametrizations of various physical quantities in relativistic field theory and compare them with those used in the NRQM.

In Ref. 1 we introduced a unitary transformation V connecting the exact states of a hadron to its model states. Using this transformation, we obtained general parametrizations for the magnetic moments of the lowest baryon octet ("general parametrizations" means those obtained from the relativistic field theory, say QCD) as a sum of several terms with different spin-flavor structures, each multiplied by some coefficient. We also gave general parametrizations for the baryon masses[1] and the semileptonic decays.[2] Because these general parametrizations have a form similar to the formulas usually written in a nonrelativistic quark model[4] (NRQM), we could see that the formulas of the NRQM represent a selection of certain terms in the complete parametrizations; in the cases examined it emerged that the terms considered in the NRQM are in fact the dominant ones. In particular, the parametrizations for the baryon magnetic moments, masses, and semileptonic decays have clarified the meaning of using the NRQM also for systems of "light" quarks, even if v/c is not necessarily much smaller than one.

Here we will apply the method to the lowest mesonic nonets, find the general parametrizations for their masses, and compare the results with the NRQM.

II. REDUCTION OF THE FIELD THEORY CALCULATION TO A TWO-BODY PARAMETRIZATION

We define the model states ϕ_M for mesons in a way similar to the model states ϕ_B for baryons in Ref. 1. We choose the model Hamiltonian $\mathcal{H}$ for the mesons, operating only in the subspace of the Fock states with one quark and one antiquark, no gluon, as an operator having a nonrelativistic, exceedingly simple structure; it is defined in a way similar to that followed for the baryons[1] (where $\mathcal{H}$ acted, of course, in the subspace of the Fock states with three quarks, no gluon); $\mathcal{H}$ conserves the orbital angular momentum and is spin and flavor independent. The model states of the pseudoscalar and vector nonets are therefore degenerate states of $\mathcal{H}$:

$$\mathcal{H}|\phi_{Mi}\rangle|\text{no gluons}\rangle = M_0^0|\phi_{Mi}\rangle|\text{no gluons}\rangle \quad (i=\pi,\eta,\eta',K;\rho,\omega,\phi,K^*)\ . \tag{1}$$

In (1) $|\phi_i\rangle$ is a state of a meson at rest and M_0^0 is the common value of the masses of the 18 model states of the 0^- and 1^-, lowest $L=0$ mesons. From now on we omit writing the factor $|\text{no gluon}\rangle$ in (1) and also suppress the index M in ϕ_{Mi}. Because of the spin and flavor independence of the model Hamiltonian $\mathcal{H}$, the wave function ϕ_i of a meson has the form

$$\phi_i = S_c W_i \varphi(r)\ , \tag{2}$$

where S_c is the color-singlet factor for a meson (that can be forgotten in what follows), W_i is the $q\bar{q}$ spin-flavor wave function of the ith meson, and $\varphi(r)$ is a function of the relative distance r of the quark and antiquark.

The exact states $|\psi_i\rangle$ of the nonet mesons are obtained by applying to the states $|\phi_i\rangle$ the unitary operator V:

$$|\psi_i\rangle = V|\phi_i\rangle\ . \tag{3}$$

As described in Ref. 1, V is constructed in terms of the exact field-theoretical Hamiltonian H (say, the QCD Hamiltonian); we assume that H (and thus V) is written in terms of the renormalized quark masses, to be identified with the masses of the constituent quarks, possibly in the range of some hundreds of MeV for the light quarks; as for the baryons, the precise values assigned to these masses $m_{\mathcal{P}}=m_{\mathcal{N}}$ and m_λ will not intervene, except to ensure that the flavor-breaking parameter $\Delta m/m_\lambda \equiv (m_\lambda - m_{\mathcal{P}})/m_\lambda$ is decently small and can be treated perturbatively. Note also that ϕ_i, the eigenstates of the model Hamiltonian $\mathcal{H}$, are typical nonrelativistic states written with two-component Pauli spinors, whereas V is constructed in terms of creation and destruction operators of four-component Dirac fields. To define the operation of V on ϕ_i we simply imagine the Pauli spinors of the nonrelativistic wave function completed with zeros in the lower components (of course, also $\mathcal{H}$ is formally extended to act on four-component spinors, with no matrix element connecting the upper and lower components and being zero when operating on the latter).

