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Equivalence theorem redux

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The equivalence theorem states that amplitudes involving longitudinal vector bosons are equal to those with the corresponding unphysical scalars in the limit $M_W^2/s \to 0$. There are two ways to approach this limit, depending on whether $M_W/M_H \to 0$ or $M_H^2/s \to 0$. We show that the theorem has a different physical interpretation in each limit, but its validity in both depends only on the wave-function renormalization of the unphysical Goldstone bosons. We derive a condition that the renormalization parameters must satisfy in order for the theorem to hold. We show that this condition is satisfied in the first limit, appropriate to the heavy-Higgs-boson regime, if momentum subtraction at a scale $m \ll M_H$ is used. With this prescription, the theorem is true to lowest nonzero order in g and to all orders in the Higgs-boson coupling.

I. INTRODUCTION

The standard $SU(3)\times SU(2)\times U(1)$ model of the strong, weak, and electromagnetic interactions has been very successful in describing almost all experimental results to date. Despite this tremendous success, however, little is known about the mechanism that drives the weak-interaction symmetry breaking. The careful investigation of this symmetry breaking has become an essential task for both experimental and theoretical physics.

One intriguing possibility is that $SU(2)\times U(1)$ is broken in such a way that the weak interactions actually become strong at high energies.[1-5] The new strong interactions manifest themselves through the interactions of the Higgs particle H and the longitudinal polarizations of the W and the Z. This possibility led to the discovery of the *equivalence theorem* for the weak interactions.[1,3,4]

In essence, the equivalence theorem states that S-matrix elements involving longitudinal vector bosons are equal, in the limit $M_W^2 \ll s$, to S-matrix elements with the longitudinal vector mesons replaced by their associated Goldstone bosons. At first glance, this seems to be a straightforward consequence of the Higgs mechanism. After all, vector bosons obtain their mass by absorbing Goldstone bosons, so the longitudinal parts of the vector mesons should retain a memory of the scalar interactions. At high energies, where the vector-boson masses are negligible, one might expect that the longitudinal parts of the vector bosons could be replaced by the Goldstone scalars from which they came.

The equivalence theorem can be written formally as

$$S[W_L\text{'s, physical}] = i^n \times S[\rho\text{'s, physical}] , \qquad (1.1)$$

where $M_W^2 \ll s$. The left-hand side of this equation is the true S-matrix element for the scattering of n longitudinal vector bosons W_L with any other physical particles (including the Higgs and the transverse vector bosons). The right-hand side contains the same S-matrix element, with the external longitudinal vector bosons replaced by unphysical Goldstone bosons ρ, computed as if the Goldstone bosons were real physical particles.

There are, however, two problems with this statement of the theorem. The first is that we must be more specific about how the large-s limit is taken. There are three physical quantities of interest, which we choose to be M_W and the dimensionless couplings g and $\lambda \equiv g^2 M_H^2/8M_W^2$. [For simplicity we consider just an SU(2) theory.] Since we are performing a perturbation expansion in both couplings, we need to know their relative strengths:

$$g^2/8\lambda = \frac{M_W^2/s}{M_H^2/s} . \qquad (1.2)$$

As we take $M_W^2/s \to 0$ we must also take either (1) $g^2/\lambda \to 0$, or (2) $M_H^2/s \to 0$. Although this is a simple point, it is crucial to understanding the physical intuition behind the theorem. At the tree level the theorem holds in either limit, but at higher loops the situation is more subtle.

The second problem in the above statement of the equivalence theorem is that it ignores the renormalization condition for the unphysical Goldstone bosons. Of course, in all physical calculations this is irrelevant, but in computing the right-hand side of (1.1), it makes a big difference. For example, if we change renormalization conditions, the right-hand side gains a factor of $(Z'_\rho/Z_\rho)^{-n/2}$, where Z_ρ is the old wave-function renormalization constant and Z'_ρ the new. [Note that Z_W is completely determined by the fact that the left-hand side of (1.1) is a physical S-matrix element.] If the equivalence theorem is to hold for higher loops, it can only be true for one choice of renormalization prescription.

In Sec. II we will develop our intuition about the equivalence theorem by briefly discussing it from a physical point of view. We find that the physical interpretation of the theorem is different in each of the two limits, and that the two problems mentioned above are related. We use this discussion to gain some insight into the proper renormalization prescriptions for the Goldstone bosons. We find that momentum subtraction at a scale m, where $m \to 0$ as $g^2/\lambda \to 0$, is a physically sensible prescription for limit (1). We find no natural prescription for limit (2).

In Sec. III we examine the equivalence theorem to all orders in perturbation theory. We follow the arguments of Chanowitz and Gaillard,[4] but we correctly amputate the Green's functions and renormalize all physical parameters in a physical way. Using Ward identities derived in the Appendix, we obtain

$$S[W_L\text{'s, physical}] = (iC)^n \times S[\rho\text{'s, physical}] , \quad (1.3)$$

a result which is true in either of the above limits. At the tree level, $C=1$ and the equivalence theorem holds, but with quantum corrections, the value of C depends on the choice of the renormalization constant Z_ρ.

