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Radiative K_{l3} decays and chiral symmetry

Barry R. Holstein
Department of Physics and Astronomy, University of Massachusetts, Amherst, Massachusetts 01003
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Chiral perturbation theory is used in order to analyze the $K_{l3\gamma}$ process and results are compared with the PCAC (partial conservation of axial-vector current) and Low-theorem analyses of Fearing, Fischbach, and Smith. These predictions are found to be consistent with but much stronger than those of the earlier analysis, and verification of these results can be used as a test of chiral symmetry.

I. INTRODUCTION

Interest in the theory of radiative K_{l3} decay peaked nearly two decades ago with the careful work of Fearing, Fischbach, and Smith (FFS) in which they calculated the spectra of both charged and neutral modes using the Low theorem and PCAC (partial conservation of axial-vector current) (Ref. 1). They concluded that, unfortunately, observation of a structure-dependent effect was difficult for $K_{\mu 3\gamma}$ and nearly impossible for $K_{e3\gamma}$ decay. Since that time there have been a number of experimental studies which are quite consistent with these calculations, and there is no evidence as yet that any such structure dependence has been detected.[2] From a strictly experimental perspective then there is at present little interest in such radiative decays. However, from a theoretical vantage point this is *not* the case and it is of interest to analyze the radiative-K_{l3}-decay sector from the viewpoint of contemporary particle physics.

Below we shall undertake such an analysis and will demonstrate that QCD, or more specifically chiral symmetry, makes unambiguous predictions for $K_{l3\gamma}$ decay spectra which are well beyond what FFS were able to do two decades earlier. In principle verification of such predictions would provide a test of the QCD framework and would be of great significance. In Sec. II then we shall outline the chiral perturbation formalism on which our analysis will be based. In Sec. III we shall utilize this formalism to analyze the $K_{l3\gamma}$ decay sector and will compare with earlier theoretical work. Finally our results will be summarized in a concluding Sec. IV.

II. CHIRAL FORMALISM

Since the chiral perturbation formalism of Gasser and Leutwyler has been detailed elsewhere,[3] we shall be content to present here only a brief synopsis in order to define notation used in the remainder of the paper.

In the limit of vanishing quark mass QCD is invariant under separate left- and right-handed (global) rotations

$$\begin{aligned} q_L &\to \exp\left[i\sum_{j=1}^{8}\lambda_j\alpha_j\right]q_L \equiv Lq_L\ , \\ q_R &\to \exp\left[i\sum_{j=1}^{8}\lambda_j\beta_j\right]q_R \equiv Rq_R\ , \end{aligned} \tag{1}$$

i.e., under chiral $SU(3)_L \otimes SU(3)_R$. Here λ_j are the usual Gell-Mann matrices. This symmetry is spontaneously broken to $SU(3)_V$ and Goldstone's theorem demands the existence of eight massless pseudoscalar bosons.[4] In the real world, of course, since quark masses are nonvanishing these Goldstone bosons, identified with the π, K, and η mesons, have nonzero but small masses, and it is the interactions among these particles which will be described below. The restrictions which chiral symmetry places upon these interactions are best described in terms of the nonlinear representation

$$U = \exp\left[i\sum_{j=1}^{8}\frac{1}{F_\pi}\lambda_j\phi_j\right]\ , \tag{2}$$

where ϕ_j are the pseudoscalar fields and $F_\pi = 94$ MeV is the pion decay constant. Under chiral rotations U transforms as

$$U \to LUR^\dagger\ . \tag{3}$$

Assuming that the nonzero quark mass breaking of chiral invariance can be treated perturbatively, the simplest Lagrangian constructed from U which is consistent with both Lorentz and chiral invariance in addition to U(1) gauge invariance is

$$\mathcal{L}^{(2)} = \tfrac{1}{4}F_\pi^2\,\mathrm{Tr}[D_\mu U D^\mu U^\dagger + m(U+U^\dagger)] - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\ , \tag{4}$$

where

$$D_\mu U \equiv \partial_\mu U + ie[Q,U]A_\mu \tag{5}$$

is the covariant derivative and

$$m = C\begin{bmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{bmatrix},\quad Q = \begin{bmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}. \tag{6}$$

The first term of Eq. (4) is chiral invariant and includes the meson kinetic energy, while the second piece transforms as $(3_L,\bar{3}_R)+(\bar{3}_L,3_R)$ and describes the pseudoscalar masses provided

$$C = \frac{2m_K^2}{m_d+m_s} = \frac{2m_\pi^2}{m_u+m_d} = \frac{6m_\eta^2}{m_u+m_d+4m_s}\ . \tag{7}$$

The third component is simply the free-photon Lagrangian.

