Constraints from anomaly cancellation on strong, weak, and electromagnetic interactions

X.-G. He, G. C. Joshi, and R. R. Volkas
Research Centre for High Energy Physics, School of Physics, University of Melbourne, Parkville 3052, Australia
(Received 25 August 1989)

We examine the extent to which the four known forces in nature are interwoven through the use of anomaly cancellation. We discover significant interconnections, though uniqueness is not attained. Possible other criteria necessary to obtain uniqueness are discussed.

For each of the four known forces of nature (gravity, strong interactions, weak interactions, electromagnetism) there is a well-defined and phenomenologically successful theory. These theories have great similarities with each other, as well as significant differences. Consider, for instance, the following.

(i) All the theories are based on a covariance principle.

(ii) General relativity has not been successfully quantized (except possibly in superstring theory), whereas the other forces have well-defined quantum-field-theoretic descriptions.

(iii) Gravitons have spin 2, whereas gluons, the weak bosons and the photon have spin 1.

(iv) Weak and electromagnetic interactions are intimately related through symmetry breaking, while the strong and gravitational interactions are not related in this manner to any known forces.

(v) The graviton, gluons, and the photon are massless, whereas the weak bosons are massive.

(vi) Quarks feel all four forces, charged leptons feel three of them, and neutrinos feel either two or one of them depending on chirality. (ν_R need not exist, of course.)

(vii) All the interactions are described using local fields.

One may extend this list if one wishes.

The fact that there are both similarities and differences allows us to study the self-consistency of the standard theory with the standard matter particles. Put in another way, one can enquire as to how much input one needs before internal consistency, whose principles are related to the common theoretical structure of the interactions, is sufficient to allow us to deduce the rest of the theory. In this paper we will use the consistency requirements of (i) gauge anomaly cancellation,[1] (ii) mixed gauge gravitational anomaly cancellation,[2] and (iii) the absence of the Witten anomaly,[3] to investigate the interrelations between the known forces of nature.

Specifically, our procedure will be to input as much knowledge of quantum gravity (assumed to exist) as required to calculate the mixed anomaly, plus complete knowledge of two of the other forces, and then see to what extent anomaly cancellation allows us to deduce the properties of remaining interaction. Our study is in a similar vein, and complementary to, other recent work.[4-6] In this paper we break new ground by examining constraints on the non-Abelian sector of the theory, and we also clarify some of the subtleties of the U(1) case.

Consider first a fermion generation from the minimal standard model (SM), and suppose that the $SU(3)_c \otimes SU(2)_L$ assignments are known:

$$l_L=(1,2)(y_1),\quad l_R=(1,1)(y_2),\quad q_L=(3,2)(y_4)\,,\quad q_{1R}=(3,1)(y_5),\quad q_{2R}=(3,1)(y_6)\,. \tag{1}$$

Note that we have explicitly omitted the putative right-handed neutrino, and that we are also assuming that the remaining force has a U(1) gauge structure for which the specific charges are unknown. Witten anomaly cancellation is automatic, while gauge and mixed anomaly cancellation yields

$$\begin{aligned}&[SU(3)_c]^2U(1)_Y:\ 2y_4-y_5-y_6=0\,,\\&[SU(2)_L]^2U(1)_Y:\ y_1+3y_4=0\,,\\&[U(1)_Y]^3:\ 2y_1^3-y_2^3+6y_4^3-3y_5^3-3y_6^3=0\,,\\&\text{mixed:}\ 2y_1-y_2+6y_4-3y_5-3y_6=0\,.\end{aligned} \tag{2}$$

These equations yield two types of solutions:[5,7]

$$\text{(A)}\ y_1\neq 0,\quad y_2=2y_1,\quad y_4=-\tfrac{1}{3}y_1\,,\quad y_5=-\tfrac{4}{3}y_1,\quad y_6=\tfrac{2}{3}y_1\,, \tag{3a}$$

$$\text{(B)}\ y_1=y_2=y_4=0,\quad y_5=-y_6\,. \tag{3b}$$

Case (A) yields the usual weak hypercharge, while case (B) represents a peculiar U(1) symmetry.[7] The latter theory does not contain a U(1) group which can be identified with conventional $U(1)_{em}$. It is, however, a simpler solution than case (A) because one may omit l_R on the grounds that it does not contribute to any anomaly, thus leaving a 14-fermion generation rather than the standard 15-fermion generation. (l_L cannot be omitted for an odd number of generations because of the Witten anomaly.) Anomaly cancellation does not imply that the simplest chiral generation under the SM gauge group is the standard one. One should also note that there is further ambiguity within both cases, because the spectrum with left and right interchanged is also acceptable (mirror generation).