We show that it is always possible to *define* Z_ρ so that $C=1$; however, one would rather choose a prescription *a priori,* and check that the prescription preserves the theorem. In Sec. IV we prove that in limit (1) the theorem is indeed preserved if we use a momentum subtraction scheme at a scale $m \ll M_H$. This is just the prescription suggested by the physical arguments. It is also the prescription used by Yao and Yuan in their verification of the equivalence theorem at one loop.[6] Thus, using this prescription in the limit appropriate to heavy-Higgs-boson studies, the equivalence theorem is correct to lowest nonzero order in g and to all orders λ (Ref. 7). This is of critical importance to any calculation which uses the equivalence theorem including radiative corrections.[8]

Section V contains our conclusions. We extend the theorem to include the standard model by adding an extra U(1) gauge boson, as well as fermions. We also examine the gauge dependence of the theorem, and suggest that, despite appearances, the theorem in limit (1) does not rely on the use of the R_ξ gauge.

II. PHYSICAL INTERPRETATIONS

In this section we will try to understand the origin of the equivalence theorem by making physical interpretations of the quantities on both sides of Eq. (1.1). Certainly, this should not take the place of the rigorous arguments which we will supply in the next two sections. However, it will serve as a useful guide, indicating where subtle points will arise in the rigorous arguments.

Let us first discuss limit (1), where $g^2/\lambda \to 0$ with M_H^2/s fixed. In this limit, $S[W_L\text{'s, physical}]$ is an S-matrix element of a spontaneously broken gauge theory at very weak gauge coupling. The longitudinal vector bosons are physical particles, while the Goldstone bosons are unphysical particles, included to ensure renormalizability. At $g=0$, $S[\rho\text{'s, physical}]$ is an S-matrix element in a theory where the Goldstone bosons are completely decoupled from the gauge sector. In this theory the Goldstone bosons are physical particles, while the longitudinal polarizations of the vector bosons are unphysical, included to maintain manifest covariance. Now, if we assume that the physical observables of these two theories are the same in the $g \to 0$ limit, we must identify the physical longitudinal vector bosons of the first theory with the physical Goldstone bosons of the second. This implies (1.1) at $g=0$. We shall see in the next two sections that (1.1) also holds to lowest nonzero order in g for those amplitudes which vanish at $g=0$. This might be expected because to lowest order in g there are no gauge bosons in loops, so they can be treated as purely classical fields.

This analysis also suggests the proper renormalization procedure for the Goldstone bosons. Since at $g=0$, $S[\rho\text{'s, physical}]$ can be interpreted as an S-matrix element in a theory with physical Goldston bosons, the theorem should hold if we use a renormalization prescription which preserves this interpretation. Such a prescription is given by momentum subtraction at a scale m, where $m \to 0$ as $g^2/\lambda \to 0$. In Sec. IV we shall see that this physically sensible renormalization for the Goldstone bosons ensures the validity of the theorem.

The version (2) of the equivalence theorem is relevant to the limit where $M_H^2/s \to 0$. This is the high-energy limit, where hard-scattering processes do not feel any effects of the vacuum expectation value. In this regime, $S[W_L\text{'s, physical}]$ is an S-matrix element of a spontaneously broken gauge theory, in the limit of vanishing vacuum expectation value (VEV). The longitudinal vector bosons are physical particles, while the Goldstone bosons are unphysical. In contrast, we can think of $S[\rho\text{'s, physical}]$ as an S-matrix element in an unbroken gauge theory, in the limit where all scalar masses are zero. In this theory, the Goldstone bosons are physical particles, while the longitudinal polarization of the vector bosons are unphysical. As above, one might expect that the first theory (in the limit of zero VEV) and the second theory (in the limit of zero mass) should produce the same S-matrix elements. At the tree level, where the bare states are the asymptotic states, this intuition is indeed correct, and the theorem holds. With radiative corrections, however, the situation is more subtle: The renormalization of the asymptotic states depends on whether or not the symmetry is broken. Nevertheless, we shall see in the next section that the equivalence theorem does hold up to a constant wave-function renormalization, at least in the R_ξ gauge. However, we have no clue as to which renormalization prescription will ensure that the constant is one.

III. RIGOROUS ARGUMENTS

Although the physical arguments of the last section are helpful in understanding why the equivalence theorem should work, they are no substitute for a careful analysis. Therefore, it is necessary to examine the theorem from a more rigorous point of view. For simplicity, we consider an SU(2) gauge theory, spontaneously broken by the vacuum expectation value of an ordinary Higgs doublet.