Tree-level evaluation of $\mathcal{L}^{(2)}$ yields, at $O(p^2,m^2)$, lowest-order current-algebra–PCAC results. For example, in the π^4 sector we find the well-known Weinberg scattering lengths[5]

$$a_0^0=\frac{7m_\pi}{32\pi^2F_\pi^2},\quad a_0^2=\frac{-m_\pi}{16\pi^2F_\pi^2}\ . \tag{8}$$

However, such a tree-level calculation violates unitarity. Inclusion of loops solves this problem but introduces divergences at $O(p^4,p^2m^2,m^4)$. Such infinities can be absorbed into renormalizing phenomenological couplings of an order-four Lagrangian, such as given by Gasser and Leutwyler:[3]

$$\begin{aligned}\mathcal{L}^{(4)}=&L_1(\mathrm{Tr}D_\mu U D^\mu U^\dagger)^2+L_2(\mathrm{Tr}D_\mu U D_\nu U^\dagger)^2+L_3\mathrm{Tr}(D_\mu U D^\mu U^\dagger)^2+L_4\mathrm{Tr}(D_\mu U D^\mu U^\dagger)\mathrm{Tr}m(U+U^\dagger)\\&+L_5\mathrm{Tr}D_\mu U D^\mu U^\dagger(mU+U^\dagger m)+L_6[\mathrm{Tr}m(U+U^\dagger)]^2+L_7[\mathrm{Tr}m(U-U^\dagger)]^2+L_8\mathrm{Tr}(mUmU+mU^\dagger mU^\dagger)\\&-iL_9\mathrm{Tr}(F^L_{\mu\nu}D^\mu U D^\nu U^\dagger+\mathrm{Tr}F^R_{\mu\nu}D^\mu U^\dagger D^\nu U)+L_{10}\mathrm{Tr}F^L_{\mu\nu}UF^{\mu\nu R}U^\dagger+L_{11}\mathrm{Tr}F_{\mu\nu}+L_{12}\mathrm{Tr}m^2\ .\end{aligned} \tag{9}$$

Here $F^L_{\mu\nu},F^R_{\mu\nu}$ are external field-strength tensors defined via

$$F^{L,R}_{\mu\nu}=\partial_\mu F^{L,R}_\nu-\partial_\nu F^{L,R}_\mu-i[F^{L,R}_\mu,F^{L,R}_\nu],\quad F^L_\mu=V_\mu-A_\mu,\quad F^R_\mu=V_\mu+A_\mu \tag{10}$$

and $L_1,\dots,L_{12}$ are arbitrary. The L_i coefficients are themselves unphysical inasmuch they can be used to absorb divergent loop corrections from the lowest-order chiral Lagrangian $\mathcal{L}^{(2)}$. However, renormalized coefficients can be defined via

$$L_i^r(\mu)=L_i+\frac{\Gamma_i}{32\pi^2}\left[\frac{1}{\epsilon}+\ln\frac{4\pi}{\mu^2}+1-\gamma\right]\ . \tag{11}$$

Here $L_i^r(\mu)$ are the physical (renormalized) couplings measured at scale μ and

$$\begin{aligned}&\Gamma_1=\tfrac{1}{32},\quad \Gamma_2=\tfrac{3}{16},\quad \Gamma_3=0,\quad \Gamma_4=\tfrac{1}{8},\quad \Gamma_5=\tfrac{3}{8},\quad \Gamma_6=\tfrac{11}{144}\ ,\\&\Gamma_7=0,\quad \Gamma_8=\tfrac{5}{48},\quad \Gamma_9=\tfrac{1}{4},\quad \Gamma_{10}=-\tfrac{1}{4},\quad \Gamma_{11}=-\tfrac{1}{8},\quad \Gamma_{12}=\tfrac{5}{24}\end{aligned} \tag{12}$$

are constants which cancel the $\mathcal{L}^{(2)}$ divergences. We shall also require the piece of the Lagrangian which arises from the anomaly, which has the form[6]

$$\begin{aligned}\mathcal{L}^{(4)}_{\mathrm{anom}}=-\frac{N_c}{48\pi^2}\int d^4x\,\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}[&-F^R_\mu R_\nu R_\rho R_\sigma-F^L_\mu L_\nu L_\rho L_\sigma+\tfrac{1}{2}F^R_\mu R_\nu F^R_\rho R_\sigma-\tfrac{1}{2}F^L_\mu L_\nu F^L_\rho L_\sigma\\&-F^R_\mu U^{-1}F^L_\nu UR_\rho R_\sigma+F^L_\mu UF^R_\nu U^{-1}L_\rho L_\sigma+\partial_\mu F^R_\nu U^{-1}F^L_\rho UR_\sigma+\partial_\mu F^L_\nu UF^R_\rho U^{-1}L_\sigma\\&+(F^R_\mu\partial_\nu F^R_\rho+\partial_\mu F^R_\nu F^R_\rho)R_\sigma+(F^L_\mu\partial_\nu F^L_\rho+\partial_\mu F^L_\nu F^L_\rho)L_\sigma+\cdots]\ .\end{aligned} \tag{13}$$

Here

$$R_\sigma=U^{-1}\partial_\sigma U,\quad L_\sigma=\partial_\sigma U\,U^{-1} \tag{14}$$

and only the terms of $\mathcal{L}^{(4)}_{\mathrm{anom}}$ relevant to our calculation have been included.