In the above notation the mass of the ith meson is

$$M_i=\langle \psi_i|H|\psi_i\rangle=\langle \phi_i|V^\dagger H V|\phi_i\rangle \ . \tag{4}$$

Because $|\phi_i\rangle$ is, by construction, a two-body state (one quark, one antiquark) only the projection $\tilde{H}$ of the operator $V^\dagger H V$ in the subspace of these two-body states intervenes in the calculation of (4):

$$\tilde{H}\equiv\sum|1q,1\bar{q}\rangle\langle 1q,1\bar{q}|V^\dagger H V|1q',1\bar{q}\,'\rangle\langle 1q',1\bar{q}\,'| \ , \tag{5}$$

where by $|1q,1\bar{q}\rangle$ we mean a complete set of states of one quark, one antiquark (with total three-momentum zero, since we consider a meson at rest) and the sums in (5) are over such a set of states.

The above statement that the only part of $V^\dagger H V$ intervening in the calculation of M_i (4) is the two-body part $\tilde{H}$ is trivial but essential; in fact calculating the expectation value of the field operator $V^\dagger H V$ in the state $|\phi_i\rangle$ *becomes equivalent to calculating the expectation value of a certain quantum-mechanical two-body operator* $\tilde{H}$ on the wave function ϕ_i in ordinary nonrelativistic two-body quantum mechanics:

$$M_i=\langle \phi_i|\tilde{H}|\phi_i\rangle \ . \tag{6}$$

The most general parametrization of the meson masses amounts then to write in ordinary nonrelativistic two-body quantum mechanics the most general operator $\tilde{H}$ of the relative coordinate and momentum $\mathbf{r},\mathbf{p}$ of the spins $\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2$ (1=quark, 2=antiquark) and of the flavor operators f (see below) invariant with respect to translations and rotations.

III. THE LIST OF THE SPIN FLAVOR TERMS IN THE PARAMETRIZATION

The most general operator $\tilde{H}$ of the space and spin-flavor variables of the quark-antiquark pair can be written in the form

$$\tilde{H}=\sum_\nu R_\nu(\mathbf{r},\mathbf{p})G_\nu(\boldsymbol{\sigma},f) \ . \tag{7}$$

Here $G_\nu(\boldsymbol{\sigma},f)$ is the set of independent operators (specified by the index ν) constructed in terms of $\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2$ and the flavor operators f of the quark and antiquark; $R_\nu(\mathbf{r},\mathbf{p})$ are operators (the expression of which we do not need to know) constructed in terms of $\mathbf{r}$ and $\mathbf{p}$ of the quark-antiquark pair.

On performing the expectation value (6) of $\tilde{H}$, the masses M_i become

$$M_i=\sum_\nu g_\nu\langle W_i|G_\nu(\boldsymbol{\sigma},f)|W_i\rangle \ , \tag{8}$$

where g_ν stands for

$$g_\nu=\langle \varphi(r)|R_\nu(\mathbf{r},\mathbf{p})|\varphi(r)\rangle \ . \tag{9}$$

Because of the fact that we have chosen the model Hamiltonian $\mathcal{H}$ independent from the flavor and from J, the space wave function $\varphi(r)$ in (9) is independent of the meson index i and also the coefficients g_ν in (8) are independent of i. Therefore the mass operator (of which the expectation value has to be taken purely on the spin-flavor part W_i of the wave function) is

$$M=\sum_\nu g_\nu G_\nu(\boldsymbol{\sigma},f) \ . \tag{10}$$

Note that this is a general result from field theory; its interest (as for the analogous expression for the baryons in Ref. 1) is in the fact that it appears in a notation directly related to that of the nonrelativistic quark model.