We take our Lagrangian to be

$$\mathcal{L}_{\text{total}} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{ghost}} , \quad (3.1)$$

written in terms of bare fields and parameters. The gauge field Lagrangian $\mathcal{L}_{\text{gauge}}$ contains the standard SU(2) kinetic energy. The scalar Lagrangian is simply

$$\mathcal{L}_{\rm scalar}=|(\partial^\mu-ig_0T^aW_0^{a\mu})\phi_0|^2-\lambda_0(|\phi_0|^2-u_0^2/2)^2 . \tag{3.2}$$

After shifting the Higgs field, we use the replacement

$$\phi_0=\frac{1}{\sqrt{2}}\begin{bmatrix}-\rho_0^2-i\rho_0^1\\ H_0+v_0+i\rho_0^3\end{bmatrix} , \tag{3.3}$$

where $v_0/\sqrt{2}$ is the full vacuum expectation value of the field ϕ_0, obtained by minimizing the quantum effective potential. Note that $v_0\neq u_0$.

The gauge-fixing Lagrangian is given by

$$\mathcal{L}_{\rm gf}=-\frac{1}{2\xi_0}(G_0^aG_0^a) , \tag{3.4}$$

where G_0^a denotes the gauge-fixing condition, and ξ_0 is the gauge-fixing parameter. The R_ξ gauge is defined by

$$G_0^a=\partial^\mu W_{0\mu}^a-\tfrac{1}{2}g_0\xi_0z_0\rho_0^a , \tag{3.5}$$

where z_0 is completely arbitrary. Usually, z_0 is chosen so that $\langle W_{0\mu}^a(-k)\rho_0^b(k)\rangle=0$ at the tree level.[9] We will briefly discuss gauge dependence in Sec. V.

The ghost Lagrangian $\mathcal{L}_{\rm ghost}$ is just

$$\mathcal{L}_{\rm ghost}=-g_0\eta_0^{*a}(\delta G_0^a/\delta\theta^b)\eta_0^b , \tag{3.6}$$

where $\delta G_0^a/\delta\theta^b$ is the change of the gauge-fixing condition under a gauge transformation parametrized by θ^b. For our particular gauge condition, this becomes

$$\mathcal{L}_{\rm ghost}=-\eta_0^{*a}\partial^\mu(\partial_\mu\eta_0^a+g_0\epsilon^{abc}W_{0\mu}^b\eta_0^c) +\tfrac{1}{4}g_0^2\xi_0z_0\eta_0^{*a}[\epsilon^{abc}\rho_0^b\eta_0^c-(H_0+v_0)\eta_0^a] . \tag{3.7}$$

In general there is mixing between the vector-boson fields W_μ^a and the Goldstone fields ρ^a. For this reason it is useful to define a five-dimensional field $\mathbf{W}_{0M}^a$ as a notational convenience,[4] where $\mathbf{W}_{0M}^a=(W_{0\mu}^a,\rho_0^a)$. The five-dimensional mixed propagator[10] is then

$$\mathbf{D}_{0,MN}^{ab}(k)=\begin{bmatrix}D_{0WW,\mu\nu}^{ab}(k) & D_{0W\rho,\mu}^{ab}(k)\\ D_{0\rho W,\nu}^{ab}(k) & D_{0\rho\rho}^{ab}(k)\end{bmatrix} . \tag{3.8}$$

In this notation, the R_ξ gauge condition can be written as $G_0^a(k)=\mathbf{K}^M\mathbf{W}_{0M}^a(k)$, where $\mathbf{K}^M=(ik^\mu,-g_0\xi_0z_0/2)$.

Now we are ready to study the equivalence theorem. For the moment, we restrict our attention to the case of a single longitudinal vector boson. We begin by noting that the Lagrangian (3.1) is invariant under the set of Becchi-Rouet-Stora-Tyutin (BRST) transformations given in the Appendix, and that the physical states are defined to be BRST invariant. For the case at hand, this reduces to the statement

$$\langle 0|G_0^a(k)|\text{physical}\rangle\equiv\mathbf{K}^M\langle 0|\mathbf{W}_{0M}^a(k)|\text{physical}\rangle=0 , \tag{3.9}$$

where "physical" refers to any physical state, such as the Higgs and longitudinal or transverse gauge bosons.[4,7,11]

To relate this to the S matrix, we extract the five-dimensional propagator and evaluate the residue on the W mass shell. Extracting the propagator gives

$$\mathbf{D}_{0,NM}^{ab}(k)\mathbf{K}^M\langle\mathbf{W}_0^{bN}(k)|\text{physical}\rangle=0 . \tag{3.10}$$

We can now use the Ward identity (A7) from the Appendix to find

$$\mathbf{X}_N^{ab}(k)\langle\mathbf{W}_0^{bN}(k)|\text{physical}\rangle=0 , \tag{3.11}$$

where we set $k^2=M_W^2$, and the five-dimensional vector $\mathbf{X}_N^{ab}$ is defined in Eqs. (A8) and (A9).[12]