Gasser and Leutwyler have obtained empirical values for the phenomenological couplings $L_1,\dots,L_{10}$ in Ref. 3 and this Lagrangian $\mathcal{L}^{(4)}$ can be used at the tree level together with $\mathcal{L}^{(2)}$ at the tree and one-loop levels and with $\mathcal{L}^{(4)}_{\mathrm{anom}}$ in order to generate a picture of low-energy mesonic interactions consistent with QCD. In order to see how this program is carried out we give a simple example. Imagine one is calculating the pion electromagnetic form factor

$$\langle\pi^+(p')|J^{\mathrm{em}}_\mu|\pi^+(p)\rangle\equiv F_\pi(q^2)(p+p')_\mu\ . \tag{15}$$

Contributions will arise from both tree and loop diagrams from which one finds[6]

$$\begin{aligned}&\mathcal{L}^{(2)}_{\mathrm{tree}}:\ F_\pi(q^2)=1\ ,\\&\mathcal{L}^{(2)}_{\mathrm{loop}}:\ F_\pi(q^2)=\frac{1}{32\pi^2F_\pi^2}\left[2m_\pi^2H\left[\frac{q^2}{m_\pi^2}\right]+m_K^2H\left[\frac{q^2}{m_\pi^2}\right]-\frac{1}{3}q^2\ln\frac{m_\pi^2}{\mu^2}-\frac{1}{6}q^2\ln\frac{m_K^2}{\mu^2}+\frac{q^2}{2}\left[\frac{1}{\epsilon}+\ln4\pi+1-\gamma\right]\right],\\&\mathcal{L}^{(2)}_{\mathrm{tree}}:\ F_\pi(q^2)=\frac{2q^2}{F_\pi^2}L_9\ ,\end{aligned} \tag{16}$$

where

$$H(x)=-\frac{4}{3}+\frac{5x}{18}+\left[\frac{2}{3}-\frac{1}{6}x\right]\left[\frac{x-4}{x}\right]^{1/2}\ln\frac{\sqrt{x-4}+\sqrt{x}}{\sqrt{x-4}-\sqrt{x}}\ . \tag{17}$$

At low q^2 we find

$$F_\pi(q^2) \underset{q^2 \ll m_\pi^2}{\sim} 1+q^2\left[\frac{2}{F_\pi^2}L_9^r-\frac{1}{32\pi^2F_\pi^2}\left[\frac{1}{2}+\frac{1}{3}\ln\frac{m_\pi^2}{\mu^2}+\frac{1}{6}\ln\frac{m_K^2}{\mu^2}\right]\right] . \tag{18}$$

We observe that divergences have been absorbed into the phenomenological parameter L_9^r and from the experimental pion charge radius[7]

$$\langle r_\pi^2\rangle = 0.439\pm 0.008 \text{ fm}^2 \tag{19}$$

we determine

$$L_9^r(\mu=m_\eta)=(7.7\pm0.2)\times10^{-3} . \tag{20}$$

Similarly the remaining parameters $L_1^r(\mu),\ldots,L_{10}^r(\mu)$ have been fit empirically. Despite the apparently large number—ten—of such parameters the model *is* predictive, as we shall see.

A perspicacious reader will no doubt be asking what will happen if loop effects from $\mathcal{L}^{(4)}$ are included. The answer is that such diagrams are divergent and must be canceled by counterterms of order $\mathcal{L}^{(6)}$. Why can such loop diagrams be neglected? The key to understanding this point is realizing that chiral perturbation theory represents an expansion in derivatives of the Goldstone fields and hence in momentum. The parameter Λ which determines the scale of this expansion is of order[8]

$$\Lambda \sim 4\pi F_\pi \sim 1 \text{ GeV} ; \tag{21}$$

i.e., the coefficients of terms in $\mathcal{L}^{(2)},\mathcal{L}^{(4)},\mathcal{L}^{(6)},\ldots$ should be of order

$$c^{(2)}:c^{(4)}:c^{(6)}:\cdots \sim 1:\Lambda^{-2}:\Lambda^{-4}:\cdots \tag{22}$$

so at low enough energies—say, $s<m_K^2$—the chiral expansion should be realistic.