To list the possible operators $G_\nu(\boldsymbol{\sigma},f)$ in (10), we first discuss their flavor dependence. As in Ref. 1 we restrict[3] ourselves to a field theory (such as QCD) where the only flavor λ matrix in the strong Hamiltonian is λ_8; in fact in QCD λ_8 intervenes in the quark mass difference term and the flavor-invariant part of the Hamiltonian does not contain λ-flavor matrices at all. Instead of λ_8 we consider the projection operator P^λ,

$$P^\lambda=\tfrac{1}{3}(1-\lambda_8) \ , \tag{11}$$

that gives zero when applied to the $\mathcal{P}$ or $\mathcal{N}$ quarks (or antiquarks) and is 1 when applied to the λ quark (or antiquark); it is $(P^\lambda)^n=P^\lambda$. For a system of one quark and one antiquark, the list of flavor operators acting on states with isospin $I\neq 0$ is

$$1, \ \ P_1^\lambda, \ \ P_2^\lambda \ . \tag{12}$$

In (12) and in the following, the index 1 refers to the quark and the index 2 to the antiquark. For the mesons with $I=0$ the list (12) is incomplete. Indeed the flavor operators in (12) connect only a $q\bar{q}$ pair to a $q\bar{q}$ of the same type (say $\mathcal{P}\bar{\mathcal{P}}$ with $\mathcal{P}\bar{\mathcal{P}}$, or $\mathcal{N}\bar{\mathcal{N}}$ with $\mathcal{N}\bar{\mathcal{N}}$ or $\mathcal{P}\bar{\lambda}$ with $\mathcal{P}\bar{\lambda}$, and so on); the sum in Eq. (5), however, includes, for mesons with isospin $I=0$, also matrix elements of $\tilde{H}$ that connect a $\mathcal{P}\bar{\mathcal{P}}$ state with a $\mathcal{N}\bar{\mathcal{N}}$ state or a $\lambda\bar{\lambda}$ state with a $\mathcal{P}\bar{\mathcal{P}}+\mathcal{N}\bar{\mathcal{N}}$ state (in QCD they are due to the gluon annihilation diagrams). Thus there are other flavor operators in addition to (12); to list them we first introduce the $I=0$ flavor states:

$$|z\rangle=|\mathcal{P}_1\bar{\mathcal{P}}_2+\mathcal{N}_1\bar{\mathcal{N}}_2+\lambda_1\bar{\lambda}_2\rangle \ , \quad |w\rangle=|\lambda_1\bar{\lambda}_2\rangle \ . \tag{13}$$

Note that in (13) $|z\rangle$ is not normalized; for normalization a factor $(1/\sqrt{3})$ should be inserted; while keeping this in mind, (13) is more convenient for calculations. Then to the list (12) we must add the operators

$$|z\rangle\langle z| \,, \tag{14a}$$

$$|z\rangle\langle w|+|w\rangle\langle z| \,, \tag{14b}$$

$$|w\rangle\langle w|\equiv P_1^\lambda P_2^\lambda \,. \tag{14c}$$

The first of these operators [(14a)] is a unitary singlet; the second [(14b)] transforms linearly in λ_8 [precisely as $P^\lambda=(1-\lambda_8)/3$] under $SU(3)_{\rm flavor}$. The third [(14c)] transforms quadratically in P^λ. We should have included in (14) also $i(|z\rangle\langle w|-|w\rangle\langle z|)$; we did not because the expectation value of this operator vanishes on states with a real wave function (as ϕ_i's are).