The next step in proving the theorem is to replace the bare parameters with their physically renormalized counterparts. To do this we define the wave-function renormalizations

$$\begin{aligned}&W_{0\mu}^a=Z_W^{1/2}W_\mu^a,\quad \rho_0^a=Z_\rho^{1/2}\rho^a ,\\ &H_0=Z_H^{1/2}H,\quad \eta_0^a=Z_\eta^{1/2}\eta^a ,\end{aligned} \tag{3.12}$$

and coupling-constant renormalizations,[13]

$$\begin{aligned}&g_0=g\mu^\epsilon(1+\delta_g) ,\\ &\xi_0=Z_W\xi ,\\ &\tfrac{1}{2}g_0v_0=M_W(1+\delta_{M_W}) ,\\ &\tfrac{1}{2}g_0z_0=M_W(1+\delta_z) ,\\ &2\lambda_0v_0^2=M_H^2(1+\delta_{M_H}+\delta_v) ,\\ &2\lambda_0u_0^2=M_H^2(1+\delta_{M_H}+3\delta_v) .\end{aligned} \tag{3.13}$$

The parameter μ is the renormalization scale introduced for purposes of dimensional regularization, with the dimension $d=4-2\epsilon$. For ease of notation, we also introduce the parameters Δ_1, Δ_2, and Δ_3:

$$\begin{aligned}&\Delta_1\delta^{ab}=\frac{g\mu^\epsilon}{2M_W}\int\frac{d^{4-2\epsilon}}{(2\pi)^{4-2\epsilon}}\langle 0|H(-k-q)\eta^a(q)|\eta^{*b}(k)\rangle ,\\ &\Delta_2\delta^{ab}=-\frac{g\mu^\epsilon}{2M_W}\epsilon^{acd}\int\frac{d^{4-2\epsilon}q}{(2\pi)^{4-2\epsilon}}\times\langle 0|\rho^c(-k-q)\eta^d(q)|\eta^{*b}(k)\rangle ,\\ &\Delta_3\delta^{ab}ik_\mu=-g\mu^\epsilon\epsilon^{acd}\int\frac{d^{4-2\epsilon}q}{(2\pi)^{4-2\epsilon}}\times\langle 0|W_\mu^c(-k-q)\eta^d(q)|\eta^{*b}(k)\rangle ,\end{aligned} \tag{3.14}$$

where the antighost states are amputated, and we take $k^2=M_W^2$. In defining Δ_3, we use the fact that k_μ is the only available four-vector. It is easy to see that the parameters (3.14) vanish at the tree level. At one loop, they are given by the Feynman diagrams shown in Fig. 1.

We now substitute (3.12)–(3.14) into (3.11), and divide through by the ghost propagator, to find

$$-ik^\mu\langle W_\mu^a|\text{physical}\rangle=CM_W\langle\rho^a|\text{physical}\rangle , \tag{3.15}$$

where

$$C=(Z_W/Z_\rho)^{1/2}\times\left[\frac{1+\delta_{M_W}+\Delta_1Z_H^{1/2}(1+\delta_g)+\Delta_2Z_\rho^{1/2}(1+\delta_g)}{1+\Delta_3Z_W^{1/2}(1+\delta_g)}\right] . \tag{3.16}$$

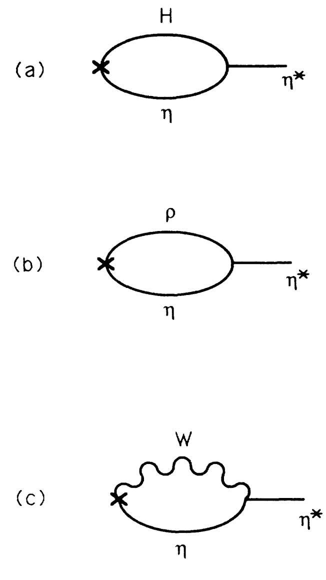

FIG. 1. The one-loop Feynman diagrams associated with the parameters (a) Δ_1, (b) Δ_2, and (c) Δ_3.

Finally, we obtain our desired result: Using (3.15) and the fact that the longitudinal-polarization vector satisfies $\epsilon_L^\mu(k) \to k^\mu / M_W$ as $M_W^2/s \to 0$, we find

$$S[W_L\text{'s, physical}] = (iC)^n \times S[\rho\text{'s, physical}] , \tag{3.17}$$

as $M_W^2/s \to 0$. This obviously follows for the case of a single longitudinal vector boson. For the more subtle case of more than one longitudinal vector boson, the arguments of Ref. 4 can be used to derive (3.17) from a generalization of (3.15), in each of the limits discussed in the previous section.

Thus, the theorem is valid if $C=1$. At the tree level, this is obviously the case. At higher loops, the theorem depends on the renormalization constants introduced above. Most of these constants are fixed. The parameters Δ_1, Δ_2, and Δ_3 are derived quantities that cannot be varied. The parameter δ_{M_W} is determined by the fact that M_W is the physical W mass. The parameters Z_W, Z_H, and δ_g are determined by the fact that we are considering S-matrix elements for the scattering of physical W's and H's. The only free parameter is Z_ρ, the wave-function renormalization of the unphysical Goldstone bosons. Since this parameter is not determined by any physical argument, it can be *defined* so that $C=1$. When this is done, the equivalence theorem holds in both of the limits discussed above.