III. APPLICATION TO $K_{l3\gamma}$

We can now apply this formalism to the $K_{l3\gamma}$ sector. We shall quote here only the tree-level results, as loop effects are in general considerably smaller.[9] We begin with charged-kaon decay: $K^+\to\pi^0 l^+\nu_l$. Since the weak current consists of both vector and axial-vector components there are contributions to the radiative decay process from both anomalous and conventional components of the weak Lagrangian. For the conventional (vector-current) amplitude we find

$$\begin{aligned}A_{\mu\nu}(p_1,q_1,q_2)&=\int d^4x\, e^{iq_1\cdot x}\langle\pi^0(p_2)|T(V_\mu^{\text{em}}(x)V_\nu^{K^-}(0))|K^+(p_1)\rangle\\&=A_{\mu\nu}^{\text{Born}}(p_1,q_1,q_2)+\frac{2\sqrt{2}L_9}{F_\pi^2}(-g_{\mu\nu}p_1\cdot q_1+p_{1\mu}q_{1\nu}+g_{\mu\nu}p_2\cdot q_2-p_{2\nu}q_{2\mu}+p_{1\nu}p_{2\mu}-p_{1\mu}p_{2\nu})\\&\quad-2\sqrt{2}\frac{L_{10}}{F_\pi^2}(g_{\mu\nu}q_1\cdot q_2-q_{1\nu}q_{2\mu})-\left[\frac{1}{2}\right]^{1/2}\frac{F_K}{F_\pi} ,\end{aligned} \tag{23}$$

where we have used the identity

$$\frac{F_K}{F_\pi}=1+\frac{4}{F_\pi^2}(m_K^2-m_\pi^2)L_5^r . \tag{24}$$

Here the Born amplitude represents the simple kaon-pole diagram

$$A_{\mu\nu}^{\text{Born}}(p_1,q_1,q_2)=\frac{\langle\pi^0(p_2)|V_\nu^{K^-}|K^+(p_1-q_1)\rangle\langle K^+(p_1-q_1)|V_\mu^{\text{em}}|K^+(p_1)\rangle}{(p_1-q_1)^2-m_K^2} \tag{25}$$

with

$$\langle K^+(p_2)|V_\mu^{\text{em}}|K^+(p_1)\rangle=(p_1+p_2)_\mu\left[1+\frac{2L_9}{F_\pi^2}(p_1-p_2)^2\right]-(p_1-p_2)_\mu\frac{2L_9}{F_\pi^2}(p_1^2-p_2^2) , \tag{26a}$$

$$\sqrt{2}\langle\pi^0(p_2)|V_\nu^{K^-}|K^+(p_1)\rangle=(p_1+p_2)_\mu\left[1+\frac{2L_9}{F_\pi^2}(p_1-p_2)^2\right]+(p_1-p_2)_\mu\left[\frac{F_K}{F_\pi}-1-\frac{2L_9}{F_\pi^2}(p_1^2-p_2^2)\right] . \tag{26b}$$

For the anomalous (axial-vector-current) amplitude, we find

$$\begin{aligned}B_{\mu\nu}(p_1,q_1,q_2)&=\int d^4x\, e^{iq_1\cdot x}\langle\pi^0(p_2)|T(V_\mu^{\text{em}}(x)A_\nu^{K^-}(0))|K^+(p_1)\rangle\\&=-\epsilon_{\mu\nu\alpha\beta}\frac{1}{\sqrt{2}24\pi^2F_\pi^2}q_1^\alpha(3p_1^\beta+9p_2^\beta)+B_{\mu\nu}^{K^-\text{pole}}\end{aligned} \tag{27}$$

with

$$B_{\mu\nu}^{K^- \text{pole}}=\epsilon_{\mu\alpha\beta\gamma}p_1^\alpha p_2^\beta q_1^\gamma \frac{F_K}{\sqrt{2}4\pi^2F_\pi^2}\frac{(p_1-p_2-q_1)_\nu}{m_K^2-(p_1-p_2-q_1)^2} .$$

However, the kaon-pole term is higher in momentum and is proportional to the lepton mass when contracted with the lepton current. This piece was thus dropped by FFS, and similarly we shall not consider it further. It is elementary to verify that both axial-vector- and vector-current amplitudes satisfy the gauge-invariance requirement

$$q_1^\mu A_{\mu\nu}(p_1,q_1,q_2)=-\langle\pi^0(p_2)|V_\nu^{K^-}|K^+(p_1)\rangle,\quad q_1^\mu B_{\mu\nu}(p_1,q_1,q_2)=-\langle\pi^0(p_2)|A_\nu^{K^-}|K^+(p_1)\rangle=0 . \tag{28}$$

On the other hand, if the soft-pion limit is taken and PCAC is used, we require, for the vector-current amplitude,

$$\lim_{p_2\to 0}A_{\mu\nu}(p_1,q_1,q_2)=\frac{1}{2F_\pi}M_{\mu\nu}(p_1,q_1) , \tag{29}$$

where

$$M_{\mu\nu}(p_1,q_1)=\int d^4x\, e^{iq_1\cdot x}\langle 0|T(V_\mu^{\text{em}}(x)A_\nu^{K^-}(0))|K^+(p_1)\rangle \tag{30}$$

is the axial-vector-current amplitude for radiative decay. We have, from Eq. (23),