As to the $\boldsymbol{\sigma}$ dependence of the $G_v(\boldsymbol{\sigma},f)$'s, only the following expressions (15) are possible:

$$1, \quad \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 \,. \tag{15}$$

IV. THE MESON MASSES

We start discussing the *mesons* with $I\neq 0$ $(\pi,K;\rho,K^*)$. For them [compare the Eq. (12)] the flavor-breaking part of the $G_v(\boldsymbol{\sigma},f)$'s contains only P_1^λ and P_2^λ. Multiplying the spin operators (15) with the flavor operators (12) and noting that the charge-conjugation invariance of the Hamiltonian implies that the G_v's must be invariant for the exchange of 1 and 2, the list of G_v's is $1,\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$, $P_1^\lambda+P_2^\lambda$, $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2(P_1^\lambda+P_2^\lambda)$. Therefore the *most general expression* of the mass of a $I\neq 0$ meson, *correct to all orders in flavor breaking,* is

$$M_{I\neq 0}=A+B\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+C(P_1^\lambda+P_2^\lambda)+D\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2(P_1^\lambda+P_2^\lambda)\,, \tag{16}$$

where A,B,C,D are four real coefficients. Because $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2=-3$ for $J=0$ and $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2=+1$ for $J=1$, the masses (indicated with the meson symbols) are

$$\pi=A-3B\ (=138),\quad K=A-3B+C-3D\ (=495)\,, \tag{17}$$
$$\rho=A+B\ (=770),\quad K^*=A+B+C+D\ (=894)\,.$$

Equations (17) imply (in MeV)

$$A=612,\quad B=158,\quad C=182,\quad D=-58\,. \tag{18}$$

Note finally that, restricting ourselves to the $I\neq 0$ mesons, an equation of the form (16) holds not only for the masses, but also for their squares or for any power of the masses [of course with values of the coefficients different from (18)].

We now *consider the mesons with* $I=0$. For them the most general mass formula is constructed adding to (16) [with the coefficients A,B,C,D just determined from the masses of $(\pi,K;\rho,K^*)$] another part obtained by multiplying the flavor operators (14) with the spin operators (15); that is,

$$\begin{aligned}M_{I=0}=&A+B\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+C(P_1^\lambda+P_2^\lambda)+D\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2(P_1^\lambda+P_2^\lambda)\\&+(E+F\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)|z\rangle\langle z|\\&+(H+G\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(|z\rangle\langle w|+|w\rangle\langle z|)\\&+(N+T\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)P_1^\lambda P_2^\lambda\,.\end{aligned} \tag{19}$$

Equation (19) contains in addition to A,B,C,D [determined (18) from $\pi,K;\rho,K^*$] six parameters E,F,H,G,N,T; one might determine them from the masses of $\eta,\eta';\omega,\phi$ plus the two mixing angles (the vector θ_V and the pseudoscalar θ_P), if the angles are known in a way free of theoretical uncertainties. Alternatively we can limit to the first-order flavor-breaking approximation, as we did for the baryons in Ref. 1; in such a case the last term in Eq. (19), which is of second order in flavor breaking [being proportional to $P_1^\lambda P_2^\lambda$, it is of order $(\Delta m/m_\lambda)^2$] can be disregarded $(N=0,\ T=0)$. In this first-order flavor-breaking approximation the masses of $\eta,\eta';\omega,\phi$ fix E,F,H,G; one can then determine, to first order in flavor breaking, the mixing angles.

We restrict ourselves here to the determination of the parameters for the pseudoscalar (P) mesons η,η'; for them $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2=-3$. For the vector mesons one proceeds similarly, simply putting $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2=+1$ and therefore replacing below $A-3B$ with $A+B;C-3D$ with $C+D;E-3F$ with $E+F;H-3G$ with $H+G$. As is well know, the parametrization with A,B,C,D alone is almost sufficient for the $I=0$ vector mesons (the vector mixing angle is nearly ideal); this amounts to saying that in (19) $E+F\approx 0$ and $H+G\approx 0$; of course there would be no problem in calculating the exact parameters and angles also for the vector mesons.