It is interesting that anomaly cancellation, using the above input, is sufficiently constraining so as to allow only two choices for the U(1) charges.[7] One must now wonder whether or not nature makes use of both possibilities, or whether an additional criterion must be found to eliminate solution (3b). It is interesting that the naive simplicity criterion is not a candidate for this, as discussed above.

Let us denote the U(1) given in Eq. (3a) by $U(1)_Y$, and the U(1) in Eq. (3b) by $U(1)_Z$. The next obvious question to ask is whether one would be allowed to gauge $U(1)_Y \otimes U(1)_Z$. Interestingly, the answer is "no" because $[U(1)_Y]^2 U(1)_Z$ and $[U(1)_Z]^2 U(1)_Y$ gauge anomalies cannot cancel, due to the fact that $y_6 \neq -y_5$ in Eq. (3a).

One therefore cannot resolve the ambiguity of Eq. (3) by saying that nature does in fact employ both solutions in the manner suggested above, but it is an accident of symmetry breaking that the $U(1)_Z$ gauge boson has remained unobserved.

Is there something deeper behind solution (3b)? It would appear that there is, because Z charge actually corresponds to the diagonal generator of $SU(2)_R^q$, under which q_{1R} and q_{2R} form a doublet while all other fermions are singlets. In fact $SU(3)_c \otimes SU(2)_L \otimes U(1)_Z$ can be embedded in $SU(3)_c \otimes SU(2)_L \otimes SU(2)_R^q$ without affecting gauge and mixed anomaly cancellation or introducing more fermions. This therefore may be a hint that $U(1)_Z$ has something to do with an underlying left-right-symmetric model which has been butchered by the omission of a right-handed neutrino. One should note that for an odd number of generations $SU(2)_R^q$ is not a viable gauge symmetry because of a Witten anomaly. However this does not affect the mathematical result that $U(1)_Z$ is a subgroup of $SU(2)_R^q$.

The next reasonable step seems to be to introduce ν_R and thereby hope to gain more insight into, and possibly resolve, the ambiguity. To the fermion spectrum of Eq. (1) let us add

$$\nu_R = (1,1)(y_3) \ . \tag{4}$$

One can easily amend Eqs. (2) to obtain the solutions[5]

$$\text{(A)} \quad y_1 \text{ and/or } y_3 \neq 0, \quad y_2 = 2y_1 - y_3, \quad y_4 = -\tfrac{1}{3}y_1 \ ,$$
$$y_5 = -\tfrac{4}{3}y_1 + y_3, \quad y_6 = \tfrac{2}{3}y_1 - y_3 \ , \tag{5a}$$

$$\text{(B)} \quad y_1 = 0, \quad y_2 = -y_3, \quad y_4 = 0, \quad y_6 = -y_5 \ . \tag{5b}$$

Although this looks like two U(1)'s, one can actually find three independent U(1)'s from Eq. (5). Putting $y_3 = 0$ in Eq. (5a) yields $U(1)_Y$. Putting $y_3 = 0$ in Eq. (5b) yields $U(1)_Z$, while taking $y_5 = 0$ in Eq. (5b) yields $U(1)_{Z'}$ which is the leptonic analogue of $U(1)_Z$. [Note: taking $y_1 = 0$ in Eq. (5a) yields a subgroup of $U(1)_Z \otimes U(1)_{Z'}$, which we will denote by $U(1)_R$, and is thus not a new solution.]