$$\begin{aligned}\lim_{p_2\to 0}A_{\mu\nu}(p_1,q_1,q_2)=&\frac{1}{q_1^2-2p_1q_1}\left[\frac{1}{2}\right]^{1/2}\left[(2p_1-q_1)_\mu\left[1+\frac{2L_9}{F_\pi^2}q_1^2\right]+q_{1\mu}\frac{2L_9}{F_\pi^2}(q_1^2-2p_1\cdot q_1)\right]\frac{F_K}{F_\pi}(p_1-q_1)_\nu\\&+\frac{2\sqrt{2}}{F_\pi^2}(-g_{\mu\nu}p_1\cdot q_1+p_{1\mu}q_{1\nu})-\left[\frac{1}{2}\right]^{1/2}\frac{F_K}{F_\pi}\\&-\frac{2\sqrt{2}}{F_\pi^2}L_{10}[g_{\mu\nu}q_1\cdot(p_1-q_1)-q_{1\nu}(p_1-q_1)_\mu] ,\end{aligned} \tag{31}$$

while from direct calculation we find

$$\begin{aligned}M_{\mu\nu}(p_1,q_1)=&M_{\mu\nu}^{\text{Born}}(p_1,q_1)-\sqrt{2}F_Kg_{\mu\nu}\\&+\frac{4\sqrt{2}}{F_\pi}(L_9+L_{10})[(p_1-q_1)_\mu q_{1\nu}\\&\qquad-g_{\mu\nu}(p_1-q_1)\cdot q_1]\\&-\frac{4\sqrt{2}}{F_\pi}L_9(g_{\mu\nu}q_1^2-q_{1\mu}q_{1\nu}) ,\end{aligned} \tag{32}$$

where $M_{\mu\nu}^{\text{Born}}(p_1,q_1)$ represents the kaon-pole term

$$\begin{aligned}M_{\mu\nu}^{\text{Born}}(p_1,q_1)=&\frac{1}{q_1^2-2p_1\cdot q_1}\sqrt{2}F_\pi(p_1-q_1)_\nu\\&\times\left[(2p_1-q_1)_\mu\left[1+\frac{2L_9}{F_\pi^2}q_1^2\right]\right.\\&\left.+q_{1\mu}\frac{2L_9}{F_\pi^2}(q_1^2-2p_1\cdot q_1)\right] .\end{aligned} \tag{33}$$

Comparison with Eqs. (29) and (31) reveals that the PCAC condition is precisely satisfied.

Similarly we may examine the axial-vector-current term, for which we expect

$$\lim_{p_2\to 0}B_{\mu\nu}(p_1,q_1,q_2)=\frac{1}{2F_\pi}N_{\mu\nu}(p_1,q_1) . \tag{34}$$

Here

$$\begin{aligned}N_{\mu\nu}(p_1,q_1)=&\int d^4x\, e^{iq_1\cdot x}\\&\times\langle 0|T(V_\mu^{\text{em}}(x)V_\nu^{K^-}(0))|K^+(p_1)\rangle\end{aligned} \tag{35}$$

is the polar-vector amplitude for radiative kaon decay and is related by a V-spin rotation to the amplitude for $\pi^0\to 2\gamma$:

$$N_{\mu\nu}(p_1,q_1)=\frac{1}{4\sqrt{2}\pi^2F_\pi}\epsilon_{\mu\nu\alpha\beta}p_1^\alpha q_1^\beta . \tag{36}$$

On the other hand, taking the $p_2\to 0$ limit of Eq. (27) we find

$$\begin{aligned}\lim_{p_2\to 0}B_{\mu\nu}(p_1,q_1,q_2)&=\frac{3}{\sqrt{2}24\pi^2F_\pi^2}\epsilon_{\mu\nu\alpha\beta}q_1^\alpha p_1^\beta\\&=\frac{1}{2F_\pi}N_{\mu\nu}(p_1,q_1) .\end{aligned} \tag{37}$$

Thus the soft-pion condition is indeed satisfied by the axial-vector-current contribution. This agreement appears to be accidental since it is easily verified that the soft-kaon limit is *not* obeyed. Also in the related neutral radiative K_{l3} amplitude discussed below the soft-*pion* condition is violated while the soft-*kaon* limit is obeyed. However, these results are perhaps not surprising since the origin of such terms is from $\mathcal{L}_{\text{anom}}$.

It is also of interest to compare these chiral-symmetry predictions [Eqs. (23) and (27)] with the PCAC–Low-theorem analysis of FFS. These authors, for the vector-current piece, use the definition

$$\begin{aligned}A_{\mu\nu}(p_1,q_1,q_2)=&Ag_{\mu\nu}-Bq_{1\mu}q_{1\nu}-Cp_{2\mu}p_{2\nu}\\&-Dp_{1\mu}p_{1\nu}-Eq_{1\mu}p_{1\nu}-Fp_{1\mu}q_{1\nu}\\&-Gp_{1\mu}p_{2\nu}-Hp_{2\mu}p_{1\nu}\\&-Ip_{2\mu}q_{1\nu}-Jq_{1\mu}p_{2\nu} .\end{aligned} \tag{38}$$