Let us proceed with the P mesons; we introduce the abbreviations

$$\begin{aligned}&b=A-3B\ (=138\ \text{MeV}),\quad d=C-3D\ (=357\ \text{MeV}),\\&f=E-3F,\quad g=H-3G\,,\end{aligned} \tag{20}$$

where b and d are known from (18). Thus Eq. (19) for the P mesons with $I=0$ takes the form

$$\begin{aligned}M_{I=0}(P)=&b+d(P_1^\lambda+P_2^\lambda)+f|z\rangle\langle z|\\&+g(|z\rangle\langle w|+|w\rangle\langle z|)+O((\Delta m/m_\lambda)^2)\,.\end{aligned} \tag{21}$$

The diagonalizing equation

$$\begin{vmatrix}b+f-M & f & f+g\\ f & b+f-M & f+g\\ f+g & f+g & b+2d+f+2g-M\end{vmatrix}=0 \tag{22}$$

determines the coefficients f and g from the masses of η and η'. One solution of (22) is the π^0 mass; of course we get $\pi^0=b\equiv A-3B$ [compare (17)]; as to the masses of η and η', they are the two roots of

$$(b+2f-M)(b+2d+f+2g-M)-2(f+g)^2=0\,. \tag{23}$$

The solutions of (23) are $M=b+k\pm(k^2+2g^2-4fd)^{1/2}$ where $2k\equiv 3f+2d+2g$. Equating the two solutions with the masses of η(549 MeV) and η'(958 MeV) we obtain $\eta'+\eta=2b+3f+2d+3g$ and $(\eta'-\eta)/2=[2g^2-4fd+\frac{1}{4}(3f+2d+2g)^2]^{1/2}$ where b and d have the values of (20). Solving for g and f we obtain two possible solutions $(g;f)_1$ and $(g;f)_2$; in MeV they are

$$(g;f)_1=(-133,+261)\ ,$$
$$(g;f)_2=(-343,+400)\ . \tag{24}$$

The ratio $(g/f\sqrt{3})$ between the coefficient $g\sqrt{3}$ of the (normalized) flavor-breaking term and the coefficient $3f$ of the (normalized) unitary singlet term is substantially smaller for the solution $N.1$ than for the solution $N.2$. Thus we must choose the solution $N.1$ to be consistent with the assumption of neglecting terms of order higher than the first in $(\Delta m/m_\lambda)$. On choosing $(g;f)_1$ it is straightforward to check that the diagonalization (22) implies [in addition to $\pi^0=(1/\sqrt{2})(\mathcal{P}\bar{\mathcal{P}}-\mathcal{N}\bar{\mathcal{N}})$]

$$\eta=0.603(\mathcal{P}\bar{\mathcal{P}}+\mathcal{N}\bar{\mathcal{N}})-0.522\lambda\bar{\lambda}\ , \tag{25}$$

$$\eta'=0.367(\mathcal{P}\bar{\mathcal{P}}+\mathcal{N}\bar{\mathcal{N}})+0.854\lambda\bar{\lambda}\ . \tag{26}$$

With the usual definition $\eta=\eta_1\sin\theta_P+\eta_8\cos\theta_P$ and $\eta'=\eta_1\cos\theta_P+\eta_8\sin\theta_P$, where $\eta_1=(1\sqrt{3})(\mathcal{P}\bar{\mathcal{P}}+\mathcal{N}\bar{\mathcal{N}}+\lambda\bar{\lambda})$ and $\eta_8=(1/\sqrt{6})(\mathcal{P}\bar{\mathcal{P}}+\mathcal{N}\bar{\mathcal{N}}-2\lambda\bar{\lambda})$, we have, from (25) and (26),

$$\sin\theta_P\simeq-0.39\quad(\text{that is, }\theta_P\simeq-23^\circ)\ . \tag{27}$$