IV. HEAVY-HIGGS-BOSON LIMIT

We have just seen that a renormalization condition exists for the unphysical Goldstone bosons which guarantees that the equivalence theorem holds in the presence of radiative corrections. However, this involves defining Z_ρ order by order in perturbation theory, using (3.16) with $C=1$. It would be better if we could find a simple renormalization condition which automatically ensures that $C=1$.

In the M_W^2/s, $M_H^2/s \to 0$ limit of the equivalence theorem, the ratio $g^2/\lambda \sim 1$. This implies that we must set $C=1$ to all orders in g^2 and λ. As discussed in Sec. II, we know of no physical renormalization prescription that enforces this condition.

In the limit M_W^2/s, $g^2/\lambda \to 0$, however, we only need $C=1$ to all orders in λ. Previously, we suggested that a reasonable prescription might be momentum subtraction at a scale m, with $m \to 0$ as $g \to 0$. In this section we shall see that this prescription gives $C=1$ in the limit $g \to 0$.

The first point to note is that $\Delta_1 \sim g^2$. From Eq. (3.14) and Fig. 1(a), is it easy to see that evaluating the Green's function at any order requires at least one insertion of the $H\eta^{*a}\eta^a$ vertex. Thus,

$$\Delta_1 \sim (g\xi M_W)\left[\frac{g}{M_W}\right] \sim g^2\xi . \tag{4.1}$$

At higher loops, there may be additional factors of g^2 and λ (as well as logarithms), but there are no poles in g^2. Similar arguments can be used to show that $\Delta_2 \sim g^2\xi$ and $\Delta_3 \sim g^2$. Hence, the terms can all be neglected.

We can also neglect the radiative corrections to Z_W. To see this, note that any wave-function renormalization diagram must have two factors of g, one from each of the external vector bosons. A typical one-loop diagram is shown in Fig. 2(a). This yields a leading contribution of the form $g^2 g_{\mu\nu} k^2$. Again there are additional factors of g^2 and λ at higher loops, as well as logarithms. However, these terms all give at most $O(g^2)$ corrections to Z_W, and can therefore be ignored.

We cannot, however, neglect the vector-boson mass renormalization that comes from Figs. 2(a) and 2(b). It is of the form $g^2 g^{\mu\nu} M_H^2 \sim \lambda g^{\mu\nu} M_W^2$, which implies that there are strong corrections to δ_{M_W}. Strong corrections to Z_ρ also occur, as shown by the one-loop diagram of Fig. 3. Thus, we find

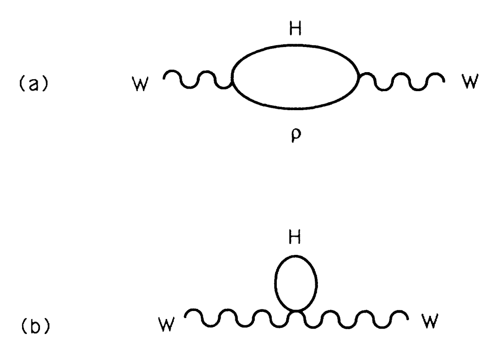

FIG. 2. The one-loop contributions to the W inverse propagator, to lowest order in g^2. Figures (a) and (b) contribute to mass renormalization, while only (a) is relevant to wave-function renormalization.

$$C=(1+\delta_{M_W})Z_\rho^{-1/2}+O(g^2)\ , \tag{4.2}$$

to all orders in λ.

To compute the value of C, we will use a Ward identity, valid to $O(g^2)$, which relates the W and ρ inverse propagators. To obtain the identity, we need the effective action. We start by writing the generating functional in terms of the bare fields:[14]

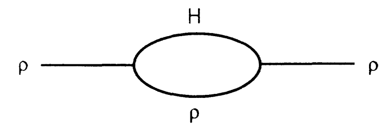

FIG. 3. The one-loop contribution to the wave-function renormalization of the ρ inverse propagator, to lowest order in g^2.