Which of these three U(1)'s can one gauge simultaneously? We have already seen that $U(1)_Y \otimes U(1)_Z$ is anomalous. Similarly $U(1)_Y \otimes U(1)_{Z'}$ is anomalous. It is easy to check that the $U(1)_R$ subgroup of $U(1)_Z \otimes U(1)_{Z'}$ has no anomalies with $U(1)_Y$. Trivially $U(1)_Z \otimes U(1)_{Z'}$ is also anomaly free. So there are again two solutions:

$$\text{(A)} \quad G_1 = SU(3)_c \otimes SU(2)_L \otimes U(1)_R \otimes U(1)_Y \ , \tag{6a}$$

$$\text{(B)} \quad G_2 = SU(3)_c \otimes SU(2)_L \otimes U(1)_Z \otimes U(1)_{Z'} \ . \tag{6b}$$

One can embed both G_1 and G_2 in larger groups without spoiling gauge and mixed anomaly cancellation and without introducing new fermions:

$$\text{(A)} \quad G_1' = SU(3)_c \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L} \ , \tag{7a}$$

$$\text{(B)} \quad G_2' = SU(3)_c \otimes SU(2)_L \otimes SU(2)_R^q \otimes SU(2)_R^l \ . \tag{7b}$$

$SU(2)_R^l$ is the leptonic analogue of $SU(2)_R^q$. Solution (7a) is the usual left-right-symmetric model, while solution (7b) is an extension of the model of Eq. (3b). Again, G_2' has Witten anomalies for an odd number of generations, though its subgroup G_2 is of course acceptable. Thus introducing ν_R has not allowed us to resolve the ambiguity, although solution (3b) is in a sense used in the left-right-symmetric gauge group. Why does nature choose either (3a) or possibly (7a)? Perhaps there are as yet undiscovered gauge interactions which are incompatible with solutions (3b) or (6b) or (7b)? Perhaps the structure of the symmetry breaking and mass generation sector has to be invoked?[5,6] We will now pursue these and other possibilities here.

We will now leave this line of thinking and instead ask the following. Suppose we know the $SU(3)_c \otimes U(1)_Y$ properties of fermions. What can we deduce about the missing gauge force?

Our spectrum is

$$l_L = (1,N_1)(-1), \quad l_R = (1,N_2)(-2) \quad [\nu_R = (1,N_3)(0)] \ ,$$
$$q_L = (3,N_4)(\tfrac{1}{3}), \quad u_R = (3,N_5)(\tfrac{4}{3}), \quad d_R = (3,N_6)(-\tfrac{2}{3}) \ . \tag{8}$$

The right-handed neutrino is in brackets because it is optional. $[SU(3)_c]^3$, $[SU(3)_c]^2 U(1)_Y$, $[U(1)_Y]^3$, and mixed anomalies allow us to show that

$$N_1 = 2N_2, \quad N_4 = 2N_5 = 2N_6 \ , \tag{9}$$

whether or not ν_R is included. It is interesting that compatibility with strong and weak hypercharge forces requires the left-handed fermion representation to have twice the dimension of the right-handed fermion representation.

If we assume that the missing gauge group G_W is SU(2) it is easy to show using the $[SU(2)]^2 U(1)_Y$ anomaly that

$$N_2 = N_5 \ . \tag{10}$$

The minimal solution (as expected) is as given by the SM, although there are obviously a countably infinite number of other solutions.

If we do not assume that the group G_W is SU(2) then the remaining constraints are the vanishing of $[G_W]^2 U(1)_Y$ and $[G_W]^3$ anomalies, as well as the requirement that if N_2 and N_5 are dimensions of representations then so must $2N_2$ and $2N_5$ be, as can be seen from Eq. (9). For example, if $G_W = SU(3)_L$ then $N_2 = N_5 = 3$ (and $N_3 = 3$) works because there exists a six-dimensional representation and the remaining anomalies cancel if the quarks are placed in complex-conjugate $SU(3)_L$ representations with respect to the leptons.

To summarize then, anomaly cancellation, though greatly restrictive in this case, still allows a countably infinite number of solutions. Interestingly, one is able to derive that left-handed fermion G_W representations should have twice the dimension of the right-handed fermion representations. If one adds the additional criterion of simplicity, then both SU(2) and $N_2 = N_5 = 1$ emerge as the preferred solution. The putative right-handed neutrino plays little role if G_W is taken to be SU(2), although it introduces more arbitrariness because N_3 is unconstrained. If G_W is some other group then it generally plays a role in $[G_W]^3$ anomaly cancellation.