V. COMPARISON OF THE GENERAL PARAMETRIZATIONS WITH THOSE OF THE NRQM

A. Mesons with $I\neq 0$

Equation (16) has the same form as that currently used in the NRQM to fit the $L=0$ meson masses (compare, e.g., Ref. 5). It now emerges that (independently of the NRQM) the four parameter equation (16) is the most general expression possible for the lowest nonet meson masses in any (relativistic field) theory where the only λ $SU(3)_{\text{flavor}}$ that intervenes is λ_8. It is an exact equation to all orders in flavor breaking. This fact [that Eq. (16) is an exact formula, no more an approximate NRQM result] has, in our view, an important implication for the pion: The small mass of the pion appears now just an accident that depends rather critically on the values of the parameters A and B in (18); assume, for instance, that B in (16) had a value 100 MeV instead of 158 MeV [Eq. (18)]; then the pion, if A keeps its value (18), would have a mass=312 MeV, no longer so small (the ρ mass would then become 712 MeV). Now, taking, e.g., QCD as the basic theory, a reduction of the parameter B in (16) by the above amount of $\approx 40\%$ may possibly be obtained with an even smaller percentage variation of the quark-gluon coupling. Thus it seems difficult to attribute any profound meaning (as still often done) to the smallness of the pion mass. This remark does not question at all, of course, the phenomenological treatments or results related to PCAC (partial conservation of axial-vector current), insofar as they simply take note of the empirical smallness of the pion masses.

We also compare Eq. (16) with the NRQM formula for the meson masses of De Rújula, Georgi, and Glashow[6] [their formula (17)] derived from QCD in the one-gluon-exchange approximation. A feature of that formula (see also Ref. 7) is that $\sigma_1\cdot\sigma_2$ appears only multiplied by the factor $(m_1m_2)^{-1}$ where m_1 and m_2 are the masses of the quark and antiquark; this factor $(m_1m_2)^{-1}$ cannot be obtained by general arguments, as far as we can see. One can check, however, that the general parametrization Eq. (16) above reduces to the formula (17) of Ref. 6, to first order in $(m_\lambda-m_{\mathcal{P}})/m_\lambda$, if we choose $D/B=(m_{\mathcal{P}}/m_\lambda)-1$. From the experimental values of D and B [Eq. (18) above] one gets, as noted already in Ref. 6, $(m_{\mathcal{P}}/m_\lambda)\simeq 0.63$, that is, $(m_\lambda-m_{\mathcal{P}})/m_\lambda\simeq 0.37$ to be compared with an equal value derived by similar arguments from the baryon masses and $(m_\lambda-m_{\mathcal{P}})/m_\lambda\simeq 0.35$ from the baryon magnetic moments (compare also Ref. 1; the comparison should not be made with the value $(15.4/49.2)\simeq 0.31$ derived there from Eq. (87), but with the value $\simeq 0.37$ that one obtains fitting the data without the parameter E).

B. Mesons with $I=0$

(1) The P mixing angle. Having suppressed terms of order $(\Delta m/m_\lambda)^2$ in Eq. (19), the results for $I=0$ mesons are correct only to first-order flavor breaking, as already stated; it should be stressed, however, that to that order they are exact results; what we mean by this is that if we were able to solve the equations of the correct field theory (say, QCD) and were allowed to neglect terms of order higher than the first in $\Delta m/m_\lambda$, we would obtain, in particular, the value $\theta_P\simeq-23^\circ$ for the P mixing angle. A remark on the question of the quadratic versus linear mixing angle is appropriate at this stage. We have parametrized, above, $M_i=\langle\phi_i|V^\dagger HV|\phi_i\rangle$, but clearly, as stated in Ref. 1 (Sec. XI), we might have parametrized as well $M_i^n=\langle\phi_i|V^\dagger H^nV|\phi_i\rangle$ for any real n, positive or negative, integer or fractional. The form of the parametrization is the same independently of n; it is in all cases an equation of the form (16) or (19); simply one has to replace on the left-hand side of these equations M by M^n; the coefficients A,B,C,D,E,F,H,G,N,T have of course different numerical values for each different n. If we do not neglect N and T [the second-order terms in flavor breaking in (19)] then the exact mixing angle is always the same, independently of n; the same mixing corresponds, say, to a quadratic (M^2) or square-root $(M^{1/2})$ mass formula; indeed the exact eigenstates of H^2 or $H^{1/2}$ are the same. However, as already noted, keeping N and T there is no connection between the mixing angle and the masses; a connection arises only neglecting N and T. However, then we get different values of the mixing angle corresponding to different choices of n; for instance, one can check that for θ_P we have