$$Z_{\text{total}}[J_W^\mu,J_\rho,J_H]=-i\ln\int[\mathcal{D}\rho_0][\mathcal{D}H_0]\exp\left\{i\int d^4x\left[\mathcal{L}_{\text{scalar}}(\phi_0,W_0)+\mathcal{L}_{\text{gauge}}(W_0)-\frac{1}{2\xi_0}(\partial^\mu W_{0\mu}^a)^2+\tfrac{1}{2}g_0z_0\partial^\mu W_{0\mu}^a\rho_0^a+J_W^{a\mu}W_{0\mu}^a+J_\rho^a\rho_0^a+J_HH_0\right]\right\}\ , \tag{4.3}$$

where ϕ_0 is defined in Eq. (3.3). In (4.3), we integrate over ρ_0 and H_0 only, for graphs with internal gauge or ghost loops are suppressed by powers of g^2. We neglect the ρ mass since its effects are also suppressed by $O(g^2)$. The effective action is given by the Legendre transform of Z_{total},

$$\Gamma_{\text{total}}[W_0,\rho_0,H_0]=\Gamma_{\text{scalar}}[W_0,\rho_0,H_0+v_0]+\int d^4x\left[\mathcal{L}_{\text{gauge}}(W_0)-\frac{1}{2\xi_0}(\partial^\mu W_{0\mu}^a)^2+\tfrac{1}{2}g_0z_0\partial^\mu W_{0\mu}^a\rho_0^a\right]\ , \tag{4.4}$$

where Γ_{scalar} is the Legendre transform of Z_{scalar}, the generating functional for an unshifted scalar theory coupled to a background gauge field:

$$Z_{\text{scalar}}[J_\rho,J_H]=-i\ln\int[\mathcal{D}\rho_0][\mathcal{D}H_0]\times\exp\left\{i\int d^4x[\mathcal{L}_{\text{scalar}}(\phi_0,W_0)+J_\rho^a\rho_0^a+J_H(H_0+v_0)]\right\}\ . \tag{4.5}$$

The effective action Γ_{scalar} is manifestly gauge invariant. Using this, it is straightforward to derive the Ward identity

$$k^\mu k^\nu\Gamma_{0W,\mu\nu}^{ab}(k^2)=\tfrac{1}{4}(g_0v_0)^2\Gamma_{0\rho}^{ab}(k^2)-\frac{ik^4}{\xi_0}\delta^{ab}\ , \tag{4.6}$$

where $\Gamma_{0W,\mu\nu}^{ab}(k^2)$ and $\Gamma_{0\rho}^{ab}(k^2)$ are the inverse propagators of the bare W and ρ, respectively. In terms of renormalized quantities, this becomes

$$\begin{aligned}k^\mu k^\nu\Gamma_{W,\mu\nu}^{ab}(k^2)&=M_W^2Z_\rho^{-1}(1+\delta_{M_W})^2\Gamma_\rho^{ab}(k^2)-\frac{ik^4}{\xi}\delta^{ab}\\&=M_W^2C^2\Gamma_\rho^{ab}(k^2)-\frac{ik^4}{\xi}\delta^{ab}\ .\end{aligned} \tag{4.7}$$

We will now evaluate C, using (4.7), the physical properties of the W inverse propagator, and the momentum subtraction condition for the ρ inverse propagator. At low energies, the W inverse propagator can be written

$$\Gamma_{W,\mu\nu}^{ab}(k^2)=-i\delta^{ab}\left\{g_{\mu\nu}[k^2-M_W^2-g^2M_H^2F_1(k^2/M_H^2)]-k_\mu k_\nu\left[1-\frac{1}{\xi}+g^2F_2(k^2/M_H^2)\right]\right\}\ , \tag{4.8}$$

to $O(g^2)$. The dimensionless functions F_1 and F_2 are analytic for $k^2\ll M_H^2$, and can be expanded in power series:

$$\begin{aligned}F_1(k^2/M_H^2)&=a_0+a_1(k^2/M_H^2)+a_2(k^2/M_H^2)^2+\cdots\ ,\\F_2(k^2/M_H^2)&=b_0+b_1(k^2/M_H^2)+b_2(k^2/M_H^2)^2+\cdots\ ,\end{aligned} \tag{4.9}$$

where the coefficients a_i and b_i depend implicitly on λ and $\ln(M_H/\mu)$. The physical mass condition, $F_1(M_W^2/M_H^2)=0$ to $O(g^2/\lambda)$, implies that $a_0=0$.

If we now substitute (4.8) and (4.9) into (4.7), we obtain

$$\Gamma_\rho^{ab}(k^2)C^2=i\delta^{ab}\{k^2+(g^2k^4/M_W^2)[(a_1+b_0)+(a_2+b_1)(k^2/M_H^2)+\cdots]\}\ . \tag{4.10}$$

The momentum subtraction condition for wave-function renormalization at a scale m is

$$\left.\frac{\partial\Gamma_\rho^{ab}(k^2)}{\partial k^2}\right|_{k^2=m^2}=i\delta^{ab}\ . \tag{4.11}$$