Comparison with the chiral-symmetry form Eq. (23) yields then

$$\begin{aligned}
A &= -\left[\frac{1}{2}\right]^{1/2}\left[\frac{F_K}{F_\pi}+\frac{4}{F_\pi^2}(L_9+L_{10})q_1\cdot(p_1-p_2-q_1)+\frac{4}{F_\pi^2}L_9[q_1^2-p_2\cdot(p_1-p_2-2q_1)]\right],\\
B &= \frac{2\sqrt{2}}{F_\pi^2}L_{10}-\left[\frac{1}{2}\right]^{1/2}\left[1+\frac{4L_9}{F_\pi^2}p_1\cdot q_1\right]\left[\frac{F_K}{F_\pi}+\frac{4L_9}{F_\pi^2}[m_\pi^2-p_2\cdot(p_1-q_1)]\right],\\
C &= -\frac{2\sqrt{2}}{F_\pi^2}L_9 ,\\
D &= -\sqrt{2}\left[1+\frac{2L_9}{F_\pi^2}q_1^2\right]\left[\frac{F_K}{F_\pi}+\frac{4L_9}{F_\pi^2}[m_\pi^2-p_2\cdot(p_1-q_1)]\right]\frac{1}{q_1^2-2p_1\cdot q_1},\\
E &= \left[\frac{1}{2}\right]^{1/2}\left[1+\frac{4L_9}{F_\pi^2}p_1\cdot q_1\right]\left[\frac{F_K}{F_\pi}+\frac{4L_9}{F_\pi^2}[m_\pi^2-p_2\cdot(p_1-q_1)]\right]\frac{1}{q_1^2-2p_1\cdot q_1},\\
F &= -\frac{2\sqrt{2}}{F_\pi^2}(L_9+L_{10})+\sqrt{2}\left[1+\frac{2L_9}{F_\pi}q_1^2\right]\left[\frac{F_K}{F_\pi}+\frac{4L_9}{F_\pi^2}[m_\pi^2-p_2\cdot(p_1-q_1)]\right]\frac{1}{q_1^2-2p_1\cdot q_1},\\
G &= \frac{4\sqrt{2}}{F_\pi^2}L_9-\sqrt{2}\left[1+\frac{2L_9}{F_\pi}q_1^2\right]\left[2-\frac{F_K}{F_\pi}+\frac{4L_9}{F_\pi^2}[(p_1-q_1)^2-p_2\cdot(p_1-q_1)]\right]\frac{1}{q_1^2-2p_1\cdot q_1},\\
H &= -\frac{2\sqrt{2}}{F_\pi^2}L_9 ,\\
I &= \frac{2\sqrt{2}}{F_\pi^2}L_{10} ,\\
J &= -\frac{2\sqrt{2}}{F_\pi^2}L_9+\left[\frac{1}{2}\right]^{1/2}\left[1+\frac{4L_9}{F_\pi^2}p_1\cdot q_1\right]\left[2-\frac{F_K}{F_\pi}+\frac{4L_9}{F_\pi^2}[(p_1-q_1)^2-p_2\cdot(p_1-q_2)]\right]\frac{1}{q_1^2-2p_1\cdot q_1} .
\end{aligned} \tag{39}$$

Not all ten structure functions are independent, however, and D,F,G can be eliminated using the gauge-invariance requirement. Of the remaining seven terms, B,E,J do not contribute to $K_{l3\gamma}$ decay but only to $K_{l3e^+e^-}$. Thus, we may write the most general vector-current matrix element for $K_{l3\gamma}$ in terms of A,C,H,I for which FFS gave, based upon the Low theorem and PCAC,

$$\begin{aligned}
A^{\mathrm{FFS}} &= -\left[\frac{1}{2}\right]^{1/2}\left[\frac{F_K}{F_\pi}+\frac{4L_9}{F_\pi^2}(m_\pi^2-p_1\cdot p_2)\right.\\
&\qquad\left.+\frac{4(L_9+L_{10})}{F_\pi^2}q_1\cdot p_1\right],\\
C^{\mathrm{FFS}} &= H^{\mathrm{FFS}} = -\frac{2\sqrt{2}}{F_\pi^2}L_9,\quad I^{\mathrm{FFS}}=0 .
\end{aligned} \tag{40}$$

Comparing with the chiral-symmetry predictions in Eq. (39) we find

$$\begin{aligned}
A^{\mathrm{chiral}}-A^{\mathrm{FFS}} &= -2\sqrt{2}\frac{L_9-L_{10}}{F_\pi^2}q_1\cdot p_2 ,\\
C^{\mathrm{chiral}}-C^{\mathrm{FFS}} &= H^{\mathrm{chiral}}-H^{\mathrm{FFS}}=0 ,\\
I^{\mathrm{chiral}}-I^{\mathrm{FFS}} &= \frac{2\sqrt{2}}{F_\pi^2}L_{10} .
\end{aligned} \tag{41}$$

Thus imposition of the full requirements of chiral invariance yields additional constraints over the results of FFS. However, any such differences vanish in the limit as $p_2\to 0$, as might be expected from the PCAC origin of the derivation.