$$(M^{1/2})\ \theta_P\simeq-31^\circ,\quad(M)\ \theta_P\simeq-23^\circ,\quad(M^2)\ \theta_P\simeq-10^\circ\ .$$

Note that in all cases the procedure takes into account exactly first-order flavor breaking; but note also that the expansion of the mixing angle in terms of the flavor-breaking parameter $\Delta m/m_\lambda$ starts with a term of first order in flavor breaking:

$$\sin\theta_P^{\text{exact}}=\alpha_n\frac{\Delta m}{m_\lambda}+O((\Delta m/m_\lambda)^2)\ ,$$

where α_n is some coefficient depending on the selected value of n. Because $\Delta m/m_\lambda \simeq 0.37$ is not so small, terms $O((\Delta m/m_\lambda)^2)$ may contribute significantly to θ_P, with a different relative weight for different n's. Which is the best angle? No theorem can be stated in general; only solving the problem in QCD, say, can one get the correct angle. However the following remark seems relevant: If one were to solve the problem in QCD, namely, determine the mixing angle as an expansion in the flavor-breaking parameter, one would certainly never dream of using the square or square root of the QCD Hamiltonian (indeed to perform calculations with H^2 or $H^{1/2}$ the question arises, in the first place, of defining properly such operators, not only via their eigenvalues M_i^2 or $M_i^{1/2}$).

Thus in performing a field-theoretical calculation, one would use H and get $\theta_P \simeq -23°$ plus terms $O((\Delta m/m_\lambda)^2)$. In this sense the linear mixing angle is certainly the most natural one; it represents the choice made, without saying, practically in any perturbative diagonalization problem of a Hamiltonian.

(2) Comparison with the NRQM results. We first compare our parametrization for the $I=0$ mesons with the NRQM work of Sacharov and Zeldovich[5] (SZ). Specifically we compare our Eq. (19) with the expression for the masses proposed in SZ [their Eq. (3) $H=H_a+H_b$, with H_a and H_b given, respectively, by their Eqs. (4) and (1)]. The structure and the number of parameters contained in the quoted equations of SZ is identical to that of our Eq. (19) in the approximation $N=T=0$; the only minor difference is that SZ keep terms of second order in flavor breaking [in their Eqs. (1) and (4) ignore α^2 and replace $(1-\beta)^2$ with $1-2\beta$]. With this modification the expression for the masses of SZ becomes equal to our Eq. (22) with the following correspondence of our parameters b,d,f,g with those of SZ: $b=(a_0-\frac{3}{4}c_0)_{\text{SZ}}$, $d=(b_0+\frac{3}{4}c_0\alpha)_{\text{SZ}}$, $f=(d_0)_{\text{SZ}}$, $g=-(d_0\beta)_{\text{SZ}}$ (Ref. 8).

We next compare our general parametrization (19) for the $I=0$ mesons with that of De Rújula, Georgi, and Glashow[6] (DGG). The mass matrix of DGG [appearing before their Eq. (23)] differs from ours (22) because DGG neglect in their matrix the flavor-breaking correction (our term with coefficient g). In fact if we identify our f with the β appearing in the mass matrix of DGG and include the first-order flavor-breaking corrections g (that DGG should have been included because the quark masses appear in the quark lines of the annihilation diagrams) our mass matrix and that of DGG coincide.