We now suppose that $SU(2)_L \otimes U(1)_Y$ is known. The spectrum is

$$l_L = (N_1, 2)(-1), \quad l_R = (N_2, 1)(-2),$$
$$[\nu_R = (N_3, 1)(0)], \quad q_L = (N_4, 2)(\tfrac{1}{3}), \tag{11}$$
$$u_R = (N_5, 1)(\tfrac{4}{3}), \quad d_R = (N_6, 1)(-\tfrac{2}{3}).$$

$[SU(2)_L]^2U(1)_Y$, $[U(1)_Y]^3$, and mixed anomaly cancellation yields

$$N_4 = 3N_1, \quad N_5 = 4N_2 - N_1, \quad N_6 = 5N_2 - 2N_1. \tag{12}$$

The remaining anomalies are $[G_s]^2U(1)_Y$ and $[G_s]^3$.

Equations (12) again severely restrict the allowed groups and representations, although uniqueness is not obtained. The simplest solutions are

$$G_s = SU(2), \quad N_1 = N_2 = 1, \quad N_4 = N_5 = N_6 = 3, \tag{13a}$$

$$G_s = SU(3), \quad N_1 = N_2 = N_3 = 1, \quad N_4 = N_5 = N_6 = 3. \tag{13b}$$

(Note the importance of multiples of three.) Both solutions (13a) and (13b) have zero $[G_s]^2U(1)_Y$ and $[G_s]^3$ anomalies. Case (13a) has the simplest gauge group, but case (13b) has the simplest representation content in that 3 is the fundamental representation of SU(3) but not of SU(2). Within $G_s = SU(3)$ there are other solutions, such as $N_1 = N_2 = N_3 = 8$, $N_4 = N_5 = N_6 = 24$.

In conclusion then, we have found that the criterion of anomaly cancellation, which involves all four of the forces due to their theoretical similarity, interweaves these forces in an interesting way. An intriguing ambiguity remains for the U(1) case, with or without ν_R. What additional principle will resolve this? The SU(3) and SU(2) cases yield more solutions than the U(1) case, but the criterion of simplicity eliminates almost all of them.

This work was supported by the Australian Research Council.

[1] S. Adler, Phys. Rev. **177**, 2426 (1969); J. S. Bell and R. Jackiw, Nuovo Cimento **60A**, 49 (1969); S. L. Adler and W. Bardeen, Phys. Rev. **182**, 1517 (1969); H. Georgi and S. L. Glashow, Phys. Rev. D **6**, 429 (1972); C. Bouchiat, J. Iliopoulos, and P. Meyer, Phys. Lett. **38B**, 519 (1972); D. Gross and R. Jackiw, Phys. Rev. D **6**, 447 (1972).

[2] R. Delbourgo and A. Salam, Phys. Lett. **40B**, 381 (1972); T. Eguchi and P. Freund, Phys. Rev. Lett. **37**, 1251 (1976); L. Alvarez-Gaumé and E. Witten, Nucl. Phys. **B234**, 269 (1983).

[3] E. Witten, Phys. Lett. **117B**, 324 (1982).

[4] C. Q. Geng and R. E. Marshak, Commun. Nucl. Part. Phys. **18**, 331 (1989); Phys. Rev. D **39**, 693 (1989).

[5] R. Foot, G. C. Joshi, H. Lew, and R. R. Volkas, Mod. Phys. Lett. A (to be published); R. Foot, University of Wisconsin Report No. MAD/TH/89-6, 1989 (unpublished); X.-G. He, G. C. Joshi, and B. H. J. McKellar, Europhys. Lett. (to be published); X.-G. He, G. C. Joshi, H. Lew, B. H. J. McKellar, and R. R. Volkas, Phys. Rev. D **40**, 3140 (1989); N. G. Deshpande, University of Oregon Report No. OITS107, 1979 (unpublished); L. E. Ibañez, Report No. CERN-TH.5237/88, 1988 (unpublished).

[6] K. S. Babu and R. N. Mohapatra, Phys. Rev. Lett. **63**, 938 (1989); this issue, Phys. Rev. D **41**, 271 (1990).

[7] J. A. Minahan, P. Ramond, and R. C. Warner, Phys. Rev. D (to be published).