In the case of the axial-vector-current contribution, FFS defined a similar set of structure constants

$$B_{\mu\nu}(p_1,q_1,q_2)=\epsilon_{\mu\nu\alpha\beta}(bp_1^\alpha q_1^\beta+cp_2^\alpha q_1^\beta+dp_1^\alpha p_2^\beta) \tag{42}$$

for which chiral symmetry predicts

$$\begin{aligned}
b &= \frac{3}{24\pi^2F_\pi^2\sqrt{2}} ,\\
c &= \frac{-9}{24\pi^2F_\pi^2\sqrt{2}} ,\\
d &= 0 ,
\end{aligned} \tag{43}$$

while FFS used a K^*-pole model together with the soft-pion theorem to give

$$b\simeq\frac{1}{2F_\pi}\frac{f_{K^*}G_{K^*K\gamma}}{m_{K^*}^2-m_K^2}\approx m_K^{-2},\quad d=0 . \tag{44}$$

(Of course, no information could be gained about c, since its contribution to $B_{\mu\nu}$ vanishes as $p_2\to 0$.) Since $3/24\sqrt{2}\pi^2F_\pi^2\sim 0.24m_K^{-2}$ we see that the order of magnitude is the same in the FFS and chiral approaches. However, the chiral model is completely predictive, with no need for model-dependent assumptions.

Similarly we can analyze the neutral radiative mode $K^0\to\pi^- l^+\nu_l$ for which the chiral-symmetry prediction is, using the FFS notation,

$$
\begin{aligned}
A&=2-\frac{F_K}{F_\pi}-4\frac{L_9}{F_\pi^2}[q_1\cdot p_2-p_1\cdot(p_1-p_2-q_1)]-4\frac{L_{10}}{F_\pi^2}q_1\cdot(p_1-p_2-q_1)\,,\\
B&=4\frac{L_{10}}{F_\pi^2}+\left[2-\frac{F_K}{F_\pi}+\frac{4L_9}{F_\pi^2}[m_K^2-p_1\cdot(p_2+q_1)]\right]\left[1-\frac{4L_9}{F_\pi^2}p_2\cdot q_1\right]\frac{1}{q_1^2+2p_2\cdot q_1}\,,\\
C&=2\left[1+\frac{2L_9}{F_\pi^2}q_1^2\right]\left[2-\frac{F_K}{F_\pi}+\frac{4L_9}{F_\pi^2}[m_K^2-p_1\cdot(p_2+q_1)]\right]\frac{1}{q_1^2+2p_2\cdot q_1}\,,\\
D&=4\frac{L_9}{F_\pi^2}\,,\\
E&=-4\frac{L_9}{F_\pi^2}+\left[1-\frac{4L_9}{F_\pi^2}p_2\cdot q_1\right]\left[\frac{F_K}{F_\pi}+\frac{4L_9}{F_\pi^2}[(p_2+q_1)^2-p_1\cdot(p_2+q_1)]\right]\frac{1}{q_1^2+2p_2\cdot q_1}\,,\\
F&=-4\frac{L_{10}}{F_\pi^2}\,,\\
G&=4\frac{L_9}{F_\pi^2}\,,\\
H&=-8\frac{L_9}{F_\pi^2}+2\left[1-\frac{2L_9}{F_\pi^2}q_1^2\right]\left[\frac{F_K}{F_\pi}+\frac{4L_9}{F_\pi^2}[(p_2+q_1)^2-p_1\cdot(p_2+q_1)]\right]\frac{1}{q_1^2+2p_2\cdot q_1}\,,\\
I&=-4\frac{L_9-L_{10}}{F_\pi^2}+2\left[1+\frac{2L_9}{F_\pi^2}q_1^2\right]\left[2-\frac{F_K}{F_\pi}+\frac{4L_9}{F_\pi^2}[m_K^2-p_1\cdot(p_2+q_1)]\right]\frac{1}{q_1^2+2p_2\cdot q_1}\,,\\
J&=\left[1+\frac{4L_9}{F_\pi^2}p_2\cdot q_1\right]\left[2-\frac{F_K}{F_\pi}+\frac{4L_9}{F_\pi^2}[m_K^2-p_1\cdot(p_2+q_1)]\right]\frac{1}{q_1^2+2p_2\cdot q_1}
\end{aligned}
\tag{45}
$$

for the vector current and

$$
\begin{aligned}
b&=\frac{3}{24\pi^2F_\pi^2}\,,\\
c&=\frac{-3}{24\pi^2F_\pi^2}\,,\\
d&=0
\end{aligned}
\tag{46}
$$

for the axial-vector current. Once more it is easy to verify that both gauge invariance and PCAC structures are satisfied by these forms, except for the violation of the current-algebra constraint for $B_{\nu\mu}$ as discussed above. However, in comparing with the FFS analysis, although the Born terms agree between the two approaches, FFS could not relate their non-Born contributions to experimental observables. Instead these were calculated in a simple vector–axial-vector meson-pole model, yielding

$$
\begin{aligned}
&|m_K^2 I^{\text{non-Born}}|\simeq 1.55\,,\\
&\frac{m_K^2}{p_1\cdot q_1}|A^{\text{non-Born}}+I^{\text{non-Born}}p_2\cdot q_1|\simeq 2.5\,,\\
&|m_K^2 b|\simeq 2.5\,.
\end{aligned}
\tag{47}
$$