Using this condition on (4.10) gives us

$$C^2=1+(g^2m^2/M_W^2)[2(a_1+b_0)+3(a_2+b_1)(m^2/M_H^2)+\cdots]\ . \tag{4.12}$$

Thus, if we choose the subtraction at the physically reasonable scale $m^2\sim M_W^2\ll M_H^2$, we obtain the desired result

$$C^2=1+O(g^2)\ . \tag{4.13}$$

V. CONCLUSIONS

In this paper we have seen that the equivalence theorem can be considered in two distinct limits, depending on the ratio g^2/λ. The two limits have different physical interpretations, which are very helpful in elucidating the fine points of the theorem. In both limits, we have found the theorem to hold to within a constant factor for an SU(2) gauge theory in R_ξ gauge. This factor, given in Eq. (3.16), can be set to one by an appropriate choice of Z_ρ. Furthermore, in the heavy-Higgs-boson regime, where $g^2/\lambda \to 0$, we have shown that this factor approaches unity when the unphysical Goldstone bosons are renormalized using a momentum subtraction scheme at a scale $m \ll M_H$. In this case, the equivalence theorem is true to lowest nonzero order in g and to all orders in λ.[15]

The simplicity of scalar amplitudes makes the equivalence theorem a useful tool for calculating cross sections in the heavy-Higgs-boson limit. However, since the full symmetry of the electroweak interactions is SU(2)×U(1), it is necessary to add a U(1) gauge boson. Certainly, the physical arguments of Sec. II do not depend on the symmetry group of the theory, so the equivalence theorem should still hold. The rigorous arguments require a straightforward extension of what we have done. Using the rediagonalized physical gauge fields, Z^μ and A^μ, it is easy to derive the analog of (3.15) from the BRST transformations associated with the broken generators. Then, because we are neglecting all gauge-boson loops, the arguments of Sec. IV carry through for the SU(2)×U(1) theory.

Of course, the standard model also contains fermions, so we must examine their effects. When fermions are included, there is an extra perturbation expansion in the Yukawa coupling $f \equiv gM_F/2M_W$. As might be expected, the statement of the theorem depends on the ratio g/f. When $g/f \ll 1$, arguments similar to those of Sec. IV can be used to show that the theorem is true in the heavy-Higgs-boson limit to lowest nonzero order in g and to all orders in λ and f, provided the Goldstone bosons are renormalized at a scale $m^2 \ll M_F^2$. On the other hand, when $g/f \sim 1$, the theorem is correct to lowest nonzero order in g and f, and to all orders in λ.

Finally, we note that the theorem in the heavy-Higgs-boson limit does not require use of the R_ξ gauge. The physical arguments of Sec. II do not depend on the gauge, except for the requirement that the unphysical scalars become Goldstone bosons in the limit $g^2/\lambda \to 0$. More formally, the arguments of Secs. III and IV hold for any "axial" gauge condition of the form $G_0^a(k) = \mathbf{N}^M \mathbf{W}_{0M}^a(k)$, where $\mathbf{N}_M$ is a five-vector of the form $\mathbf{N}_M = (n_\mu, g_0 n_5)$, with n_μ and n_5 arbitrary.[16] The steps that lead to (3.11) follow from the BRST transformations (A1), and do not depend on the choice of gauge. The equivalence theorem is a direct result of the gauge invariance of the physical theory.

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APPENDIX: WARD-TAKAHASHI IDENTITIES

In this appendix we will derive a set of Ward identities for an SU(2) gauge theory in terms of the bare fields and parameters. The Lagrangian is given in Eq. (3.1). It is invariant under the following BRST transformations of the fields in momentum space:

$$\begin{aligned}
\delta W_{0\mu}^a(k) &= \frac{1}{g_0} i k_\mu \eta_0^a(k) \\
&\quad + \epsilon^{abc} \int \frac{d^4q}{(2\pi)^4} W_{0\mu}^b(k-q)\eta_0^c(q) , \\
\delta \rho_0^a(k) &= \tfrac{1}{2}\epsilon^{abc} \int \frac{d^4q}{(2\pi)^4} \rho_0^b(k-q)\eta_0^c(q) - \tfrac{1}{2} v_0 \eta_0^a(k) \\
&\quad - \tfrac{1}{2} \int \frac{d^4q}{(2\pi)^4} H_0(k-q)\eta_0^a(q) , \\
\delta H_0(k) &= \tfrac{1}{2} \int \frac{d^4q}{(2\pi)^4} \rho_0^a(k-q)\eta_0^a(q) , \\
\delta \eta_0^a(k) &= -\tfrac{1}{2}\epsilon^{abc} \int \frac{d^4q}{(2\pi)^4} \eta_0^b(k-q)\eta_0^c(q) , \\
\delta \eta_0^{*a}(k) &= -\frac{1}{g_0 \xi_0} G_0^a(k) .
\end{aligned} \tag{A1}$$

The Ward identities we desire are obtained from the equation

$$\begin{aligned}
\Delta Z(J_i) &= \sum_j \int \frac{d^4q}{(2\pi)^4} J_j(-q) \langle 0|\delta\Phi_j(q)|0\rangle_{J_i} \\
&= 0 .
\end{aligned} \tag{A2}$$