VI. CONCLUSION

The main conclusions of the above analysis are as follows.

(1) Almost all the features of the mass formulas for the lowest meson nonets currently used in a NRQM description are general consequences of the field-theoretical parametrization, independent of the NRQM; the only exception is the $(m_1m_2)^{-1}$ factor multiplying the $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$ term in the DGG expression[7] for the meson masses, a result that depends specifically on the one-gluon-exchange potential.[9] Stated differently, the meson mass formulas used in the NRQM are more general than one might have thought; for the mesons with $I\neq 0$ they possess a number of free parameters as large as that of a complete relativistic parametrization; for the mesons with $I=0$, the same is true neglecting a term of second order in flavor breaking. In this respect the situation for the meson masses is quite different, for instance, from that found for the baryon magnetic moments;[1] there, we recall, the general field-theoretical expression of the magnetic moments (correct to first order in flavor breaking) had seven parameters, whereas the successful parametrization of the NRQM had only two.

(2) The procedure described above to derive the meson mass formulas clarifies the old question of the mixing angle for the $I=0$ meson nonets. The mixing angle (say θ_P) can be determined from the general parametrization (that is in a model-independent way) knowing the masses of the P mesons provided terms of order higher than the first in the flavor-breaking expansion parameter $\Delta m/m_\lambda$ are negligible and are neglected. According to this procedure the linear angle is much more natural than the quadratic or square-root ones; it would be the angle resulting from a QCD calculation on expanding the exact result in series of $\Delta m/m_\lambda$ and neglecting terms $O((\Delta m/m_\lambda)^2)$.

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[1]G. Morpurgo, Phys. Rev. D **40**, 2997 (1989).

[2]G. Morpurgo, Phys. Rev. D **40**, 3111 (1989).

[3]In Refs. 1 and 2 the assumption just stated has been phrased repeatedly in an unclear and confusing way, although it was used correctly. Statements such as "provided the flavor breaking is due only the mass difference between the quarks" or "the Lagrangian is flavor independent except for the mass terms" should have been phrased as "The only λ-SU(3)$_{\text{flavor}}$ matrix present in the strong Hamiltonian is λ_8 (in the flavor-breaking mass term)." As I repeat, this is the property (compare Sec. VI of Ref. 1) that was in fact always used.

[4]G. Morpurgo, Physics (N.Y.) **2**, 95 (1965).

[5]Ya. B. Zeldovich and A. D. Sakharov, Yad. Fiz. **4**, 395 (1966) [Sov. J. Nucl. Phys. **4**, 283 (1967)] (indicated as SZ).

[6]A. De Rújula, H. Georgi, and S. Glashow, Phys. Rev. D **12**, 147 (1975) (indicated as DGG).

[7]A. D. Sacharov, Pis'ma Zh. Eksp. Teor. Fiz. **21**, 554 (1975) [JETP Lett. **21**, 258 (1975)].

[8]There seems to be a numerical mistake in the Eqs. (5) and (5a) of SZ for the η and η': (a) solving the Eqs. (5) and (5a) of SZ (with their input parameters) we find $d_0=267$ instead of their value $d_0=580$ and $\beta=0.49$ instead of their 0.75; (b) this is confirmed by our solution $(g;f)_1$ that gives $f=(d_0)_{\text{SZ}}=261$ and $\beta=-(g/f)=0.51$; the differences 261 and 267, 0.51 and 0.49 are due to slight differences of input data, the α^2 term, and the use of $(1-\beta)^2$ instead of $(1-2\beta)$.

[9]Needless to say, a fundamental calculation from the basic Lagrangian (say QCD), if feasible, evaluates in all cases all the parameters in terms, say, of the gluon-quark coupling constant and few others. In a sense the derivation of the above $(m_1 m_2)^{-1}$ factor is an example of this, although of course in this case the (old) question arises of why one should trust a Fermi-Breit one-gluon-exchange treatment of QCD at low Q^2.