The corresponding chiral-Lagrangian predictions are

$$
\begin{aligned}
m_K^2 I^{\text{non-Born}}&=-4\frac{L_{10}}{F_\pi^2}m_K^2\\
&=\frac{m_K^2}{p_1\cdot q_1}(A^{\text{non-Born}}+I^{\text{non-Born}}p_2\cdot q_1)\\
&\approx 0.66\,,\\
m_K^2 b&=\frac{3m_K^2}{24\pi^2F_\pi^2}=0.35\,,
\end{aligned}
\tag{48}
$$

which are somewhat smaller than the FFS predictions. We see again that the chiral model is completely predictive, with no model-dependent assumptions required except for that of chiral symmetry, which follows from QCD.

In principle then careful measurement of radiative K_{l3} spectra could provide a definitive test of the chiral predictions. Unfortunately, such a program does not appear to be currently practical experimentally. For the electron radiative mode, inclusion of structure dependence makes virtually no change in the calculated spectrum while for muon decay the effect of inclusion of structure-dependent (i.e., non-Born) terms is only slightly larger, as discussed in Ref. 1. In both cases, the spectrum is dominated by the characteristic $1/k$ dependence of the inner bremsstrahlung (i.e., Born) terms. Numerically, we have, for the electron mode,

$$\frac{\Gamma(K^+\to\pi^0e^+\nu_e\gamma, E_\gamma>30\text{ MeV})}{\Gamma(K^+\to\pi^0e^+\nu_e)} = (2.0981-0.006m_K^2f-0.003m_K^2b)\times 10^{-2}\,,$$

$$\frac{\Gamma(\bar K^0\to\pi^+e^-\bar\nu_e\gamma, E_\gamma>30\text{ MeV})}{\Gamma(\bar K^0\to\pi^+e^-\bar\nu_e)} = (2.3889+0.007m_K^2f+0.002m_K^2b)\times 10^{-2}\,, \tag{49}$$

where

$$f=\frac{1}{p_1\cdot q_1}(A^{\text{non-Born}}+p_2\cdot q_1 I^{\text{non-Born}})\,. \tag{50}$$

Thus changing any of the structure-dependent terms affects the above ratios at only the tenth of a percent level. In the case of the muonic modes effects are larger:

$$\frac{\Gamma(K^+\to\pi^0\mu^+\nu_\mu\gamma, E_\gamma>30\text{ MeV})}{\Gamma(K^+\to\pi^0e^+\nu_e)} = (0.509\,15-0.007\,46m_K^2f-0.007\,58m_K^2b)\times 10^{-3}\,,$$

$$\frac{\Gamma(\bar K^0\to\pi^+\mu^-\bar\nu_\mu\gamma, E_\gamma>30\text{ MeV})}{\Gamma(\bar K^0\to\pi^+e^-\bar\nu_e)} = (1.466+0.017m_K^2f+0.005m_K^2b)\times 10^{-3}\,. \tag{51}$$

We see then that changes in the structure-dependent terms now can affect results at the percent level. However, the overall branching ratios are an order of magnitude lower than in the case of the electron counterparts.

IV. CONCLUSIONS

We have demonstrated that the assumption of chiral invariance allows a model-independent prediction to be made for the matrix elements for radiative K_{l3} decay in terms of $\mathcal{L}^{(4)}$ parameters L_9, L_{10} which are well determined from the pion charge radius, pion radiative decay, respectively. Comparison with earlier PCAC–Low-theorem methods are much more powerful and do not require model-dependent assumptions. While the updating of these predictions is interesting from a theoretical perspective, the verification of the chiral results is probably beyond the reach of present experiments because of the dominance of the inner bremsstrahlung component of the radiative spectrum.

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[9]Indeed such loop contributions are $O(1/N)$ with respect to their tree-level counterparts. For example, in the pion form factor at the renormalization point $\mu=m_\eta$ we find $\langle r_\pi^2\rangle = \langle r_\pi^2\rangle_{\mathcal{L}^{(2)}\,\text{loop}} + \langle r_\pi^2\rangle_{\mathcal{L}^{(4)}\,\text{tree}}$ with $\langle r_\pi^2\rangle_{\mathcal{L}^{(4)}\,\text{tree}} \simeq -16.5\langle r_\pi^2\rangle_{\mathcal{L}^{(2)}\,\text{loop}}$.