The expectation value is taken in the presence of the sources J_i, one for each of the fields Φ_i above. To obtain our first identity, we differentiate (A2) as follows:

$$\left. \frac{\delta^2 \Delta Z(J_i)}{\delta J_W(-p)\delta J_{\eta^*}(-k)} \right|_{J_i=0} = 0 . \tag{A3}$$

This gives us

$$\begin{aligned}
\langle 0| W_{0\mu}^a(-k) G_0^b(k)|0\rangle
&= -ik_\mu \xi_0 \langle 0|\eta_0^a(-k)\eta_0^{*b}(k)|0\rangle + g_0\xi_0\epsilon^{acd} \\
&\quad \times \int \frac{d^4q}{(2\pi)^4} \langle 0| W_{0\mu}^c(-k-q)\eta_0^d(q)\eta_0^{*b}(k)|0\rangle .
\end{aligned} \tag{A4}$$

Similarly,

$$\left. \frac{\delta^2 \Delta Z(J_i)}{\delta J_\rho(-p)\delta J_{\eta^*}(-k)} \right|_{J_i=0} = 0 . \tag{A5}$$

yields

$$\langle 0|\rho_0^a(-k)G_0^b(k)|0\rangle = -\tfrac{1}{2}g_0\xi_0 v_0\langle 0|\eta_0^a(-k)\eta_0^{*b}(k)|0\rangle - \tfrac{1}{2}g_0\xi_0\int\frac{d^4q}{(2\pi)^4}\langle 0|H_0(-k-q)\eta_0^a(q)\eta_0^{*b}(k)|0\rangle + \tfrac{1}{2}g_0\xi_0\epsilon^{acd}\int\frac{d^4q}{(2\pi)^4}\langle 0|\rho_0^c(-k-q)\eta_0^d(q)\eta_0^{*b}(k)|0\rangle \ . \quad (A6)$$

Equations (A4) and (A6) are the Ward identities that we need. If we use the five-dimensional notation of Sec. III and we restrict to R_ξ gauge, these equations can be combined into the single equation[12]

$$\mathbf{D}_{0,MN}^{ab}(k)\mathbf{K}^N = \mathbf{X}_M^{ab}(k) \ , \quad (A7)$$

where $\mathbf{K}^N$ and $\mathbf{D}_{0,MN}^{ab}$ are defined in (3.8),

$$\mathbf{X}_\mu^{ab} = -ik_\mu\xi_0\langle 0|\eta_0^a(-k)\eta_0^{*b}(k)|0\rangle + g_0\xi_0\epsilon^{acd}\int\frac{d^4q}{(2\pi)^4}\langle 0|W_{0\mu}^c(-k-q)\eta_0^d(q)\eta_0^{*b}(k)|0\rangle \ , \quad (A8)$$

and

$$\mathbf{X}_5^{ab} = -\tfrac{1}{2}g_0\xi_0 v_0\langle 0|\eta_0^a(-k)\eta_0^{*b}(k)|0\rangle - \tfrac{1}{2}g_0\xi_0\int\frac{d^4q}{(2\pi)^4}\langle 0|H_0(-k-q)\eta_0^a(q)\eta_0^{*b}(k)|0\rangle + \tfrac{1}{2}g_0\xi_0\epsilon^{acd}\int\frac{d^4q}{(2\pi)^4}\langle 0|\rho_0^c(-k-q)\eta_0^d(q)\eta_0^{*b}(k)|0\rangle \ . \quad (A9)$$

As a check, we have verified that the equivalent Ward identities in an Abelian theory are satisfied at one loop.

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[9]Three obvious choices for z_0 come to mind: One could choose $z_0 = u_0$ or $z_0 = v_0$, or one could choose z_0 so that $\langle W_{0\mu}^a(-k)\rho_0^b(k)\rangle = 0$ to all orders in perturbation theory, for some particular value of k^2. This last choice is similar to that used by Yao and Yuan (Ref. 6).

[10]In Landau gauge, where $\xi_0 = 0$, the off-diagonal elements are zero,[8] but this is not necessary for the validity of the theorem.

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[12]This relation differs from that derived in Ref. 4 by the addition of the last term in (A8) and the last two terms in (A9).

[13]This is a generalization of the method used in J. van der Bij and M. Veltman, Nucl. Phys. **B231**, 205 (1984).

[14]The arguments that follow are similar to those used by Einhorn and Wudka in their discussion of the "screening theorem:" M. Einhorn and J. Wudka, Phys. Rev. D **39**, 2758 (1989).

[15]Because the theorem holds order by order in λ, but not diagram by diagram, it is necessary to include *all* diagrams of a given order in λ. Thus there is no reason to expect the theorem to hold when using a propagator with a Breit-Wigner width, which is equivalent to summing a subset of diagrams of higher order in λ. Indeed, this expectation is confirmed by explicit calculations in Ref. 5.

[16]In limit (2), with $M_H^2/s \to 0$, this is not necessarily the case: there are terms proportional to n_μ, suppressed by g^2, that potentially modify (3.15).